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Research Article

Existence of Positive Solution for Fractional Differential Systems with Multipoint Boundary Value Conditions

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In this paper, we apply the fixed-point theorems of γ concave and $(-\gamma)$ convex operators to establish the existence of positive solutions for fractional differential systems with multipoint boundary conditions. Two examples are given to support our results.

1. Introduction

Consider the following system of nonlinear fractional differential equations

$$\begin{cases} D_{0+}^{\alpha}u(t)+\mu_1 f(t,v(t))+\mu_2 g(t,v(t))=0, & 0<t<1, \\ D_{0+}^{\alpha}v(t)+\mu_1 f(t,u(t))+\mu_2 g(t,u(t))=0, & 0<t<1, \end{cases} \tag{1}$$

$$\begin{cases} D_{0+}^{\alpha}u(t)+\mu_1 f(t,u(t))+\mu_2 g(t,v(t))=0, & 0<t<1, \\ D_{0+}^{\alpha}v(t)+\mu_1 f(t,v(t))+\mu_2 g(t,u(t))=0, & 0<t<1, \end{cases} \tag{2}$$

with the multipoint boundary conditions with the multipoint boundary conditions

$$\begin{cases} D_{0+}^{\beta}u(1)=\sum_{i=1}^{m-2}\xi_i D_{0+}^{\beta}u(\eta_i), & u(0)=0, \\ D_{0+}^{\beta}v(1)=\sum_{i=1}^{m-2}\xi_i D_{0+}^{\beta}v(\eta_i), & v(0)=0, \end{cases} \tag{3}$$

where D_{0+}^{α} is the standard Riemann–Liouville fractional derivative, $f: J\times[0,\infty)\longrightarrow[0,\infty)$ is continuous, $1<\alpha\le 2$, $0\le\beta\le 1$, $0\le\alpha-\beta-1$, $0<\xi_i,\eta_i<1$, $i=1,2,\dots,m-2$, $\sum_{i=1}^{m-2}\xi_i\eta_i^{\alpha-\beta-1}<1$, $J=[0,1]$, and $\mu_1,\mu_2\in(0,+\infty)$, $\mu_1\ge\mu_2$.

Differential equations with fractional order have been applied in various areas of science and engineering. For their applications, there has been a sharp increase in studying fractional differential equations (see [1–18] and references therein). Meanwhile, the theory of boundary value problems with multipoint boundary conditions has various applications in applied fields, which have been studied by many authors (cf., e.g., [19–26]). Many authors have studied these problems by using different methods, such as monotone iterative technique, the method of upper and lower solutions, fixed-point theorems in cones, nonlinear alternatives of Leray–Schauder, and coincidence degree theory. However, concave (convex) operators are a class of important operators, which can be used in nonlinear differential and integral equations (cf., e.g., [27–31]). Moreover, few papers can be reported on the existence of solutions for coupled systems of fractional differential equations with multipoint boundary conditions by using fixed-point theorems of γ concave and $(-\gamma)$ convex operators.

In [25], we considered the following m-point boundary value problem for fractional differential equation

$$D_{0+}^{\alpha}u(t)+f(t,u(t))=0,\quad 0<t<1, \tag{4}$$

with the multipoint boundary conditions $u(0)=0$, $D_{0+}^{\beta}u(1)=\sum_{i=1}^{m-2}\xi_i D_{0+}^{\beta}u(\eta_i)$, where D_{0+}^{α} is the standard Riemann–Liouville fractional derivative, $n=[\alpha]+1$, $f: [0,1]\times[0,\infty)\longrightarrow[0,\infty)$ is continuous, $1<\alpha\le 2$, $0\le\beta\le 1$, $0\le\alpha-\beta-1$, $0<\xi_i,\eta_i<1$, $i=1,2,\dots,m-2$, and $\sum_{i=1}^{m-2}\xi_i\eta_i^{\alpha-\beta-1}<1$.

In [26], Henderson and Luca studied the following system of nonlinear second-order ordinary differential equation

$$\begin{cases} u''(t) + c(t)f(v(t)) = 0, & t \in (0,T), \\ v''(t) + d(t)g(u(t)) = 0, & t \in (0,T), \end{cases} \quad (5)$$

with the multipoint boundary conditions $\alpha u(0) - \beta u'(0) = 0$, $u(T) = \sum_{i=1}^{m-2} {}_i u(\xi_i) + a_0 a, m \in \mathbb{N}, m \geq 3$ and $\gamma v(0) - \delta v'(0) = 0$, $v(T) = \sum_{i=1}^{m-2} b_i v(\eta_i) + b_0$, $n \in \mathbb{N}, m \geq 3$. By using the Schauder fixed-point theorem, the existence of positive solutions was investigated.

Motivated by above papers, in this paper, we investigate the existence of positive solutions for systems (1)–(3).

