Research Article

Common Best Proximity Coincidence Point Theorem for Dominating Proximal Generalized Geraghty in Complete Metric Spaces

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In this paper, we introduce a new concept of dominating proximal generalized Geraghty for two mappings and prove the existence and uniqueness of a common best proximity coincidence point in complete metric spaces. And also, we give an example for the main theorems. The main theorem is a generalization and improvement of some well-known theorems.

1. Introduction

The best proximity point problems have been attracted to many researchers as there are various applications in real-world problems. The optimization problem is one of the applications that benefit from the best proximity point theory. In other words, it helps finding an approximate solution to the fixed point problems even the mapping itself does not have a fixed point (see [1–23]). In literature, most works focus on suggesting suitable conditions to promise the existence of approximate optimal solutions. These results give the best proximity point theorem in a variety of approaches.

For instance, the work of Geraghty [24] is one of several important results inspired by the Banach contraction principle for the existence of fixed points for self mappings in metric spaces. In fact, this result generalizes previous concepts by introducing the class Θ of all mappings $\theta : [0, \infty) \rightarrow [0, 1]$ such that

$$\lim_{n \rightarrow \infty} \theta(t_n) = 1 \implies \lim_{n \rightarrow \infty} t_n = 0. \quad (1)$$

In 2012, Basha [25] proposed a result on common best proximity points with a property called proximal commutativity of mappings. Later, Kumam and Mongkolekeha [26]

considered common best proximity point theorems for proximity commuting mappings. In addition, this study has been done according to Geraghty's work in complete metric spaces. After that, Chen [27] established the definition of a mapping T generally dominates a mapping S and accomplished theorems of existence and uniqueness of common best proximity points for a pair of nonself mappings. Lately, Ayari [28] improved the class Θ of Geraghty and defined a new class $\mathcal{B}$ of the mappings $\beta : [0, \infty) \rightarrow [0, 1]$ such that

$$\lim_{n \rightarrow \infty} \beta(t_n) = 1 \implies \lim_{n \rightarrow \infty} t_n = 0. \quad (2)$$

Accordingly, the existence and uniqueness of best proximity points is guaranteed for α -proximal Geraghty nonself mappings on a closed subset of a complete metric space.

To generalize previous results, we are interested to extend our study to common best proximity coincidence points for two mappings under certain conditions. Specifically, we investigate the existence and uniqueness of common best proximity coincidence points for any pairs of two mappings that are dominating proximal generalized Geraghty on a complete metric space. In particular, this work is organized into three sections. First, the motivation of the present study is given as described above. Next, we recall some essential

definitions needed in our work. In Section 3, a new concept of dominating proximal generalized Geraghty for two mappings is introduced. Then, we show that a common best proximity coincidence point of these mappings uniquely exists under some additional assumptions. Moreover, an example is provided to support the main result. Lastly, we consider some further results following from our main theorem.

2. Preliminaries

In this section, we review some notations and important definitions to be used in the next section. Let (A, B) be a pair of nonempty subsets of a metric space (X, d) . We adopt the following notations:

$$\begin{aligned} d(A, B) &:= \inf \{d(a, b) : a \in A, b \in B\}, \\ A_0 &:= \{a \in A : \text{there exists } b \in B \text{ such that } d(a, b) = d(A, B)\}, \\ B_0 &:= \{b \in B : \text{there exists } a \in A \text{ such that } d(a, b) = d(A, B)\}. \end{aligned} \quad (3)$$

Definition 1 (see [1, 26, 29]). Let $S, T : A \longrightarrow B$ and $g : A \longrightarrow A$ be mappings.

An element $x^* \in A$ is said to be

(i) A best proximity point of T if

$$d(x^*, Tx^*) = d(A, B) \quad (4)$$

(ii) A best proximity coincidence point of the pair (g, T) if

$$d(gx^*, Tx^*) = d(A, B) \quad (5)$$

(iii) A common best proximity coincidence point of the pair (S, T) if

$$d(x^*, Sx^*) = d(A, B) = d(x^*, Tx^*) \quad (6)$$

Definition 2 (see [29]). Let $S, T : A \longrightarrow B$ be mappings. A pair (S, T) is said to commute proximally if for each $x, u, v \in A$,

$$d(u, Sx) = d(v, Tx) = d(A, B) \text{ implies } Sv = Tu. \quad (7)$$

3. Main Results

In this section, we introduce a class of pairs of some proximal generalized Geraghty contractions with dominating property and prove common best proximity theorem for this class.

