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Research Article

New Contractive Mappings and Their Fixed Points in Branciari Metric Spaces

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In this paper, we introduce the notion of generalized $\mathscr{L}$-contractions which enlarge the class of $\mathscr{L}$-contractions initiated by Cho in 2018. Thereafter, we also, define the notion of $\mathscr{L}^*$-contractions. Utilizing our newly introduced notions, we establish some new fixed-point theorems in the setting of complete Branciari's metric spaces, without using the Hausdorff assumption. Moreover, some examples and applications to boundary value problems of the fourth-order differential equations are given to exhibit the utility of the obtained results.

1. Introduction

In 2000, Branciari [1] initiated the concept of a generalized metric by replacing the natural triangle inequality of a metric with a relatively more general inequality termed as rectangular (or quadrilateral) inequality which involves four points instead of three. In the literature, such a metric is known as the Branciari metric. Branciari [1] assumed that each of the Branciari metric space becomes a Hausdorff topological space, and the Branciari metric is a continuous function in each coordinate. Sarma et al. [2] and Samet [3] showed that these assumptions were not correct. Meanwhile, several authors obtained various fixed-point results in the Branciari metric space with the assumption that the space is a Hausdorff and (or) the Branciari metric is continuous. However, it was shown (e.g. [4, 5]) that in general, neither the Hausdorff topological property nor the continuity of the metric is required in the proofs. For a recent update or extension of the Branciari metric spaces, we refer to [6].

On the other hand, in 2014, Jleli and Samet [7] introduced the concept of Θ-contractions and proved some fixed-point results in the setting of the Branciari metric spaces. Very recently, Cho [8] introduced the notion of $\mathscr{L}$-contractions which unify several concepts of contractions in the existing literature including Θ-contractions in the setting of the Branciari metric spaces. For more notions and results in such spaces, we refer the reader to [9–17].

This paper is aimed at introducing two types of contraction mappings, namely, generalized $\mathscr{L}$-contractions and $\mathscr{L}^*$-contractions. For both types of contractions, we prove separate fixed-point theorems in the setting of the complete Branciari metric spaces. The obtained results extend, generalize, and improve some results of the existing literature. Moreover, applications to boundary value problems of the fourth-order are given to exhibit the utility of the obtained results.

2. Preliminaries

The following definitions and basic results are needed in the sequel.

Definition 1 [1]. Let X be a nonempty set and $d : X \times X \longrightarrow [0,\infty)$ a mapping such that for all $x, y \in X$ and all $z \neq w \in X/\{x, y\}$:

(BMS1) $d(x, y) = 0$, if and only if $x = y$

(BMS2) $d(x, y) = d(y, x)$

(BMS3) $d(x, y) \leq d(x, z) + d(z, w) + d(w, y)$.

The metric d is called a Branciari metric, and the pair (X, d) is called a Branciari metric space.

The next example shows that the topologies of the Branciari metric spaces and the usual metric spaces are different. In particular, we have the following:

(i) The Branciari metric need not be continuous in both variables

(ii) The Branciari metric space is not necessarily Hausdorff

(iii) An open ball need not be an open set

(iv) A convergent sequence is not necessarily a Cauchy sequence.

Example 2 [2]. Let $X = A \cup B$, where $A = \{0, 2\}$ and $B = \{(1/n): n \in \mathbb{N}\}$. Define $d : X \times X \rightarrow [0,\infty)$ as

$$d(x, y) = \begin{cases} 0, & \text{if } x = y, \\ 1, & \text{if } x \neq y \text{ and } \{x, y\} \subset A \text{ or } \{x, y\} \subset B, \\ y, & \text{if } x \in A \text{ and } y \in B, \\ x, & \text{if } x \in B \text{ and } y \in A. \end{cases} \quad (1)$$

Then (X, d) is a complete Branciari metric space. However, it is easy to see that

(i) $\lim_{n\to\infty} d((1/n),(1/2)) \neq d(0,(1/2))$ although $\lim_{n\to\infty} (1/n) = 0$, and hence the function d is not continuous

(ii) There does not exist $r > 0$ such that $B_r(0) \cap B_r(2) = \phi$, and hence the respective topology is not a Hausdorff

(iii) $B_{2/3}(1/3) = \{0, 2, (1/3)\}$; however, there does not exist $r > 0$ such that $B_r(0) \subseteq B_{2/3}(1/3)$, and hence an open ball need not be an open set

(iv) The sequence $\{1/n\}_{n\in\mathbb{N}}$ converges to both 0 and 2, and it is not a Cauchy sequence.

Definition 3 [7]. Let (X, d) be a Branciari metric space. A mapping $T : X \longrightarrow X$ is said to be a Θ-contraction if there exist $\Theta \in \Omega_{1,2,3}$ and $k \in (0, 1)$ such that (for all $x, y \in X$)

$$d(Tx, Ty) > 0 \Rightarrow \Theta(d(Tx, Ty)) \leq [\Theta(d(x, y))]^k, \quad (2)$$

where $\Omega_{1,2,3}$ is the family of all functions Θ: $(0,\infty) \rightarrow (1,\infty)$ which satisfy the following conditions:

(Θ_1) Θ is nondecreasing

(Θ_2) For each sequence $\{\alpha_n\} \subset (0,\infty)$, $\lim_{n\to\infty}\Theta(\alpha_n) = 1 \Leftrightarrow \lim_{n\to\infty}\alpha_n = 0^+$

(Θ_3) There exists $r \in (0, 1)$ and $l \in (0,\infty]$ such that $\lim_{\alpha\to 0^+}(\Theta(\alpha) - 1)/\alpha^r = l$.

Theorem 4 [7]. *Let (X, d) be a complete Branciari metric space and $T : X \longrightarrow X$ a Θ-contraction mapping. Then T has a unique fixed point.*

Imdad et al. [18] observed that this theorem can be proved without the condition (Θ_1). Also, Ahmad et al. [19] replaced the condition (Θ_3) by the following one:

(Θ_4)Θ is continuous.

Remark 5. It is known that every Θ-contraction mapping is continuous.

In the sequel, we adopt the following notations:

(i) $\Omega_{1,2,3}$ is the class of all functions Θ which satisfy (Θ_1)-(Θ_3)

(ii) $\Omega_{1,2,4}$ is the class of all functions Θ which satisfy (Θ_1), (Θ_2), and (Θ_4)

(iii) $\Omega_{1,2,3,4}$ is the class of all functions Θ which satisfy (Θ_1)-(Θ_4).