In this paper, we need the following assumptions that we shall use in the sequel:

(H_1) $f, g \in C[J \times \mathbb{R}^+, \mathbb{R}^+]$, $f(t,x)$ and $g(t,x)$ are increasing in x for $x \in \mathbb{R}^+$, $g(t,0) \neq 0$

(H_2) There exists a constant $\gamma \in (0,1)$ such that $f(t,\lambda x) \geq \lambda^\gamma f(t,x)$ and $g(t,\lambda x) \geq \lambda^\gamma g(t,x)$, $\forall t \in J$, $\lambda \in (0,1)$, $x \in \mathbb{R}^+$

(H_3) There exists a constant $\delta_0 > 0$ such that $f(t,x) \geq \delta_0 g(t,x)$, $t \in J, x \in \mathbb{R}^+$

(H_4) $f \in C[J \times \mathbb{R}^+, \mathbb{R}^+]$, $f(t,x)$ is increasing in x for $x \in \mathbb{R}^+$, $f(t,0) \neq 0$

(H_5) There exists a constant $\gamma \in (0,1)$ such that $f(t,\lambda x) \geq \lambda^\gamma f(t,x)$, $\forall t \in J, \lambda \in (0,1), x \in \mathbb{R}^+$

(H_6) $f, g \in C[J \times \mathbb{R}^+, \mathbb{R}^+]$, $f(t,x)$ is nondecreasing in x, and $g(t,y)$ is nonincreasing in y

(H_7) $f(t,x)$ and $g(t,y)$ are bounded in $[J \times \mathbb{R}^+]$

(H_8) There exists $0 \leq \gamma_1 < 1$ such that $f(t,kx) \geq k^{\gamma_1} f(t,x)$, and there exists $0 \leq \gamma_2 < 1$ such that $g(t,kx) \leq k^{-\gamma_2} g(t,x)$, where $k \in (0,1)$, $0 \leq \gamma_1 + \gamma_2 < 1$

Here are our main results.

Theorem 1. *Suppose that $(H_1) - (H_3)$ hold. Then, equations (1)–(3) have a unique positive solution (u^*, v^*) in $P_h \times P_h$, where $h(t) = t^{\alpha-1}, t \in J$. Moreover, for any initial value $u_0 \in P_h$ and $v_0 \in P_h$, constructing successively the sequence*

$$u_n(t) = \int_0^1 G(t,s)\left(\mu_1 f(s, v_{n-1}(s)) + \mu_2 g(s, v_{n-1}(s))\right) ds, \quad n = 0,1,2,\ldots,$$
$$v_n(t) = \int_0^1 G(t,s)\left(\mu_1 f(s, u_{n-1}(s)) + \mu_2 g(s, u_{n-1}(s))\right) ds, \quad n = 0,1,2,\ldots, \quad (6)$$

we have $(u_n(t), v_n(t)) \longrightarrow (u^*(t), v^*(t))$ as $n \longrightarrow \infty$.

Corollary 1. *Suppose that $(H_4) - (H_5)$ holds. Then, system*

$$\begin{cases} D_{0+}^\alpha u(t) + \mu_1 f(t, v(t)) = 0, & 0 < t < 1, \\ D_{0+}^\alpha v(t) + \mu_1 f(t, u(t)) = 0, & 0 < t < 1, \end{cases} \quad (7)$$

with the multipoint boundary conditions

$$\begin{cases} D_{0+}^\beta u(1) = \sum_{i=1}^{m-2} \xi_i D_{0+}^\beta u(\eta_i), & u(0) = 0, \\ D_{0+}^\beta v(1) = \sum_{i=1}^{m-2} \xi_i D_{0+}^\beta v(\eta_i), & v(0) = 0, \end{cases} \quad (8)$$

has a unique positive solution (u^*, v^*) in $P_h \times P_h$, where $h(t) = t^{\alpha-1}$, $t \in J$, D_{0+}^α is the standard Riemann–Liouville fractional derivative, $f\colon J \times [0,\infty) \longrightarrow [0,\infty)$ is continuous, $1 < \alpha \leq 2$, $0 \leq \beta \leq 1$, $0 \leq \alpha - \beta - 1$, $0 < \xi_i, \eta_i < 1$, $i = 1, 2, \ldots, m-2$, $\sum_{i=1}^{m-2} \xi_i \eta_i^{\alpha-\beta-1} < 1$, $J = [0,1]$, and $\mu_1, \mu_2 \in (0, +\infty)$, $\mu_1 \geq \mu_2$. Moreover, for any initial value $u_0 \in P_h$ and $v_0 \in P_h$, constructing successively the sequence

$$u_n(t) = \mu_1 \int_0^1 G(t,s) f(s, v_{n-1}(s)) ds, \quad n = 0,1,2,\ldots,$$
$$v_n(t) = \mu_1 \int_0^1 G(t,s) f(s, u_{n-1}(s)) ds, \quad n = 0,1,2,\ldots, \quad (9)$$

we have $(u_n(t), v_n(t)) \longrightarrow (u^*(t), v^*(t))$ as $n \longrightarrow \infty$.

Theorem 2. *Suppose that $(H_6) - (H_8)$ hold. Then, equations (1)–(3) have exactly one positive solution $(u^*, v^*) \in [u_0, v_0] \times [u_0, v_0]$, where $u_0, v_0 \in P$ with $u_0 \leq v_0$, and constructing successively the sequence*

$$u_n(t) = \int_0^1 G(t,s)\left(\mu_1 f(s, u_{n-1}(s)) + \mu_2 g(s, v_{n-1}(s))\right) ds, \quad n = 0,1,2,\ldots,$$
$$v_n(t) = \int_0^1 G(t,s)\left(\mu_1 f(s, v_{n-1}(s)) + \mu_2 g(s, u_{n-1}(s))\right) ds, \quad n = 0,1,2,\ldots, \quad (10)$$

we have $(u_n(t), v_n(t)) \longrightarrow (u^*(t), v^*(t))$ as $n \longrightarrow \infty$.