Definition 3. Let $S, T : A \longrightarrow B$ be mappings. A pair (S, T) is said to be dominating proximal generalized Geraghty if there exists $\beta \in \mathcal{B}$ such that for each $x_1, x_2, u_1, u_2, v_1, v_2 \in A$,

$$d(u_1, Sx_1) = d(u_2, Sx_2) = d(A, B) = d(v_1, Tx_1) = d(v_2, Tx_2), \quad (8)$$

implies

$$d(u_1, u_2) \leq \beta(M(v_1, v_2, u_1, u_2))M(v_1, v_2, u_1, u_2), \quad (9)$$

where $M(v_1, v_2, u_1, u_2) = \max \{d(v_1, v_2), d(v_1, u_1), d(v_2, u_2), ((d(v_1, u_2) + d(v_2, u_1))/2)\}$.

Theorem 4. Let (A, B) be a pair of nonempty subsets of a complete metric space (X, d) , and let $S, T : A \longrightarrow B$ be mappings. Suppose that the pair (S, T) is dominating proximal generalized Geraghty. Assume that A_0 and B_0 are nonempty such that A_0 is closed. If the following assertions hold:

(i) $S(A_0) \subseteq B_0$ and $S(A_0) \subseteq T(A_0)$

(ii) S and T are continuous

(iii) S and T commute proximally

then there is only one common best proximity coincidence point x^* of (S, T) in A .

Proof. Let x_0 be a fixed element in A_0 . From the assumption that $S(A_0) \subseteq T(A_0)$, we get that for each element $x \in A_0$, there is an element $y \in A_0$ such that $Sx = Ty$. Then, we obtain a sequence $\{x_n\}$ in A_0 satisfying

$$Sx_n = Tx_{n+1}. \quad (10)$$

for each $n \geq 0$. Since $S(A_0) \subseteq B_0$, there exists an element $u_n \in A_0$ such that

$$d(u_n, Sx_n) = d(A, B), \quad (11)$$

for all $n \geq 0$. Further, we obtain that

$$d(A, B) = d(u_n, Sx_n) = d(u_n, Tx_{n+1}), \quad (12)$$

for all $n \geq 0$.

Our first goal is to show that $Su = Tu$ for some $u \in A_0$.

In the case that $u_{n_0} = u_{n_0+1}$ for some $n_0 \geq 0$, by (11) and (12), we get that

$$d(u_{n_0+1}, Sx_{n_0+1}) = d(A, B) = d(u_{n_0}, Sx_{n_0}) = d(u_{n_0}, Tx_{n_0+1}). \quad (13)$$

Since S and T commute proximally, $S(u_{n_0}) = T(u_{n_0+1}) = T(u_{n_0})$, and so we are done.

Now, for the harder part, assume that $u_n \neq u_{n+1}$ for all $n \geq 0$. From (12), note that

$$d(u_n, Sx_n) = d(u_{n+1}, Sx_{n+1}) = d(A, B) = d(u_{n-1}, Tx_n) = d(u_n, Tx_{n+1}), \quad (14)$$

for all $n \geq 1$. Since (S, T) is dominating proximal generalized Geraghty, we have that

$$\begin{aligned} d(u_n, u_{n+1}) &\leq \beta(M(u_{n-1}, u_n, u_n, u_{n+1}))M(u_{n-1}, u_n, u_n, u_{n+1}) \\ &\leq M(u_{n-1}, u_n, u_n, u_{n+1}). \end{aligned} \quad (15)$$

Consider that

$$\begin{aligned} M(u_{n-1}, u_n, u_n, u_{n+1}) &= \max \left\{ d(u_{n-1}, u_n), d(u_n, u_{n+1}), \frac{d(u_{n-1}, u_{n+1})}{2} \right\} \\ &\leq \max \left\{ d(u_{n-1}, u_n), d(u_n, u_{n+1}), \frac{d(u_{n-1}, u_n) + d(u_n, u_{n+1})}{2} \right\} \\ &= \max \{d(u_{n-1}, u_n), d(u_n, u_{n+1})\} \leq M(u_{n-1}, u_n, u_n, u_{n+1}). \end{aligned} \quad (16)$$

This implies that $M(u_{n-1}, u_n, u_n, u_{n+1}) = \max \{d(u_{n-1}, u_n), d(u_n, u_{n+1})\}$ for all $n \geq 1$.