In 2018, Cho [8] introduced the notion of $\mathscr{L}$-contractions which unify several concepts of contractions in the existing literature including Θ-contractions as under one unifying concept:

Definition 6 [8]. Let (X, d) be a Branciari metric space. A mapping $T : X \longrightarrow X$ is said to be an $\mathscr{L}$-contraction with respect to $\zeta \in \mathscr{L}$ if there exists $\Theta \in \Omega_{1,2,4}$ such that (for all $x, y \in X$)

$$d(Tx, Ty) > 0 \Rightarrow \zeta[\Theta(d(Tx, Ty)), \Theta(d(x, y))] \geq 1, \quad (3)$$

where $\mathscr{L}$ is the class of all functions $\zeta : [1,\infty) \times [1,\infty) \longrightarrow \mathbb{R}$ which satisfy the following conditions (ζ_1^*):

(ζ_1^*) $\zeta(1, 1) = 1$

(ζ_2^*) $\zeta(t, s) < (s/t)$, for all $t, s > 1$

(ζ_3^*) If $\{t_n\}$ and $\{s_n\}$ are two sequences in $(1,\infty)$ with $t_n < s_n$, such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 1$, then $\limsup_{n\to\infty}\zeta(t_n, s_n) < 1$.

Example 7 [8]. Let $\zeta_k, \zeta_\psi : [1,\infty) \times [1,\infty) \longrightarrow \mathbb{R}$ be two functions defined as under:

(a) $\zeta_k(t,s) = (s^k/t)$, for all $t, s \geq 1$, where $k \in (0,1)$

(b) $\zeta_\psi(t,s) = (s/t\psi(s))$, for all $t, s \geq 1$, where $\psi : [1,\infty) \longrightarrow [1,\infty)$ is a lower semicontinuous and nondecreasing function with $\psi^{-1}(\{1\}) = 1$

Then $\zeta_k, \zeta_\psi \in \mathscr{L}$.

Example 8. Let $\zeta_{\psi,\varphi} : [1,\infty) \times [1,\infty) \longrightarrow \mathbb{R}$ be a function defined by

$$\zeta_{\psi,\varphi}(t,s) = \frac{\varphi(s)}{\psi(s)}, \quad \forall t, s \geq 1, \tag{4}$$

where $\psi, \varphi : [1,\infty) \longrightarrow [1,\infty)$ are upper semicontinuous from the right such that $\varphi(t) < t \leq \psi(t)$, for all $t > 1$. Then $\zeta_{\psi,\varphi} \in \mathscr{L}$.

Based on the above definition, the author in [8] proves the following theorem.

Theorem 9 [8]. *Let (X,d) be a complete Branciari metric space and $T : X \longrightarrow X$ an $\mathscr{L}$-contraction mapping. Then T has a unique fixed point.*

Remark 10. It is known that every $\mathscr{L}$-contraction mapping is continuous.

Remark 11. Let $\{a_n\}, \{b_n\}, \{c_n\}$ be three sequences of real numbers such that $\lim_{n\to\infty} a_n = a$, $\lim_{n\to\infty} b_n = b$, and $\lim_{n\to\infty} c_n = c$. Then

(i) $\lim_{n\to\infty} \max\{a_n, b_n, c_n\} = \max\{a, b, c\}$

(ii) $\lim_{n\to\infty} \min\{a_n, b_n, c_n\} = \min\{a, b, c\}$.

The following lemmas are useful in the sequel.

Lemma 12 [20]. *Let $\{x_n\}$ be a Cauchy sequence in a Branciari metric space (X,d) such that $x_n \neq x_m$ whenever $n \neq m$. Then $\{x_n\}$ can converge to at most one point.*

Lemma 13 [21]. *Let $\{x_n\}$ be a Cauchy sequence in a Branciari metric space (X,d) such that $\lim_{n\to\infty} d(x_n, x) = 0$, for some $x \in X$. Then, $\lim_{n\to\infty} d(x_n, y) = d(x, y)$, for all $y \in X$. In particular, $\{x_n\}$ does not converge to y if $y \neq x$.*

3. Fixed-Point Results for Generalized $\mathscr{L}$-Contractions

We begin this section by introducing the concept of generalized $\mathscr{L}$-contractions followed by the main result of this section as follows:

Definition 14. Let (X,d) be a Branciari metric space and $T : X \longrightarrow X$. Then T is said to be a generalized $\mathscr{L}$-contraction with respect to $\zeta \in \mathscr{L}$ if there exist $\Theta \in \Omega_{1,2,4}$ and constant $L \geq 0$ such that (for all $x, y \in X$)

$$d(Tx, Ty) > 0 \Rightarrow \zeta[\Theta(d(Tx,Ty)), \Theta(d(x,y) + LN(x,y))] \geq 1, \tag{5}$$

where $N(x,y) = \min\{d(x,Ty), d(y,Tx), (d(x,Tx) + d(y,Ty)/2)\}$.

Theorem 15. *Let (X,d) be a complete Branciari metric space and $T : X \longrightarrow X$. If T is a continuous generalized $\mathscr{L}$-contraction mapping with respect to $\zeta \in \mathscr{L}$, then it has a unique fixed point.*

Proof. Let x_0 be an arbitrary element of X and define a Picard sequence $\{x_n\} \subseteq X$ by $x_n = Tx_{n-1}$, for all $n \in \mathbb{N}$. If $d(x_{n_0}, x_{n_0+1}) = 0$, for some $n_0 \in \mathbb{N}$, then x_{n_0} is a fixed point of T and the proof is finished. Now, assume that $d(x_n, x_{n+1}) > 0$, for all $n \in \mathbb{N}$. Using the contractive condition (5) and (ζ_2^*), we have

$$\begin{aligned} 1 &\leq \zeta[\Theta(d(Tx_{n-1}, Tx_n)), \Theta(d(x_{n-1}, x_n) + LN(x_{n-1}, x_n))] \\ &= \zeta[\Theta(d(x_n, x_{n+1})), \Theta(d(x_{n-1}, x_n) + LN(x_{n-1}, x_n))] \\ &< \frac{\Theta(d(x_{n-1}, x_n) + LN(x_{n-1}, x_n))}{\Theta(d(x_n, x_{n+1}))}, \end{aligned} \tag{6}$$

which implies that

$$\Theta(d(x_n, x_{n+1})) < \Theta(d(x_{n-1}, x_n) + LN(x_{n-1}, x_n)), \quad \forall n \in \mathbb{N}. \tag{7}$$