The rest of this paper is organized as follows. In Section 2, we present some background materials and preliminaries. Section 3 deals with the existence results. In Section 4, two examples are given to illustrate the result.

2. Background Materials and Preliminaries

Definition 1 (see [6]). The fractional integral of order α with the lower limit t_0 for a function f is defined as

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^t (t-s)^{\alpha-1} f(s) ds, \quad t > t_0, \alpha > 0, \quad (11)$$

where Γ is the gamma function.

Definition 2 (see [6]). For a function $f\colon [0,\infty) \longrightarrow \mathbb{R}$, the Riemann–Liouville derivative of fractional order is defined as

$$D^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-\alpha-1} f(s) ds, \quad \alpha > 0, \ n = [\alpha] + 1. \quad (12)$$

Definition 3 (see [28]). Let E be a real Banach space and P be a cone in E which defined a partial ordering in E by $x \le y$ if and only if $y - x \in P$. P is said to be normal if there exists a positive constant N such that $\theta \le x \le y$ implies $\|x\| \le N\|y\|$. P is called solid if its interior $\mathring{P}$ is nonempty.

Definition 4 (see [28]). An operator A: $D \times D \longrightarrow E$ is said to be mixed monotone if $A(x, y)$ is nondecreasing in x and nonincreasing in y, i.e., $\forall x_i, y_i \in D (i = 1, 2), x_1 \le x_2$, and $y_2 \le y_1$ imply $A(x_1, y_1) \le A(x_2, y_2)$.

Definition 5 (see [28]). For all $x, y \in E$, the notation $x \sim y$ means that there exists $\lambda > 0$ and $\mu > 0$ such that $\lambda x \le y \le \mu x$. Clearly, $\sim$ is an equivalence relation. Given $h > \theta$ (i.e., $h \ge \theta$ and $h \ne \theta$), we denote by P_h the set $P_h = \{x \in E \mid x \sim h\}$. It is easy to see that $P_h \subset P$.

Definition 6 (see [30]). Let $D = P$ or $D = \mathring{P}$ and γ be a real number with $0 \le \gamma < 1$. An operator A: $P \longrightarrow P$ is said to be γ concave $((-\gamma)$convex) if it satisfies

$$\begin{gathered} A(\lambda x) \ge \lambda^{\gamma} Ax, \\ \left(A(\lambda x) \le \lambda^{-\gamma} Ax\right), \\ \forall \lambda \in (0,1), \\ x \in D. \end{gathered} \tag{13}$$

Definition 7 (see [31]). An operator A: $E \longrightarrow E$ is said to be homogeneous if it satisfies

$$A(\lambda x) = \lambda Ax, \ \forall \lambda > 0, \ x \in E. \tag{14}$$

An operator A: $P \longrightarrow P$ is said to be subhomogeneous if it satisfies

$$A(\lambda x) \ge \lambda Ax, \quad \forall \lambda \in (0,1), \ x \in P. \tag{15}$$

Lemma 1 (see [25]). *Let $y \in C[0,1]$. Then, the fractional differential equation*

$$\begin{cases} D_{0+}^{\alpha} u(t) + y(t) = 0, & 0 < t < 1, \ 1 < \alpha \le 2, \\ D_{0+}^{\beta} u(1) = \sum_{i=1}^{m-2} \xi_i D_{0+}^{\beta} u(\eta_i), & u(0) = 0, \end{cases} \tag{16}$$

has a unique solution which is given by

$$u(t) = \int_0^1 G(t,s) y(s) ds, \tag{17}$$

where

$$G(t,s) = G_1(t,s) + G_2(t,s), \tag{18}$$

in which

$$\begin{gathered} G_1(t,s) = \begin{cases} \dfrac{1}{\Gamma(\alpha)} \left(t^{\alpha-1}(1-s)^{\alpha-\beta-1} - (t-s)^{\alpha-1}\right), & 0 \le s \le t \le 1, \\ \dfrac{1}{\Gamma(\alpha)} t^{\alpha-1}(1-s)^{\alpha-\beta-1}, & 0 \le t \le s \le 1, \end{cases} \\ G_2(t,s) = \begin{cases} \dfrac{1}{A\Gamma(\alpha)} \left[\sum_{0 \le s \le \eta_i} \left(\xi_i \eta_i^{\alpha-\beta-1} t^{\alpha-1}(1-s)^{\alpha-\beta-1} - \xi_i t^{\alpha-1}(\eta_i - s)^{\alpha-\beta-1}\right)\right], & t \in [0,1], \\ \dfrac{1}{A\Gamma(\alpha)} \left(\sum_{\eta_i \le s \le 1} \xi_i \eta_i^{\alpha-\beta-1} t^{\alpha-1}(1-s)^{\alpha-\beta-1}\right), & t \in [0,1], \end{cases} \end{gathered} \tag{19}$$

where

$$A = 1 - \sum_{i=1}^{m-2} \xi_i \eta_i^{\alpha-\beta-1}. \tag{20}$$

Lemma 2. *Let $h(t) = t^{\alpha-1}$, then $G(t,s)$ in Lemma 1 has the following property:*