Next, we prove that the sequence $\{d(u_n, u_{n+1})\}$ converges to 0.

Consider the following two cases.

Case 1. $M(u_{n-1}, u_n, u_n, u_{n+1}) = d(u_{n-1}, u_n)$.

From (15), we have that

$$d(u_n, u_{n+1}) \leq \beta(d(u_{n-1}, u_n))d(u_{n-1}, u_n) \leq d(u_{n-1}, u_n), \quad (17)$$

for all $n \geq 1$. Therefore, $\{d(u_n, u_{n+1})\}$ is a nonincreasing sequence which is bounded below, and so it is convergent. To obtain that $\lim_{n \rightarrow \infty} d(u_n, u_{n+1}) = 0$, suppose on the contrary that $\lim_{n \rightarrow \infty} d(u_n, u_{n+1}) > 0$. By (17), letting $n \rightarrow \infty$ implies

$$1 = \lim_{n \rightarrow \infty} \frac{d(u_n, u_{n+1})}{d(u_{n-1}, u_n)} \leq \lim_{n \rightarrow \infty} \beta(d(u_{n-1}, u_n)) \leq 1. \quad (18)$$

Since $\lim_{n \rightarrow \infty} \beta(d(u_{n-1}, u_n)) = 1$, by the definition of β ,

$$0 < \lim_{n \rightarrow \infty} d(u_n, u_{n+1}) = \lim_{n \rightarrow \infty} d(u_{n-1}, u_n) = 0, \quad (19)$$

which is a contradiction. Thus, we get that $\lim_{n \rightarrow \infty} d(u_n, u_{n+1}) = 0$.

Case 2. $M(u_{n-1}, u_n, u_n, u_{n+1}) = d(u_n, u_{n+1})$. Similarly, by (15), we have that

$$d(u_n, u_{n+1}) \leq \beta(d(u_n, u_{n+1}))d(u_n, u_{n+1}) \leq d(u_n, u_{n+1}). \quad (20)$$

Since $d(u_n, u_{n+1}) > 0$ for all $n \geq 0$, we get that $1 \leq \beta(d(u_n, u_{n+1})) \leq 1$, and hence, $\lim_{n \rightarrow \infty} \beta(d(u_n, u_{n+1})) = 1$. By the definition of β , we also have that $\lim_{n \rightarrow \infty} d(u_n, u_{n+1}) = 0$.

Due to both cases, we obtain the desired limit

$$\lim_{n \rightarrow \infty} d(u_n, u_{n+1}) = 0. \quad (21)$$

Now, we claim that $\{u_n\}$ is a Cauchy sequence.

Suppose contradiction, that is, $\{u_n\}$ is not a Cauchy sequence. Then, there exists $\varepsilon > 0$ such that there are subsequences $\{u_{m_k}\}$ and $\{u_{n_k}\}$ of $\{u_n\}$ so that for all $k \in \mathbb{N}$ with $m_k > n_k > k$, we obtain

$$d(u_{m_k}, u_{n_k}) \geq \varepsilon. \quad (22)$$

In addition, we can choose the smallest n_k satisfying (22) for all $k \in \mathbb{N}$ so that

$$d(u_{m_k}, u_{n_{k-1}}) < \varepsilon. \quad (23)$$

By using (22) and (23), we have that

$$\varepsilon \leq d(u_{m_k}, u_{n_k}) \leq d(u_{m_k}, u_{n_{k-1}}) + d(u_{n_{k-1}}, u_{n_k}) < \varepsilon + d(u_{n_{k-1}}, u_{n_k}). \quad (24)$$

Since $\lim_{n \rightarrow \infty} d(u_n, u_{n+1}) = 0$, taking the limit as $k \rightarrow \infty$ in (24) implies

$$\lim_{k \rightarrow \infty} d(u_{m_k}, u_{n_k}) = \varepsilon. \quad (25)$$

Consider, by the triangular inequality, that

$$d(u_{m_k}, u_{n_k}) \leq d(u_{m_k}, u_{m_{k+1}}) + d(u_{m_{k+1}}, u_{n_{k+1}}) + d(u_{n_{k+1}}, u_{n_k}). \quad (26)$$

Consequently, $\varepsilon = \lim_{k \rightarrow \infty} d(u_{m_k}, u_{n_k}) \leq \lim_{k \rightarrow \infty} d(u_{m_{k+1}}, u_{n_{k+1}})$.