Notice that

$$\begin{aligned} N(x_{n-1}, x_n) &= \min\left\{ d(x_{n-1}, Tx_n), d(x_n, Tx_{n-1}), \frac{d(x_{n-1}, Tx_{n-1}) + d(x_n, Tx_n)}{2} \right\} \\ &= \min\left\{ d(x_{n-1}, x_{n+1}), d(x_n, x_n), \frac{d(x_{n-1}, x_n) + d(x_n, x_{n+1})}{2} \right\} = 0. \end{aligned} \tag{8}$$

Hence, inequality (7) becomes

$$\Theta(d(x_n, x_{n+1})) < \Theta(d(x_{n-1}, x_n)), \quad \forall n \in \mathbb{N}, \tag{9}$$

which implies (in view of (Θ_1)) that

$$d(x_n, x_{n+1}) < d(x_{n-1}, x_n), \quad \forall n \in \mathbb{N}. \tag{10}$$

Therefore, the sequence $\{d(x_{n-1}, x_n)\}$ is decreasing and bounded below by 0. This insures the existence of a number $r \geq 0$ such that $\lim_{n\to\infty} d(x_{n-1}, x_n) = r$. Assume that $r \neq 0$, then it follows from (Θ_2) that

$$\lim_{n\to\infty} \Theta(d(x_{n-1}, x_n)) > 1. \tag{11}$$

Setting $t_n = \Theta(d(x_n, x_{n+1}))$ and $s_n = \Theta(d(x_{n-1}, x_n))$. In view of (9), (11), and (Θ_4), we have $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 1$ and $t_n < s_n$, for all $n \in \mathbb{N}$. Therefore, applying the condition (ζ_3^*), we deduce

$$1 \leq \limsup_{n\to\infty} \zeta(t_n, s_n) < 1, \tag{12}$$

which is a contradiction, and hence we must have

$$\lim_{n\to\infty} d(x_{n-1}, x_n) = 0. \tag{13}$$

Now, assume that $x_m = x_n$, for some $m > n$. Then, also $x_{m+1} = x_{n+1}$. Using (9), we get

$$\begin{aligned} \Theta(d(x_m, x_{m+1})) &< \Theta(d(x_{m-1}, x_m)) < \Theta(d(x_{m-2}, x_{m-1})) \\ &< \cdots < \Theta(d(x_n, x_{n+1})) = \Theta(d(x_m, x_{m+1})), \end{aligned} \tag{14}$$

which is a contradiction. Therefore, we conclude that $x_m \neq x_n$, for all $n \neq m$.

Next, we claim that the sequence $\{x_n\}$ is a Cauchy sequence in (X, d). On the contrary, assume that it is not Cauchy, then there exists an $\varepsilon > 0$ for which we can find two subsequences $\{x_{n_k}\}$ and $\{x_{m_k}\}$ of $\{x_n\}$ such that $n_k > m_k \geq k$, for all $k \in \mathbb{N}$ and

$$d(x_{m_k}, x_{n_k}) \geq \varepsilon. \tag{15}$$

Suppose that n_k is the least integer exceeding m_k satisfying inequality (15). Then, we have

$$d(x_{m_k}, x_{n_k-1}) < \varepsilon. \tag{16}$$

Using (15), (16), and the rectangular inequality, we get

$$\begin{aligned} \varepsilon \leq d(x_{m_k}, x_{n_k}) &\leq d(x_{m_k}, x_{n_k-2}) + d(x_{n_k-2}, x_{n_k-1}) \\ &+ d(x_{n_k-1}, x_{n_k}) < \varepsilon + d(x_{n_k-2}, x_{n_k-1}) + d(x_{n_k-1}, x_{n_k}). \end{aligned} \tag{17}$$

On taking the limit as $k \longrightarrow \infty$ and making use of (13), we obtain

$$\lim_{k\to\infty} d(x_{m_k}, x_{n_k}) = \varepsilon. \tag{18}$$

Employing the rectangular inequality once again, we get

$$\begin{aligned} d(x_{m_k}, x_{n_k}) &\leq d(x_{m_k}, x_{m_k-1}) + d(x_{m_k-1}, x_{n_k-1}) + d(x_{n_k-1}, x_{n_k}) \\ &\leq 2d(x_{m_k}, x_{m_k-1}) + d(x_{m_k}, x_{n_k}) + 2d(x_{n_k-1}, x_{n_k}). \end{aligned} \tag{19}$$

On letting $k \longrightarrow \infty$ and using (13) as well as (18), we get

$$\lim_{k\to\infty} d(x_{m_k-1}, x_{n_k-1}) = \varepsilon. \tag{20}$$

Now, using (5) and (ζ_2^*), we obtain

$$\begin{aligned} 1 &\leq \zeta\Big[\Theta(d(Tx_{m_k-1}, Tx_{n_k-1})), \Theta(d(x_{m_k-1}, x_{n_k-1}) \\ &\quad + LN(x_{m_k-1}, x_{n_k-1}))\Big] = \zeta\Big[\Theta(d(x_{m_k}, x_{n_k})), \\ &\quad \Theta(d(x_{m_k-1}, x_{n_k-1}) + LN(x_{m_k-1}, x_{n_k-1}))\Big] \\ &< \frac{\Theta(d(x_{m_k-1}, x_{n_k-1}) + LN(x_{m_k-1}, x_{n_k-1}))}{\Theta(d(x_{m_k}, x_{n_k}))}, \end{aligned} \tag{21}$$

where

$$N(x_{m_k-1}, x_{n_k-1}) = \min\left\{ d(x_{m_k-1}, x_{n_k}), d(x_{n_k-1}, x_{m_k}), \frac{d(x_{m_k-1}, x_{m_k}) + d(x_{n_k-1}, x_{n_k})}{2} \right\}. \tag{22}$$

Consequently, we deduce that

$$\begin{aligned} \Theta(d(x_{m_k}, x_{n_k})) &< \Theta\Big(d(x_{m_k-1}, x_{n_k-1}) \\ &\quad + LN(x_{m_k-1}, x_{n_k-1})\Big), \quad \forall k \in \mathbb{N}. \end{aligned} \tag{23}$$