(i) $G(t,s) \ge h(t)((1/\Gamma(\alpha))((1-s)^{\alpha-\beta-1} - (1-s)^{\alpha-1}) + (1/A\Gamma(\alpha))\sum_{i=1}^{m-2}(\xi_i \eta_i^{\alpha-\beta-1}(1-s)^{\alpha-\beta-1} - \xi_i(\eta_i - s)^{\alpha-\beta-1}))$

(ii) $G(t,s) \le h(t)((1/\Gamma(\alpha))(1-s)^{\alpha-\beta-1} + (1/A\Gamma(\alpha))\sum_{i=1}^{m-2}\xi_i \eta_i^{\alpha-\beta-1}(1-s)^{\alpha-\beta-1})$

Proof. For $0 \le s \le t \le 1$, $1 < \alpha \le 2$, we have

$$\begin{aligned} t^{\alpha-1}(1-s)^{\alpha-\beta-1} - (t-s)^{\alpha-1} &= t^{\alpha-1}\left((1-s)^{\alpha-\beta-1} - \left(1 - \frac{s}{t}\right)^{\alpha-1}\right) \\ &\ge t^{\alpha-1}\left((1-s)^{\alpha-\beta-1} - (1-s)^{\alpha-1}\right) \\ &\ge 0. \end{aligned} \tag{21}$$

Thus,

$$G_1(t,s) \ge \frac{t^{\alpha-1}}{\Gamma(\alpha)}\left((1-s)^{\alpha-\beta-1} - (1-s)^{\alpha-1}\right) \ge 0. \tag{22}$$

From [25], we have

$$G_2(t,s) \ge \frac{t^{\alpha-1}}{A\Gamma(\alpha)} \sum_{i=1}^{m-2} \left(\xi_i \eta_i^{\alpha-\beta-1}(1-s)^{\alpha-\beta-1} - \xi_i(\eta_i - s)^{\alpha-\beta-1}\right) \ge 0. \quad (23)$$

This means that (i) holds. From Lemma 1, we know that (ii) is obvious. □

Theorem 3 (see [31]). *Let P be a normal cone in a real Banach space E and A: $P \longrightarrow P$ be an increasing γ − concave operator and B: $P \longrightarrow P$ be an increasing subhomogeneous operator. Assume that*

(i) There is $h > \theta$ such that $Ah \in P_h$ and $Bh \in P_h$

(ii) There exists a constant $\delta_0 > 0$ such that $Ax \ge \delta_0 Bx$, $\forall x \in P$

Then, operator equation $Ax + Bx = x$ has a unique solution x^* in P_h. Moreover, constructing successively the sequence $y_n = Ay_{n-1} + By_{n-1}, n = 1, 2, \ldots$ for any initial value $y_0 \in P_h$, we have $y_n \longrightarrow x^*$ as $n \longrightarrow \infty$.

Theorem 4 (see [30]). *Let P be a normal cone of the real Banach space E and A: $P \times P \longrightarrow P$ be a mixed monotone operator. Suppose that*

(i) For fixed y, $A(\cdot, y)$: $P \longrightarrow P$ is γ_1 concave; for fixed x, $A(x)$: $P \longrightarrow P$ is $(-\gamma_2)$ convex, where $0 \le \gamma_1 + \gamma_2 < 1$

(ii) There exist elements $u_0, v_0 \in P$ with $u_0 \le v_0$ and a real number $r_0 > 0$ such that

$$\begin{aligned} u_0 &\ge r_0 v_0, \\ u_0 &\le A(u_0, v_0), \\ A(v_0, u_0) &\le v_0. \end{aligned} \quad (24)$$

Then, A has exactly one fixed point x^* in $[u_0, v_0]$, and constructing successively the sequence

$$\begin{aligned} x_n &= A(x_{n-1}, y_{n-1}), \\ y_n &= A(y_{n-1}, x_{n-1}), \\ n &= 1, 2, \ldots, \end{aligned} \quad (25)$$

for any initial value $(x_0, y_0) \in [u_0, v_0] \times [u_0, v_0]$, we have $x_n \longrightarrow x^*, y_n \longrightarrow y^* \ (n \longrightarrow \infty)$.

3. Main Results

In this section, we shall investigate the existence of positive solutions for systems (1)–(3). We consider the space $E = C([0,1], \mathbb{R})$ equipped with the norm $\|u\| = \sup_{0\le t\le 1}|u(t)|$. Let $P = \{u \in E \mid u(t) \ge 0\}$, then P is a cone in E. Let $\mathbb{R}^+ = [0, +\infty)$.

From Lemma 1, we know that (1)–(3) can be translated into the following equation

$$\begin{cases} u(t) = \int_0^1 G(t,s)(\mu_1 f(s, v(s)) + \mu_2 g(s, v(s)))\mathrm{d}s, \\ v(t) = \int_0^1 G(t,s)(\mu_1 f(s, u(s)) + \mu_2 g(s, u(s)))\mathrm{d}s, \end{cases} \quad (26)$$

and (2)-(3) can be translated into the following equation

$$\begin{cases} u(t) = \int_0^1 G(t,s)(\mu_1 f(s, u(s)) + \mu_2 g(s, v(s)))\mathrm{d}s, \\ v(t) = \int_0^1 G(t,s)(\mu_1 f(s, v(s)) + \mu_2 g(s, u(s)))\mathrm{d}s. \end{cases} \quad (27)$$

Thus, (u, v) is a solution of (1)–(3) if and only if (u, v) is a solution of system (26), and (u, v) is a solution of (2)-(3) if and only if (u, v) is a solution of system (27).