In the same way, we get that

$$d(u_{m_{k+1}}, u_{n_{k+1}}) \leq d(u_{m_{k+1}}, u_{m_k}) + d(u_{m_k}, u_{n_k}) + d(u_{n_k}, u_{n_{k+1}}), \quad (27)$$

and so $\lim_{k \rightarrow \infty} d(u_{m_{k+1}}, u_{n_{k+1}}) \leq \lim_{k \rightarrow \infty} d(u_{m_k}, u_{n_k}) = \varepsilon$.

Thus,

$$\lim_{k \rightarrow \infty} d(u_{m_{k+1}}, u_{n_{k+1}}) = \varepsilon. \quad (28)$$

Since $\{u_{m_k}\}$ and $\{u_{n_k}\}$ satisfy equations (11) and (12), we obtain that

$$\begin{aligned} d(u_{n_{k+1}}, Sx_{n_{k+1}}) &= d(A, B) = d(u_{n_k}, Tx_{n_{k+1}}), \\ d(u_{m_{k+1}}, Sx_{m_{k+1}}) &= d(A, B) = d(u_{m_k}, Tx_{m_{k+1}}), \end{aligned} \quad (29)$$

for each $k \geq 1$. Since (S, T) is dominating proximal generalized Geraghty,

$$d(u_{n_k+1}, u_{m_k+1}) \leq \beta(M(u_{n_k}, u_{m_k}, u_{n_k+1}, u_{m_k+1})) \cdot M(u_{n_k}, u_{m_k}, u_{n_k+1}, u_{m_k+1}), \quad (30)$$

where

$$M(u_{n_k}, u_{m_k}, u_{n_k+1}, u_{m_k+1}) = \max \left\{ d(u_{n_k}, u_{m_k}), d(u_{n_k}, u_{n_k+1}), d(u_{m_k}, u_{m_k+1}), \frac{d(u_{n_k}, u_{m_k+1}) + d(u_{m_k}, u_{n_k+1})}{2} \right\}. \quad (31)$$

By (21), we observe that

$$\begin{aligned} \lim_{k \rightarrow \infty} \frac{d(u_{n_k}, u_{m_k+1}) + d(u_{m_k}, u_{n_k+1})}{2} \\ \leq \lim_{k \rightarrow \infty} \frac{d(u_{n_k}, u_{m_k}) + d(u_{m_k}, u_{m_k+1}) + d(u_{m_k}, u_{n_k}) + d(u_{n_k}, u_{n_k+1})}{2} \\ \leq \lim_{k \rightarrow \infty} d(u_{n_k}, u_{m_k}), \end{aligned} \quad (32)$$

and, as a consequence,

$$\begin{aligned} \lim_{k \rightarrow \infty} d(u_{n_k}, u_{m_k}) &\leq \lim_{k \rightarrow \infty} M(u_{n_k}, u_{m_k}, u_{n_k+1}, u_{m_k+1}) \\ &= \max \left\{ \lim_{k \rightarrow \infty} d(u_{n_k}, u_{m_k}), \lim_{k \rightarrow \infty} d(u_{n_k}, u_{n_k+1}), \lim_{k \rightarrow \infty} d(u_{m_k}, u_{m_k+1}), \right. \\ &\quad \left. \lim_{k \rightarrow \infty} \frac{d(u_{n_k}, u_{m_k+1}) + d(u_{m_k}, u_{n_k+1})}{2} \right\} \leq \lim_{k \rightarrow \infty} d(u_{n_k}, u_{m_k}). \end{aligned} \quad (33)$$

Hence, (25) implies that $\lim_{k \rightarrow \infty} M(u_{n_k}, u_{m_k}, u_{n_k+1}, u_{m_k+1}) = \lim_{k \rightarrow \infty} d(u_{n_k}, u_{m_k}) = \varepsilon > 0$. Then, by (28) and (30), we obtain that

$$\begin{aligned} 1 &= \lim_{k \rightarrow \infty} \frac{d(u_{n_k+1}, u_{m_k+1})}{M(u_{n_k}, u_{m_k}, u_{n_k+1}, u_{m_k+1})} \\ &\leq \lim_{k \rightarrow \infty} \beta(M(u_{n_k}, u_{m_k}, u_{n_k+1}, u_{m_k+1})) \leq 1. \end{aligned} \quad (34)$$

By the property of β , we obtain that

$$0 < \lim_{n \rightarrow \infty} d(u_{n_k}, u_{m_k}) = \lim_{n \rightarrow \infty} M(u_{n_k}, u_{m_k}, u_{n_k+1}, u_{m_k+1}) = 0, \quad (35)$$

a contradiction. Therefore, we can conclude that $\{u_n\}$ is a Cauchy sequence.