Let $t_k = \Theta(d(x_{m_k}, x_{n_k}))$ and $s_k = \Theta(d(x_{m_k-1}, x_{n_k-1}) + LN(x_{m_k-1}, x_{n_k-1}))$. Then in view of Remark 11 and (23), we have $\lim_{k\to\infty} t_k = \lim_{k\to\infty} s_k > 1$ and $t_k < s_k, \forall k \in \mathbb{N}$. So, on using (ζ_3^*), we obtain

$$1 \leq \limsup_{k\to\infty} \zeta(t_k, s_k) < 1, \tag{24}$$

which is a contradiction. Therefore, $\{x_n\}$ must be a Cauchy sequence in (X, d). Since (X, d) is complete, then there exists $u \in X$ such that $\lim_{t\to\infty} x_n = u$, that is,

$$\lim_{n\to\infty} d(x_n, u) = 0. \tag{25}$$

As T is continuous, then we get that (due to (25))

$$\lim_{n\to\infty} d(x_{n+1}, Tu) = \lim_{n\to\infty} d(Tx_n, Tu) = 0, \tag{26}$$

that is, $\lim_{n\to\infty} x_{n+1} = Tu$. Using Lemma 13, we conclude that $u = Tu$, that is, u is a fixed point of T.

Finally, we show that the fixed point of the mapping T is unique. On the contrary, assume that there are two fixed points such that $d(u,z) = d(Tu,Tz) > 0$. From (5), we have

$$\begin{aligned} 1 &\leq \zeta[\Theta(d(Tu,Tz)),\Theta(d(u,z)+LN(u,z))] \\ &= \zeta[\Theta(d(u,z)),\Theta(d(u,z)+LN(u,z))] \\ &< \frac{\Theta(d(u,z)+LN(u,z))}{\Theta(d(u,z))}, \end{aligned} \quad (27)$$

where $N(u,z) = \min\{d(u,Tz), d(z,Tu), (d(u,Tu)+d(z,Tz)/2)\} = 0$.

This implies that

$$\Theta(d(u,z)) < \Theta(d(u,z)), \quad (28)$$

which is a contradiction. Then T has a unique fixed point.

Next, we furnish the following illustrative example which shows that Theorem 15 is a genuine extension of Theorems 4 and 9.

Example 16. Let $X = \{1,2,3,4,5\}$ and define a mapping $d: X\times X \longrightarrow [0,\infty)$ as

(i) $d(1,2) = d(1,4) = 4$

(ii) $d(1,5) = d(2,5) = d(3,5) = 1$

(iii) $d(2,4) = d(3,4) = d(1,3) = d(2,3) = d(4,5) = 2$

(iv) $d(x,x) = 0, \forall x\in X$ and $d(x,y) = d(y,x), \forall x,y \in X$

Then it is easy to check that (X,d) is a complete Branciari metric space which is not a metric space since

$$4 = d(1,2) > d(1,5) + d(5,2) = 2. \quad (29)$$

Consider the mapping $T: X\rightarrow X$ defined by

$$Tx = \begin{cases} 3, & \text{if } x = 1,2,3,5, \\ 1, & \text{if } x = 4. \end{cases} \quad (30)$$

Notice that T is neither a Θ-contraction nor an $\mathscr{L}$-contraction. Indeed, for $x = 3$ and $y = 4$, we have

$$d(3,4) = 2 = d(T3,T4) > 0. \quad (31)$$

So, in view of (ζ_2^*) and for any $\zeta \in \mathscr{L}$ and $\Theta \in \Omega_{1,2,4}$, we obtain

$$\Theta(2) = \Theta(d(T3,T4)) \leq [\Theta(d(3,4))]^k < \Theta(d(3,4)) = \Theta(2), \quad (32)$$

and

$$\zeta[\Theta(d(T3,T4)),\Theta(d(3,4))] < \frac{\Theta(d(3,4))}{\Theta(d(T3,T4))} = 1. \quad (33)$$

Therefore, Theorems 4 and 9 cannot be used here, but Theorem 15 is applicable. In fact, we have

$$d(Tx,Ty) = \begin{cases} 2, & \text{if } (x=4 \text{ and } y\neq 4) \text{ or } (x\neq 4 \text{ and } y = 4), \\ 0, & \text{otherwise}, \end{cases} \quad (34)$$

and hence, for all $x,y\in X$,

$$d(Tx,Ty) > 0 \Leftrightarrow (x=4 \text{ and } y\neq 4) \text{ or } (x\neq 4 \text{ and } \ \& y=4). \quad (35)$$

Now, we show that T is a generalized $\mathscr{L}$-contraction with respect to ζ_k, where $\zeta_k(t,s) = (s^k/t)$, for all $t,s\geq 1$, $L=2$, and $k=(1/2)$.

Define a function $\Theta: (0,\infty) \longrightarrow [1,\infty)$ by

$$\Theta(t) = e^t. \quad (36)$$

For all $x,y\in X$ with $d(Tx,Ty) > 0$, we have two cases:

Case 1. If $(x=4 \text{ and } y\neq 1 \text{ or } 4)$ or $(x\neq 1 \text{ or } 4 \text{ and } y=4)$, then we have

$$\begin{aligned} d(x,y) &= 2 = d(Tx,Ty), \\ N(x,y) &\geq 1. \end{aligned} \quad (37)$$

Hence,

$$\begin{aligned} &\zeta[\Theta(d(Tx,Ty)),\Theta(d(x,y)+LN(x,y))] \\ &\quad = \frac{[\Theta(d(x,y)+LN(x,y))]^k}{\Theta(d(Tx,Ty))} \geq \frac{e^{(1/2)(2+2)}}{e^2} = 1. \end{aligned} \quad (38)$$

Case 2. If $(x=4 \text{ and } y=1)$ or $(x=1 \text{ and } y=4)$, then we have

$$\begin{aligned} d(x,y) &= 4, \\ d(Tx,Ty) &= 2, \\ N(x,y) &= 0. \end{aligned} \quad (39)$$

Thus,

$$\begin{aligned} &\zeta[\Theta(d(Tx,Ty)),\Theta(d(x,y)+LN(x,y))] \\ &\quad = \frac{[\Theta(d(x,y)+LN(x,y))]^k}{\Theta(d(Tx,Ty))} = \frac{e^{(1/2)(4+0)}}{e^2} = 1. \end{aligned} \quad (40)$$

Therefore, all the hypotheses of Theorem 15 are satisfied, and hence T has a unique fixed point (namely $x=3$).

Taking $L=0$ in the contractive condition (5), Theorem 15 reduces to the following fixed-point result.