For convenience, we denote

$$\begin{aligned} p_1(s) &= \frac{1}{\Gamma(\alpha)}\left((1-s)^{\alpha-\beta-1} - (1-s)^{\alpha-1}\right) \\ &\quad + \frac{1}{A\Gamma(\alpha)} \sum_{i=1}^{m-2} \left(\xi_i \eta_i^{\alpha-\beta-1}(1-s)^{\alpha-\beta-1} - \xi_i(\eta_i - s)^{\alpha-\beta-1}\right), \\ p_2(s) &= \frac{1}{\Gamma(\alpha)}(1-s)^{\alpha-\beta-1} + \frac{1}{A\Gamma(\alpha)} \sum_{i=1}^{m-2} \xi_i \eta_i^{\alpha-\beta-1}(1-s)^{\alpha-\beta-1}. \end{aligned} \quad (28)$$

Now, we prove Theorem 1, Corollary 1, and Theorem 2.

Proof of Theorem 1. Define two operators

$$\begin{aligned} Au(t) &= \mu_1 \int_0^1 G(t,s) f(s, v(s))\mathrm{d}s, \\ Bu(t) &= \mu_2 \int_0^1 G(t,s) g(s, v(s))\mathrm{d}s. \end{aligned} \quad (29)$$

Thus,

$$\begin{aligned} Au(t) &= \mu_1 \int_0^1 G(t,s) f\left(s, \int_0^1 G(s,\tau)(\mu_1 f(\tau, u(\tau)) + \mu_2 g(\tau, u(\tau)))\mathrm{d}\tau\right)\mathrm{d}s, \\ Bu(t) &= \mu_2 \int_0^1 G(t,s) g\left(s, \int_0^1 G(s,\tau)(\mu_1 f(\tau, u(\tau)) + \mu_2 g(\tau, u(\tau)))\mathrm{d}\tau\right)\mathrm{d}s. \end{aligned} \quad (30)$$

Owing to [25], we know that $G(t,s) > 0$, $t, s \in (0,1)$. By (H_1), we have A: $P \longrightarrow P$ and B: $P \longrightarrow P$. It is obvious that (u, v) is the solution of problem (26) if and only if (u, v) is the solution of $(u = Au + Bu, v = Av + Bv)$.

Step 1: A and B are two increasing operators.

Set $u, \overline{u} \in P$, $u \leq \overline{u}$. It follows from (26) that

$$\begin{aligned} v(t) &= \int_0^1 G(t,s)\left(\mu_1 f(s,u(s)) + \mu_2 g(s,u(s))\right)\mathrm{d}s, \\ \overline{v}(t) &= \int_0^1 G(t,s)\left(\mu_1 f(s,\overline{u}(s)) + \mu_2 g(s,\overline{u}(s))\right)\mathrm{d}s. \end{aligned} \tag{31}$$

From (H_1), we have

$$v(t) \leq \overline{v}(t), \quad t \in J. \tag{32}$$

According to (H_1) and (29), we obtain

$$\begin{aligned} Au(t) &= \mu_1 \int_0^1 G(t,s) f(s, v(s))\mathrm{d}s \leq A\overline{u}(t) \\ &= \mu_1 \int_0^1 G(t,s) f(s,\overline{v}(s))\mathrm{d}s. \end{aligned} \tag{33}$$

Thus, A is an increasing operator. Similarly, we can see that B is an increasing operator.

Step 2: A is a γ concave operator, and B is a subhomogeneous operator.

In fact, for $\lambda \in (0,1), \gamma \in (0,1), u \in P, t \in J$, from (30) and (H_2), we have

$$\begin{aligned} A(\lambda u(t)) &= \mu_1 \int_0^1 G(t,s) f\left(s, \int_0^1 G(s,\tau)\left(\mu_1 f(\tau,\lambda u(\tau)) + \mu_2 g(\tau,\lambda u(\tau))\right)\mathrm{d}\tau\right)\mathrm{d}s \\ &\geq \mu_1 \int_0^1 G(t,s) f\left(s, \int_0^1 G(s,\tau)\left(\mu_1 \lambda^\gamma f(\tau,u(\tau)) + \mu_2 \lambda^\gamma g(\tau,u(\tau))\right)\mathrm{d}\tau\right)\mathrm{d}s \\ &= \mu_1 \int_0^1 G(t,s) f\left(s, \lambda^\gamma \int_0^1 G(s,\tau)\left(\mu_1 f(\tau,u(\tau)) + \mu_2 g(\tau,u(\tau))\right)\mathrm{d}\tau\right)\mathrm{d}s \\ &\geq \mu_1 (\lambda^\gamma)^\gamma \int_0^1 G(t,s) f\left(s, \int_0^1 G(s,\tau)\left(\mu_1 f(\tau,u(\tau)) + \mu_2 g(\tau,u(\tau))\right)\mathrm{d}\tau\right)\mathrm{d}s \\ &\geq \mu_1 \lambda^\gamma \int_0^1 G(t,s) f\left(s, \int_0^1 G(s,\tau)\left(\mu_1 f(\tau,u(\tau)) + \mu_2 g(\tau,u(\tau))\right)\mathrm{d}\tau\right)\mathrm{d}s \\ &= \lambda^\gamma A(u(t)). \end{aligned} \tag{34}$$

Similarly, we can get

$$B(\lambda u(t)) \geq \lambda^\gamma B(u(t)). \tag{35}$$

Thus,

$$B(\lambda u(t)) \geq \lambda B(u(t)). \tag{36}$$

Therefore, we can see that A is a γ concave operator, and B is a subhomogeneous operator.