The essential observation is that $\{u_n\}$ is a Cauchy sequence in the closed subset A_0 of the complete metric space X . Then, there exists $u \in A_0$ such that $\lim_{n \rightarrow \infty} u_n = u$. Consider, by (11) and (12), that $d(u_n, Sx_n) = d(u_{n-1}, Tx_n) = d(A, B)$. Since S and T commute proximally,

$$Su_{n-1} = Tu_n, \quad (36)$$

for all $n \geq 1$. By the continuity of S and T ,

$$Su = \lim_{n \rightarrow \infty} Su_{n-1} = \lim_{n \rightarrow \infty} Tu_n = Tu. \quad (37)$$

We are now in a position to show that a best proximity coincidence point of (S, T) exists. Since $S(A_0) \subseteq B_0$, there exists $x^* \in A_0$ such that

$$d(x^*, Tu) = d(x^*, Su) = d(A, B). \quad (38)$$

By the assumption that S and T commute proximally, $Sx^* = Tx^*$. According to the assumption that $S(A_0) \subseteq B_0$, there exists $z^* \in A_0$ such that

$$d(z^*, Tx^*) = d(z^*, Sx^*) = d(A, B). \quad (39)$$

Next, we claim that $x^* = z^*$. Suppose that $x^* \neq z^*$, i.e., $d(x^*, z^*) > 0$. We observe that

$$M(x^*, z^*, x^*, z^*) = \max \{d(x^*, z^*), d(x^*, x^*), d(z^*, z^*)\} = d(x^*, z^*),$$

$$\begin{aligned} d(x^*, z^*) &\leq \beta(M(x^*, z^*, x^*, z^*))M(x^*, z^*, x^*, z^*) \\ &= \beta(d(x^*, z^*))d(x^*, z^*) \leq d(x^*, z^*). \end{aligned} \quad (40)$$

Since $d(x^*, z^*) > 0$, we have $1 \leq \beta(d(x^*, z^*)) \leq 1$. By the property of β , $d(x^*, z^*) = 0$. This contradicts the assumption that $x^* \neq z^*$. Thus, $x^* = z^*$, and hence

$$d(x^*, Sx^*) = d(A, B) = d(x^*, Tx^*). \quad (41)$$

That is, the element $x^* \in A$ is a common best proximity coincidence point of (S, T) .

Finally, we have to show that the point x^* is unique.

Let $y^* \in A$ be a common best proximity coincidence point of (S, T) . Then

$$d(x^*, Sx^*) = d(y^*, Sy^*) = d(A, B) = d(x^*, Tx^*) = d(y^*, Ty^*). \quad (42)$$

Notice that $M(x^*, y^*, x^*, y^*) = \max \{(dx^*, y^*), d(x^*, x^*), d(y^*, y^*)\} = d(x^*, y^*)$. Since (S, T) is dominating proximal generalized Geraghty, we obtain that

$$d(x^*, y^*) \leq \beta(d(x^*, y^*))d(x^*, y^*) \leq d(x^*, y^*). \quad (43)$$

If $d(x^*, y^*) > 0$, then $\beta(d(x^*, y^*)) = 1$, and so, by using the property of β ,

$$d(x^*, y^*) = 0, \quad (44)$$

a contradiction. Thus, $d(x^*, y^*)$ must be zero. As a result, $x^* = y^*$. The proof is now completed.

Example 5. Let $X = \mathbb{R}^2$ equipped with the metric d given by

$$d((x_1, y_1), (x_2, y_2)) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}. \quad (45)$$

Let $A = \{(x, 1) : 0 \leq x \leq 5\}$ and $B = \{(x, -1) : 0 \leq x \leq 5\}$. It is easy to see that $d(A, B) = 2$. Define the mappings $S, T : A \rightarrow B$ by

$$S(x, 1) = (\ln(1+x), -1) \text{ and } T(x, 1) = (x, -1), \quad (46)$$

for all $(x, 1) \in A$. Notice that S and T are continuous. To show that the pair (S, T) is dominating proximal generalized Geraghty, define the mapping $\beta : [0, \infty) \rightarrow [0, 1]$ by

$$\beta(t) = \begin{cases} 1, & t = 0, \\ \frac{\ln(1+t)}{t}, & t > 0. \end{cases} \quad (47)$$