Corollary 17 [8]. *Let (X, d) be a complete Branciari metric space and $T : X \longrightarrow X$. If T is an $\mathscr{L}$-contraction with respect to $\zeta \in \mathscr{L}$, then T has a unique fixed point.*

Next, we present the following results, which seem new to the existing literature.

Corollary 18. *Let (X, d) be a complete Branciari metric space and let $T : X \longrightarrow X$ be continuous mapping. Suppose that there exist $\Theta \in \Omega_{1,2,4}$, $L \geq 0$ and $k \in (0, 1)$ such that (for all x, $y \in X$)*

$$d(Tx, Ty) > 0 \Rightarrow \Theta(d(Tx, Ty)) \leq [\Theta(d(x, y) + LN(x, y))]^k, \tag{41}$$

where $N(x, y) = \min\{d(x, Ty), d(y, Tx), (d(x, Tx) + d(y, Ty)/2)\}$. Then T has a unique fixed point.

Proof. Observe that T is a generalized $\mathscr{L}$-contraction with respect to $\zeta_k(t, s) = (s^k/t)$. Then, the result follows immediately from Theorem 15.

Corollary 19. *Let (X, d) be a Branciari metric space and let $T : X \longrightarrow X$ be continuous mapping such that (for all $x, y \in X$)*

$$\begin{aligned} d(Tx, Ty) > 0 \Rightarrow d(Tx, Ty) &\leq d(x, y) + LN(x, y) \\ &- \varphi(d(x, y) + LN(x, y)), \end{aligned} \tag{42}$$

where $N(x, y) = \min\{d(x, Ty), d(y, Tx), (d(x, Tx) + d(y, Ty)/2)\}$ and $\varphi : [0,\infty) \longrightarrow [0,\infty)$ is nondecreasing and lower semicontinuous such that $\varphi^{-1}(\{0\}) = 0$. Then T has a unique fixed point.

Proof. Let $\Theta(t) = e^t$, for all $t > 0$. From (42), we have

$$\begin{aligned} \Theta(d(Tx, Ty)) = e^{d(Tx,Ty)} &\leq e^{d(x,y)+LN(x,y)-\varphi(d(x,y)+LN(x,y))} \\ &= \frac{\Theta(d(x, y) + LN(x, y))}{e^{\varphi(d(x,y)+LN(x,y))}}, \end{aligned} \tag{43}$$

for all $x, y \in X$ with $d(Tx, Ty) > 0$.

Now, define $\varphi(t) = \ln(\psi(\Theta(t)))$, for all $t > 0$, where $\psi : [1,\infty) \longrightarrow [1,\infty)$ is nondecreasing and lower semicontinuous such that $\psi^{-1}(\{1\}) = 1$.

From (43), we have

$$\Theta(d(Tx, Ty)) \leq \frac{\Theta(d(x, y) + LN(x, y))}{\psi(\Theta(d(x, y) + LN(x, y))}. \tag{44}$$

Taking $\zeta(t, s) = (s/t\psi(s))$ and using (44), we have

$$\begin{aligned} 1 &\leq \frac{\Theta(d(x, y) + LN(x, y))}{\Theta(d(Tx, Ty))\psi(\Theta(d(x, y) + LN(x, y)))} \\ &= \zeta[\Theta(d(Tx, Ty)), \Theta(d(x, y) + LN(x, y))]. \end{aligned} \tag{45}$$

Therefore, all the requirements of Theorem 15 are satisfied and hence T has a unique fixed point.

4. Fixed-Point Results for $\mathscr{L}^*$-Contractions

Before presenting our main result of this section, we give the following definition.

Definition 20. Let (X, d) be a Branciari metric space and $T : X \longrightarrow X$. Then T is said to be an $\mathscr{L}^*$-contraction with respect to $\zeta \in \mathscr{L}$ if there exists $\Theta \in \Omega_{1,2,4}$ such that (for all x, $y \in X$)

$$d(Tx, Ty) > 0 \Rightarrow \zeta[\Theta(d(Tx, Ty)), \Theta(M(x, y))] \geq 1, \tag{46}$$

where $M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty)\}$.

Now, we are ready to state and prove the main result of this section.

Theorem 21. *Let (X, d) be a complete Branciari metric space and $T : X \longrightarrow X$ an $\mathscr{L}^*$-contraction with respect to $\zeta \in \mathscr{L}$. Then T has a unique fixed point.*

Proof. Let x_0 be an arbitrary point of X. Define a sequence $\{x_n\} \subseteq X$ by

$$x_n = Tx_{n-1}, \quad \forall n \in \mathbb{N}. \tag{47}$$

If there exists $n_0 \in \mathbb{N}$ such that $x_{n_0} = x_{n_0+1}$, then x_{n_0} is a fixed point of T and hence the proof is done. Assume that $x_n \neq x_{n+1}$, for all $n \in \mathbb{N}$.

Using the contractive condition (46) and (ζ_2^*), we have

$$\begin{aligned} 1 &\leq \zeta[\Theta(d(Tx_{n-1}, Tx_n)), \Theta(M(x_{n-1}, x_n))] \\ &= \zeta[\Theta(d(x_n, x_{n+1})), \Theta(M(x_{n-1}, x_n))] < \frac{\Theta(M(x_{n-1}, x_n))}{\Theta(d(x_n, x_{n+1}))}. \end{aligned} \tag{48}$$

Consequently, we obtain that

$$\Theta(d(x_n, x_{n+1})) < \Theta(M(x_{n-1}, x_n)), \quad \forall n \in \mathbb{N}, \tag{49}$$

where

$$\begin{aligned} M(x_{n-1}, x_n) &= \max\{d(x_{n-1}, x_n), d(x_{n-1}, Tx_{n-1}), d(x_n, Tx_n)\} \\ &= \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\}. \end{aligned} \tag{50}$$