Step 3: $Ah \in P_h$ and $Bh \in P_h$.

Combining (29), (30), (H_1), and Lemma 2, one has, for $t \in J$,

$$\begin{aligned} Ah(t) &= \mu_1 \int_0^1 G(t,s) f\left(s, \int_0^1 G(s,\tau)\left(\mu_1 f(\tau,h(\tau)) + \mu_2 g(\tau,h(\tau))\right)\mathrm{d}\tau\right)\mathrm{d}s \\ &\geq \mu_1 \int_0^1 h(t) p_1(s) f\left(s, \int_0^1 h(s) p_1(\tau)\left(\mu_1 f(\tau,0) + \mu_2 g(\tau,0)\right)\mathrm{d}\tau\right)\mathrm{d}s. \\ Ah(t) &= \mu_1 \int_0^1 G(t,s) f\left(s, \int_0^1 G(s,\tau)\left(\mu_1 f(\tau,h(\tau)) + \mu_2 g(\tau,h(\tau))\right)\mathrm{d}\tau\right)\mathrm{d}s \\ &\leq \mu_1 \int_0^1 h(t) p_2(s) f\left(s, \int_0^1 h(s) p_2(\tau)\left(\mu_1 f(\tau,1) + \mu_2 g(\tau,1)\right)\mathrm{d}\tau\right)\mathrm{d}s. \end{aligned} \tag{37}$$

Let

$$m=\mu_1\int_0^1 p_1(s)f\left(s,\int_0^1 h(s)p_1(\tau)(\mu_1 f(\tau,0)+\mu_2 g(\tau,0))\mathrm{d}\tau\right)\mathrm{d}s,$$
$$\overline{m}=\mu_1\int_0^1 p_2(s)f\left(s,\int_0^1 h(s)p_2(\tau)(\mu_1 f(\tau,1)+\mu_2 g(\tau,1))\mathrm{d}\tau\right)\mathrm{d}s. \quad (38)$$

Noting that (H_3) and $g(t,0)\neq 0$, we obtain

$$m>0, \quad \overline{m}>0. \quad (39)$$

Therefore,

$$mh(t)\le Ah(t)\le \overline{m}h(t), \quad (40)$$

which implies that $Ah\in P_h$. By a closely similar way, we have $Bh\in P_h$.

Step 4: There exists a constant $\delta_0>0$ such that $Au\ge\delta_0 Bu$ and $Av\ge\delta_0 Bv$, $\forall u\in P$.

For $u\in P$, $t\in J$, by (H_3), we have

$$\int_0^1 G(t,s)f(s,v(s))\mathrm{d}s\ge\delta_0\int_0^1 G(t,s)g(s,v(s))\mathrm{d}s. \quad (41)$$

This means that $Au\ge\delta_0 Bu$, $u\in P$. Similarly, we have $Av\ge\delta_0 Bv$, $v\in P$.

Therefore, by simple computation, the conditions in Theorem 3 are satisfied. This implies that the operator equation $Au+Bu=u$ has a unique solution u^* in P_h, and the operator equation $Av+Bv=v$ has a unique solution v^* in P_h. Thus, $(u=Au+Bu, v=Av+Bv)$ has a unique solution (u^*,v^*) in $P_h\times P_h$. For any initial value $u_0\in P_h$ and $v_0\in P_h$, we can construct the following sequence:

$$u_{n+1}(t)=\int_0^1 G(t,s)(\mu_1 f(s,v_n(s))+\mu_2 g(s,v_n(s)))\mathrm{d}s, \quad n=0,1,2,\ldots,$$
$$v_{n+1}(t)=\int_0^1 G(t,s)(\mu_1 f(s,u_n(s))+\mu_2 g(s,u_n(s)))\mathrm{d}s, \quad n=0,1,2,\ldots \quad (42)$$

This follows that $(u_n(t),v_n(t))\longrightarrow(u^*(t),v^*(t))$ as $n\longrightarrow\infty$. □

Proof of Corollary 1. In Theorem 3, we let B be a null operator, Theorem 3 also holds. By Theorem 1, we conclude that Corollary 1 holds. □

Proof of Theorem 2. Define the following operator:

$$u_n(t)=A(u_{n-1}(t),v_{n-1}(t))=\int_0^1 G(t,s)(\mu_1 f(s,u_{n-1}(s))+\mu_2 g(s,v_{n-1}(s)))\mathrm{d}s, \quad n=0,1,2,\ldots,$$
$$v_n(t)=A(v_{n-1}(t),u_{n-1}(t))=\int_0^1 G(t,s)(\mu_1 f(s,v_{n-1}(s))+\mu_2 g(s,u_{n-1}(s)))\mathrm{d}s, \quad n=0,1,2,\ldots \quad (43)$$

From (H_6), we know that $A\colon P\times P\longrightarrow P$ is a mixed monotone operator.