If $M(x_{n-1}, x_n) = d(x_n, x_{n+1})$, then inequality (49) becomes

$$\Theta(d(x_n, x_{n+1})) < \Theta(d(x_n, x_{n+1})), \quad \forall n \in \mathbb{N}, \tag{51}$$

which is a contradiction. Hence, we must have $M(x_{n-1}, x_n) = d(x_{n-1}, x_n)$, for all $n \in \mathbb{N}$. Therefore, inequality (49) becomes

$$\Theta(d(x_n, x_{n+1})) < \Theta(d(x_{n-1}, x_n)), \quad \forall n \in \mathbb{N}, \tag{52}$$

which implies from (Θ_1) that

$$d(x_n, x_{n+1}) < d(x_{n-1}, x_n), \quad \forall n \in \mathbb{N}. \tag{53}$$

Thus, the sequence $\{d(x_{n-1}, x_n)\}$ is decreasing and bounded below by 0, so there exists $r \geq 0$ such that $\lim_{n\to\infty} d(x_{n-1}, x_n) = r$. Suppose that $r \neq 0$, then it follows from (Θ_2) that

$$\lim_{n\to\infty} \Theta(d(x_{n-1}, x_n)) > 1. \tag{54}$$

Taking $t_n = \Theta(d(x_n, x_{n+1}))$ and $s_n = \Theta(d(x_{n-1}, x_n))$, for all $n \in \mathbb{N}$. It is clear from (52), (54), and (Θ_4) that $t_n < s_n$, $\forall n \in \mathbb{N}$ and $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 1$. Hence, using (ζ_3^*) we get

$$1 \leq \limsup_{n\to\infty} \zeta(t_n, s_n) < 1, \tag{55}$$

which is a contradiction. Therefore, $r = 0$, i.e., we have

$$\lim_{n\to\infty} d(x_{n-1}, x_n) = 0, \quad \forall n \in \mathbb{N}. \tag{56}$$

Now, let us assume that $x_m = x_n$, for some $m > n$. Then, we have $x_{m+1} = x_{n+1}$. Using (52), we get

$$\begin{aligned}\Theta(d(x_m, x_{m+1})) &< \Theta(d(x_{m-1}, x_m)) < \Theta(d(x_{m-2}, x_{m-1})) \\ &< \cdots < \Theta(d(x_n, x_{n+1})) = \Theta(d(x_m, x_{m+1})),\end{aligned} \tag{57}$$

which is a contradiction. This concludes that $x_m \neq x_n$, for all $n \neq m$.

Next, we claim that the sequence $\{x_n\}$ is a Cauchy sequence in (X, d). On the contrary, assume that it is not Cauchy, then we can find two subsequences $\{x_{n_k}\}$ and $\{x_{m_k}\}$ of $\{x_n\}$ such that n_k is the smallest index for which

$$\begin{aligned} n_k &> m_k > k, \\ d(x_{m_k}, x_{n_k}) &\geq \varepsilon, \\ d(x_{m_k}, x_{n_k-2}) &< \varepsilon. \end{aligned} \tag{58}$$

By using a similar argument as in the proof of Theorem 15, we obtain

$$\lim_{k\to\infty} d(x_{m_k}, x_{n_k}) = \varepsilon = \lim_{k\to\infty} d(x_{m_k-1}, x_{n_k-1}). \tag{59}$$

Now, using (46) and (ζ_2^*), we have

$$\begin{aligned} 1 &\leq \zeta\left[\Theta\left(d\left(Tx_{m_k-1}, Tx_{n_k-1}\right)\right), \Theta\left(M\left(x_{m_k-1}, x_{n_k-1}\right)\right)\right] \\ &= \zeta\left[\Theta\left(d\left(x_{m_k}, x_{n_k}\right)\right), \Theta\left(M\left(x_{m_k-1}, x_{n_k-1}\right)\right)\right] \\ &< \frac{\Theta\left(M\left(x_{m_k-1}, x_{n_k-1}\right)\right)}{\Theta\left(d\left(x_{m_k}, x_{n_k}\right)\right)}, \end{aligned} \tag{60}$$

which implies that

$$\Theta\left(d\left(x_{m_k}, x_{n_k}\right)\right) < \Theta\left(M\left(x_{m_k-1}, x_{n_k-1}\right)\right), \quad \forall k \in \mathbb{N}, \tag{61}$$

where $M(x_{m_k-1}, x_{n_k-1}) = \max\{d(x_{m_k-1}, x_{n_k-1}), d(x_{m_k-1}, x_{m_k}), d(x_{n_k-1}, x_{n_k})\}$.

Making use of (56), (59), and Remark 11, we deduce that

$$\lim_{k\to\infty} M\left(x_{m_k-1}, x_{n_k-1}\right) = \max\{\varepsilon, 0, 0\} = \varepsilon. \tag{62}$$

Now, let $t_k = \Theta(d(x_{m_k}, x_{n_k}))$ and $s_k = \Theta(M(x_{m_k-1}, x_{n_k-1}))$, for all $k \in \mathbb{N}$. In view of (59), (61), (62), and (Θ_4), we have $t_k < s_k$, for all $k \in \mathbb{N}$ and $\lim_{k\to\infty} t_k = \lim_{k\to\infty} s_k > 1$. Therefore, using (ζ_3^*) we obtain

$$1 \leq \limsup_{k\to\infty} \zeta(t_k, s_k) < 1. \tag{63}$$

This contradiction ensures that the sequence $\{x_n\}$ is a Cauchy sequence in (X, d). As (X, d) is complete, then there exists $u \in X$ such that

$$\lim_{n\to\infty} d(x_n, u) = 0. \tag{64}$$

Without loss of generality, we can assume that $x_n \neq u$ and $Tx_n \neq Tu$, for all $n \in \mathbb{N}$. Suppose that $d(u, Tu) > 0$, it follows from (46) and (ζ_2^*) that

$$\begin{aligned} 1 &\leq \zeta[\Theta(d(Tx_n, Tu)), \Theta(M(x_n, u))] \\ &= \zeta[\Theta(d(x_{n+1}, Tu)), \Theta(M(x_n, u))] < \frac{\Theta(M(x_n, u))}{\Theta(d(x_{n+1}, Tu))}, \end{aligned} \tag{65}$$

where $M(x_n, u) = \max\{d(x_n, u), d(x_n, x_{n+1}), d(u, Tu)\}$, which implies that

$$\Theta(d(x_{n+1}, Tu)) < \Theta(M(x_n, u)). \tag{66}$$

From Remark 11 (i) and Lemma 13, we have

$$\lim_{n\to\infty} d(x_{n+1}, Tu) = \lim_{n\to\infty} M(x_n, u) = d(u, Tu) > 0. \tag{67}$$

Let $t_n = \Theta(d(x_{n+1}, Tu))$ and $s_n = \Theta(M(x_n, u))$, for all $n \in \mathbb{N}$. Then it follows from (ζ_3^*) that

$$1 \leq \limsup_{n\to\infty} \zeta(t_n, s_n) < 1, \tag{68}$$

which is a contradiction. Therefore, we conclude that $u = Tu$, that is, u is a fixed point of T. Finally, we show that the fixed point of the mapping T is unique. Assume that u and z are two distinct fixed points in X. Then $d(u, z) = d(Tu, Tz) > 0$.

Using (46) and (ζ_2^*), we deduce that

$$\begin{aligned} 1 &\le \zeta[\Theta(d(Tu, Tz)), \Theta(M(u, z))] \\ &= \zeta[\Theta(d(u, z)), \Theta(M(u, z))] < \frac{\Theta(M(u, z))}{\Theta(d(u, z))}, \end{aligned} \tag{69}$$

where $M(u, z) = \max\{d(u, z), d(u, Tu), d(z, Tz)\} = d(u, z)$, which implies that

$$\Theta(d(u, z)) < \Theta(M(u, z)) = \Theta(d(u, z)), \tag{70}$$

which is a contradiction. Therefore, T has a unique fixed point.

Remark 22. In Theorem 15, the mapping T must be continuous, while T in Theorem 21 need not be continuous.

To support Theorem 21, we give an illustrative example. Precisely, we show that Theorem 21 can be used to cover this example while Theorems 4, 9, and 15 are not applicable.

Example 23. Let $X = A \cup B$, where $A = [1, 2]$ and $B = \{(1/n): n = 2, 3, 4, 5\}$. Define a mapping $d : X \times X \longrightarrow [0,\infty)$ as follows:

(i) $d((1/2), (1/3)) = d((1/4), (1/5)) = (3/10)$

(ii) $d((1/2), (1/5)) = d((1/3), (1/4)) = (2/10)$

(iii) $d((1/2), (1/4)) = d((1/5), (1/3)) = (6/10)$

(iv) $d(x, x) = 0$, $d(x, y) = d(y, x)$, $\forall x, y \in B$, and

(v) $d(x, y) = |x - y|$ if $x, y \in A$ or $x \in A$, $y \in B$, or $x \in B$, $y \in A$.

Observe that d is not a metric on X, because the triangle inequality is not satisfied on A. To insure this, we have

$$\frac{6}{10} = d\left(\frac{1}{5}, \frac{1}{3}\right) > d\left(\frac{1}{5}, \frac{1}{4}\right) + d\left(\frac{1}{4}, \frac{1}{3}\right) = \frac{5}{10}. \tag{71}$$

It is easy to check that (X, d) is a complete Branciari metric space. Let $T : X \longrightarrow X$ be defined as

$$Tx = \begin{cases} \dfrac{1}{5}, & \text{if } x \in \left[1, \dfrac{3}{2}\right], \\ \dfrac{1}{4}, & \text{if } x \in \left(\dfrac{3}{2}, 2\right] \cup B. \end{cases} \tag{72}$$

Since T is not continuous at $x = (3/2)$, then by Remarks 10 and 5, T is neither a Θ-contraction nor an $\mathscr{L}$-contraction, and hence Theorems 4 and 9 cannot be applied here.

Observe that T is an $\mathscr{L}^*$-contraction with respect to $\zeta : [1,\infty) \times [1,\infty) \longrightarrow \mathbb{R}$, where

$$\zeta_k(t, s) = \frac{s^k}{t}, \quad \forall t, s \in [1,\infty), \text{for any } k \in \left[\frac{3}{8}, 1\right), \tag{73}$$

and $\Theta : (0,\infty) \longrightarrow (1,\infty)$, such that $\Theta(t) = e^t$, $\forall t \in (0,\infty)$.

Indeed, for $x \in [1, (3/2)]$ and $y \in ((3/2), 2] \cup B$, we have

$$d(Tx, Ty) = d\left(\frac{1}{4}, \frac{1}{5}\right) = \frac{3}{10} > 0, \tag{74}$$

and

$$\begin{aligned} \zeta[\Theta(d(Tx, Ty)), \Theta(M(x, y))] &= \frac{[\Theta(M(x, y))]^k}{\Theta(d(Tx, Ty))} \ge \frac{e^{(4k/5)}}{e^{(3/10)}} \\ &= e^{(1/5)(4k-(3/2))} \\ &\ge 1, \quad \text{for any } k \in \left[\frac{3}{8}, 1\right). \end{aligned} \tag{75}$$

Hence, all the hypotheses of Theorem 21 are satisfied, and the unique fixed point of T is $u = (1/4)$.

Notice that due to Remark 22, Theorem 15 cannot be applied here.

The corollaries that follow are deduced as consequences of Theorem 21.

Corollary 24. *Let (X, d) be a complete Branciari metric space and $T : X \longrightarrow X$. Suppose that there exist $\Theta \in \Omega_{1,2,4}$ and $k \in (0, 1)$ such that (for all $x, y \in X$)*

$$d(Tx, Ty) > 0 \Rightarrow \Theta(d(Tx, Ty)) \le [\Theta(M(x, y))]^k, \tag{76}$$

where $M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty)\}$. Then T has a unique fixed point.

Proof. Observe that T is an $\mathscr{L}^*$-contraction with respect to $\zeta_k(t, s) = (s^k/t)$. Then, the result follows immediately from Theorem 21.

Remark 25. Corollary 25 is a generalization of Theorem 4 in [17] without assuming condition (Θ_3).

Corollary 26. *Let (X, d) be a Branciari metric space and $T : X \longrightarrow X$. Assume that (for all $x, y \in X$)*

$$d(Tx, Ty) > 0 \Rightarrow d(Tx, Ty)) \le M(x, y) - \varphi(M(x, y)), \tag{77}$$

where $M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty)\}$ and $\varphi : [0,\infty) \longrightarrow [0,\infty)$ is nondecreasing and lower semicontinuous such that $\varphi^{-1}(\{0\}) = 0$. Then T has a unique fixed point.

Proof. Let $\Theta(t) = e^t$, for all $t > 0$. From (77), we have

$$\Theta(d(Tx, Ty)) = e^{d(Tx,Ty)} \le e^{M(x,y)-\varphi(M(x,y))} = \frac{\Theta(M(x,y))}{e^{\varphi(M(x,y))}}, \quad (78)$$

for all $x, y \in X$ with $d(Tx, Ty) > 0$.

Now, define $\varphi(t) = \ln(\psi(\Theta(t)))$, for all $t > 0$, where $\psi : [1,\infty) \longrightarrow [1,\infty)$ is nondecreasing and lower semicontinuous such that $\psi^{-1}(\{1\}) = 1$.