Step 1: we will prove that the condition (i) of Theorem 4 holds

From (H_8), we know that, for $\gamma_1,\gamma_2\in(0,1), 0<\gamma_1+\gamma_2<1, k_1,k_2\in(0,1)$,

$$\begin{aligned}A(ku,v)&=\int_0^1 G(t,s)(\mu_1 f(s,ku(s))+\mu_2 g(s,v(s)))\mathrm{d}s\\&\ge\int_0^1 G(t,s)(k^{\gamma_1}\mu_1 f(s,u(s))+\mu_2 g(s,v(s)))\mathrm{d}s\\&\ge k^{\gamma_1}\int_0^1 G(t,s)(\mu_1 f(s,u(s))+\mu_2 g(s,v(s)))\mathrm{d}s\\&=k^{\gamma_1}A(u,v),\\A(u,kv)&=\int_0^1 G(t,s)(\mu_1 f(s,u(s))+\mu_2 g(s,kv(s)))\mathrm{d}s\\&\le\int_0^1 G(t,s)(\mu_1 f(s,u(s))+k^{-\gamma_2}\mu_2 g(s,v(s)))\mathrm{d}s\\&\le k^{-\gamma_2}\int_0^1 G(t,s)(\mu_1 f(s,u(s))+\mu_2 g(s,v(s)))\mathrm{d}s\\&=k^{-\gamma_2}A(u,v).\end{aligned} \quad (44)$$

Step 2: we will verify that the condition (ii) of Theorem 4 holds

It follows from (H_7) that there exists $M_1>0$ and $M_2>0$ such that

$$|f(t,x)|\le M_1, \quad |g(t,x)|\le M_2, \quad (t,x)\in J\times\mathbb{R}^+. \quad (45)$$

Let

$$v_0(t)=\int_0^1 G(t,s)(\mu_1 M_1+\mu_2 M_2)\mathrm{d}s,$$
$$u_0(t)=\int_0^1 G(t,s)\mu_1 f(s,0)\mathrm{d}s, \quad (46)$$
$$r_0=\frac{\mu_1\min_{s\in J}f(s,0)}{\mu_1 M_1+\mu_2 M_2}.$$

Obviously,

$$A(v_0,u_0)=\int_0^1 G(t,s)(\mu_1 f(s,v_0(s))+\mu_2 g(s,u_0(s)))ds$$
$$\le \int_0^1 G(t,s)(\mu_1 M_1+\mu_2 M_2)ds$$
$$=v_0(t),$$
$$A(u_0,v_0)=\int_0^1 G(t,s)(\mu_1 f(s,u_0(s))+\mu_2 g(s,v_0(s)))ds$$
$$\ge \int_0^1 G(t,s)\mu_1 f(s,0)ds$$
$$=u_0(t),$$
$$u_0(t)=\int_0^1 G(t,s)\mu_1 f(s,0)ds$$
$$\ge \frac{\mu_1\min_{s\in J} f(s,0)}{\mu_1 M_1+\mu_2 M_2}\int_0^1 G(t,s)(\mu_1 M_1+\mu_2 M_2)ds$$
$$=r_0 v_0(t). \tag{47}$$

Therefore, the conditions of Theorem 4 are satisfied. This means that (2)-(3) has exactly one positive solution $(u^*,v^*)\in[u_0,v_0]\times[u_0,v_0]$, where $u_0,v_0\in P$ with $u_0\le v_0$. By constructing successively the sequence

$$u_n(t)=\int_0^1 G(t,s)(\mu_1 f(s,u_{n-1}(s))+\mu_2 g(s,v_{n-1}(s)))ds, \quad n=0,1,2,\ldots,$$
$$v_n(t)=\int_0^1 G(t,s)(\mu_1 f(s,v_{n-1}(s))+\mu_2 g(s,u_{n-1}(s)))ds, \quad n=0,1,2,\ldots, \tag{48}$$

we obtain $(u_n(t),v_n(t))\longrightarrow(u^*(t),v^*(t))$ as $n\longrightarrow\infty$. □

4. Examples

Example 1. Let $\alpha=3/2,\beta=1/2,m=4,\xi_1=\eta_1=1/4,\xi_2=\eta_2=1/2,\mu_1,\mu_2\in(0,+\infty),\mu_1\ge\mu_2$. Consider the following boundary value problem

$$\begin{cases} D_{0+}^{3/2}u(t)+\mu_1 f(t,v(t))+\mu_2 g(t,v(t))=0, & 0<t<1,\\ D_{0+}^{3/2}v(t)+\mu_1 f(t,u(t))+\mu_2 g(t,u(t))=0, & 0<t<1,\end{cases} \tag{49}$$

with the multipoint boundary conditions

$$D_{0+}^{1/2}u(1)=\sum_{i=1}^{m-2}\xi_i D_{0+}^{1/2}u(\eta_i),\quad u(0)=0,$$
$$D_{0+}^{1/2}v(1)=\sum_{i=1}^{m-2}\xi_i D_{0+}^{1/2}v(\eta_i),\quad v(0)=0. \tag{50}$$

Here,

$$f(t,x)=x^{1/3}+t^2+2,$$
$$g(t,x)=\frac{x^{1/3}}{(1+t^2)(1+x^{1/3})}+t^2+1. \tag{51}$$