From (78), we have

$$\Theta(d(Tx, Ty)) \le \frac{\Theta(M(x,y))}{\psi(\Theta(M(x,y))}. \quad (79)$$

Taking $\zeta(t,s) = (s/t\psi(s))$ and using (79), we have

$$\begin{aligned} 1 &\le \frac{\Theta(M(x,y))}{\Theta(d(Tx,Ty))\psi(\Theta(M(x,y)))} \\ &= \zeta[\Theta(d(Tx,Ty)), \Theta(M(x,y))]. \end{aligned} \quad (80)$$

Therefore, all the requirements of Theorem 21 are satisfied, and hence T has a unique fixed point.

5. An Application to Fourth-Order Differential Equation

In this section, we discuss application of the fixed-point theorems obtained in the previous sections in solving the following boundary value problem of a fourth-order differential equation:

$$\begin{cases} x''''(t) = f\left(t, x(t), x'(t), x''(t), x'''(t)\right), & t \in [0,1], \\ x(0) = x'(0) = x''(1) = x'''(1) = 0, \end{cases} \quad (81)$$

where $f : [0,1] \times \mathbb{R}^4 \longrightarrow \mathbb{R}$ is a continuous function. Let $X = C[0,1]$ be the space of all continuous functions defined on $[0,1]$. Define a metric $d : X \times X \longrightarrow \mathbb{R}$ by

$$d(x,y) = \max_{t\in[0,1]} |x(t) - y(t)|, \quad \text{for all } x, y \in X. \quad (82)$$

It is known that (X, d) is a complete Branciari distance space. The green function associated to (81) is defined by

$$G(s,t) = \begin{cases} \frac{1}{6} t^2(3s - t), & 0 \le t \le s \le 1, \\ \frac{1}{6} s^2(3t - s), & 0 \le s \le t \le 1. \end{cases} \quad (83)$$

Now, we prove the following result on the solution of the boundary value problem (81).

Theorem 27. *Assume that the following conditions are satisfied:*

(i) $f : [0,1] \times \mathbb{R}^4 \longrightarrow \mathbb{R}$ is a continuous function

(ii) There exists $\tau > 0$ such that, for all $x, y \in X$ and $s \in [0,1]$

$$\begin{aligned} \left| f\left(s, x, x'\right) - f\left(s, y, y'\right) \right| \le 8e^{-\tau} \big[\max \{ &|x(s) - y(s)|, |x(s) \\ &- Tx(s)|, |y(s) - Ty(s)| \} \big], \end{aligned} \quad (84)$$

where $T : X \longrightarrow X$ is defined by

$$Tx(t) = \int_0^1 G(t,s) f\left(s, x(s), x'(s)\right) ds. \quad (85)$$

Then (81) has a unique solution in X.

Proof. Observe that $x \in X$ is a solution of (81) if and only if $x \in X$ is a solution of the integral equation

$$x(t) = \int_0^1 G(t,s) f\left(s, x(s), x'(s)\right) ds, \quad \forall x \in X. \quad (86)$$

In view of condition (ii). For all $x, y \in X$ with $d(Tx, Ty) > 0$ and for all $t \in [0,1]$, we have

$$\begin{aligned} |Tx(t) - Ty(t)| &= \left| \int_0^1 G(t,s) f\left(s, x(s), x'(s)\right) ds \right. \\ &\quad \left. - \int_0^1 G(t,s) f\left(s, y(s), y'(s)\right) ds \right| \\ &\le \int_0^1 G(t,s) \left| f\left(s, x(s), x'(s)\right) \right. \\ &\quad \left. - f\left(s, y(s), y'(s)\right) \right| ds \\ &\le 8e^{-\tau} \int_0^1 G(t,s) \big[\max \{ |x - y|, |x \\ &\quad - Tx|, |y - Ty| \} \big] ds \\ &\le 8e^{-\tau} [M(x,y)] \left(\sup_{t\in[0,1]} \int_0^1 G(t,s) ds \right), \end{aligned} \quad (87)$$

where $M(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty)\}$. As $\int_0^1 G(t,s)ds = (t^4/24) - (t^3/6) + (t^2/4)$, for all $t \in [0,1]$, $\sup_{t\in[0,1]} \int_0^1 G(t,s)ds = (1/8)$, we obtain

$$d(Tx, Ty) \le e^{-\tau}[M(x,y)], \quad (88)$$

or

$$e^{d(Tx,Ty)} \le \left(e^{M(x,y)}\right)^{e^{-\tau}}. \quad (89)$$

Observe that $e^{-\tau} \in (0, 1)$ as $\tau > 0$. Therefore, for all $x, y \in X$, we obtain

$$\zeta[\Theta(d(Tx, Ty)), \Theta(M(x, y))] = \frac{(\Theta(M(x, y)))^k}{\Theta(d(Tx, Ty))} = \frac{\left(e^{M(x,y)}\right)^{e^{-\tau}}}{e^{d(Tx,Ty)}} \geq 1, \tag{90}$$

where $\Theta(t) = e^t$, $\zeta(t, s) = (s^k/t)$, and $k = e^{-\tau}$. Thus, all the hypotheses of Theorem 21 are satisfied, and hence T has a unique fixed point in X which is a solution of (81).

Theorem 28. *Assume that the following conditions are satisfied:*

(i) $f : [0, 1] \times \mathbb{R}^4 \longrightarrow \mathbb{R}$ *is a continuous function*

(ii) There exists $\tau > 0$ *and* $\lambda \geq 0$ *such that for all* $x, y \in X$ *and* $s \in [0, 1]$

$$\left| f\left(s, x, x'\right) - f\left(s, y, y'\right) \right| \leq 8e^{-\tau}[|x(s) - y(s)| + \lambda N(x, y)], \tag{91}$$

where

$$N(x, y) = \min \left\{ d(x, Ty), d(y, Tx), \frac{d(x, Tx) + d(y, Ty)}{2} \right\}, \tag{92}$$

and $T : X \longrightarrow X$ *is defined by*

$$Tx(t) = \int_0^1 G(t, s) f\left(s, x(s), x'(s)\right) ds. \tag{93}$$

Then (81) has a unique solution in X.

Proof. The proof can be done using similar arguments of the proof of Theorem 28 and applying Theorem 15.

Data Availability

No data were used in this study.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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