Thus,

$$\alpha-\beta-1=0,$$
$$\sum_{i=1}^{m-2}\xi_i\eta_i^{\alpha-\beta-1}=\xi_1+\xi_2=\frac{1}{4}+\frac{1}{2}<1. \tag{52}$$

Set $\gamma=1/3$. Obviously, $f,g\in C[J\times\mathbb{R}^+,\mathbb{R}^+]$ and are increasing with respect to the second argument, $g(t,0)\ge1>0$. For $\lambda\in(0,1),t\in J,x\in\mathbb{R}^+$, we can notice that

$$g(t,\lambda x)=\frac{\lambda^{1/3}x^{1/3}}{(1+t^2)\left(1+\lambda^{1/3}x^{1/3}\right)}+t^2+1$$
$$\ge\frac{\lambda^{1/3}x^{1/3}}{(1+t^2)(1+x^{1/3})}+\lambda^{1/3}\left(t^2+1\right)$$
$$=\lambda^\gamma g(t,x), \tag{53}$$
$$f(t,\lambda x)=\lambda^{1/3}x^{1/3}+t^2+2$$
$$\ge\lambda^{1/3}\left(x^{1/3}+t^2+2\right)$$
$$=\lambda^\gamma f(t,x).$$

For $t\in J,x\in\mathbb{R}^+$, we deduce that

$$f(t,x)=x^{1/3}+t^2+2$$
$$\ge\frac{x^{1/3}}{(1+t^2)(1+x^{1/3})}+t^2+1 \tag{54}$$
$$=\delta_0 g(t,x),$$

where

$$\delta_0=1. \tag{55}$$

Thus, the assumptions of $(H_1)-(H_3)$ are satisfied. By Theorem 1, system (49)-(50) has a unique positive solution in $P_h\times P_h$, where $h(t)=t^{\alpha-1}$, $t\in[0,1]$.

Example 2. Letting $\alpha=3/2,\beta=1/2,m=4,\xi_1=\eta_1=1/4,\xi_2=\eta_2=1/2,\mu_1,\mu_2\in(0,+\infty),\mu_1\ge\mu_2$, we consider the following problem

$$\begin{cases} D_{0+}^{3/2}u(t)+\mu_1 f(t,u(t))+\mu_2 g(t,v(t))=0, & 0<t<1,\\ D_{0+}^{3/2}v(t)+\mu_1 f(t,v(t))+\mu_2 g(t,u(t))=0, & 0<t<1,\end{cases} \tag{56}$$

with the multipoint boundary conditions

$$\begin{cases} D_{0+}^{1/2}u(1) = \sum_{i=1}^{m-2} \xi_i D_{0+}^{1/2} u(\eta_i), & u(0) = 0, \\ D_{0+}^{1/2}v(1) = \sum_{i=1}^{m-2} \xi_i D_{0+}^{1/2} v(\eta_i), & v(0) = 0. \end{cases} \tag{57}$$

Here,

$$\begin{aligned} f(t,x) &= \frac{1+x^{1/3}}{2+x^{1/3}} + t + 1, \\ g(t,y) &= (1+y)^{-1/2} + t^2 + 1. \end{aligned} \tag{58}$$

We deduce that

$$\begin{aligned} &\alpha - \beta - 1 = 0, \\ &\sum_{i=1}^{m-2} \xi_i \eta_i^{\alpha-\beta-1} = \xi_1 + \xi_2 = \frac{1}{4} + \frac{1}{2} < 1. \end{aligned} \tag{59}$$

Set $\gamma = 1/3, \gamma_1 = 1/3, \gamma_2 = 1/2$. It is clear that $f, g \in C[J \times \mathbb{R}^+, \mathbb{R}^+]$ and $f(t,x)$ is nondecreasing in x, and $g(t,y)$ is nonincreasing in y, $f(t,x) \le 1 + 1 + 1 = 3$, $g(t,y) \le (1+0)^{-1/2} + 1 + 1 = 3$, and $0 < \gamma_1 + \gamma_2 = 5/6$. Moreover, for $k \in (0,1)$, we can find that

$$\begin{aligned} f(t,kx) &= \frac{1+(kx)^{1/3}}{2+(kx)^{1/3}} + t + 1 \\ &\ge \frac{1+(kx)^{1/3}}{2+x^{1/3}} + t + 1 \\ &\ge k^{1/3}\left(\frac{1+x^{1/3}}{2+x^{1/3}} + t + 1\right) \\ &= k^{1/3} f(t,x), \\ g(t,ky) &= (1+ky)^{-(1/2)} + t^2 + 1 \\ &= k^{-(1/2)}\left(\frac{1}{k} + y\right)^{-(1/2)} + t^2 + 1 \\ &\le k^{-(1/2)}(1+y)^{-(1/2)} + t^2 + 1 \\ &\le k^{-(1/2)}\left((1+y)^{-(1/2)} + t^2 + 1\right) \\ &= k^{-(1/2)} g(t,y). \end{aligned} \tag{60}$$

Then, all the conditions of Theorem 2 are fulfilled. Consequently, there exist $u_0, v_0 \in P$, and system (56)-(57) has exactly one positive solution in $[u_0, v_0] \times [u_0, v_0]$.

Data Availability

No data were used to support the study.

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this paper.

Authors' Contributions

The author read and approved the final manuscript.

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