Then, $\beta \in \mathcal{B}$. Let $x_1, x_2, u_1, u_2, v_1, v_2 \in A$ satisfying

$$d(u_1, Sx_1) = d(u_2, Sx_2) = d(A, B) = d(v_1, Tx_1) = d(v_2, Tx_2). \quad (48)$$

Observe that they must have the following forms:

$$\begin{aligned} x_1 &= (\hat{x}_1, 1), x_2 = (\hat{x}_2, 1), u_1 = (\hat{u}_1, 1), \\ u_2 &= (\hat{u}_2, 1), v_1 = (\hat{v}_1, 1), v_2 = (\hat{v}_2, 1), \end{aligned} \quad (49)$$

where $\hat{u}_1 = \ln(1+\hat{x}_1)$, $\hat{u}_2 = \ln(1+\hat{x}_2)$, $\hat{v}_1 = \hat{x}_1$, $\hat{v}_2 = \hat{x}_2$, and $\hat{x}_1, \hat{x}_2 \in [0, 5]$. To obtain the inequality (9), if $u_1 = u_2$, then we are done. Assume that $u_1 \neq u_2$. Then, $\hat{u}_1, \hat{u}_2, \hat{v}_1$, and $\hat{v}_2$ are all distinct. As a consequence, $M(v_1, v_2, u_1, u_2) > 0$. Thus, we have that

$$\begin{aligned} d(u_1, u_2) &= |\hat{u}_1 - \hat{u}_2| = |\ln(1+\hat{v}_1) - \ln(1+\hat{v}_2)| \\ &= \left| \ln \left(\frac{1+\hat{v}_2+\hat{v}_1-\hat{v}_2}{1+\hat{v}_2} \right) \right| \leq \ln(1+|\hat{v}_1-\hat{v}_2|) \\ &\leq \ln(1+M(v_1, v_2, u_1, u_2)) \\ &= \left[\frac{\ln(1+M(v_1, v_2, u_1, u_2))}{M(v_1, v_2, u_1, u_2)} \right] M(v_1, v_2, u_1, u_2) \\ &= \beta(M(v_1, v_2, u_1, u_2)) M(v_1, v_2, u_1, u_2). \end{aligned} \quad (50)$$

Therefore, the pair (S, T) is dominating proximal generalized Geraghty.

Next, consider, by the definition of A_0 and B_0 , that $A_0 = A$ and $B_0 = B$. Additionally,

$$S(A_0) = \{(x, -1) : 0 \leq x \leq \ln 6\} \subseteq \{(x, -1) : 0 \leq x \leq 5\} = B_0 = T(A_0). \quad (51)$$

Now, it remains to show that S and T commute proximally. Let $x, u, v \in A$ such that

$$d(u, Sx) = d(v, Tx) = d(A, B). \quad (52)$$

Consequently, $x = (\hat{x}, 1)$, $u = (\hat{u}, 1)$, $v = (\hat{v}, 1)$, where $\hat{u} = \ln(1+\hat{x})$ and $\hat{v} = \hat{x}$. Thus,

$$Sv = (\ln(1+\hat{v}), -1) = (\ln(1+\hat{x}), -1) = (\hat{u}, -1) = Tu. \quad (53)$$

Thus, S and T commute proximally.

Finally, by Theorem 4, we can conclude that there is a unique common best proximity coincidence point of the pair (S, T) . In fact, the point $(0, 1)$ is the unique common best proximity coincidence point of (S, T) .

As a consequence of our result, the following corollaries are given. Precisely, these are the special cases of Theorem 4 when $\beta(t) = k$ for $k \in [0, 1)$, and $\beta(t) = e^{-kt}$ for $k > 0$, respectively.

Corollary 6. Let (A, B) be a pair of nonempty subsets of a complete metric space (X, d) . Assume that A_0 and B_0 are nonempty such that A_0 is closed. If $S, T : A \rightarrow B$ are mappings such that the following assertions hold:

- (1) $S(A_0) \subseteq B_0$ and $S(A_0) \subseteq T(A_0)$
- (2) S and T are continuous
- (3) S and T commute proximally
- (4) There exists $k \in [0, 1)$ such that for each $x_1, x_2, u, v, x, y \in A$

$$d(u, Sx_1) = d(v, Sx_2) = d(A, B) = d(x, Tx_1) = d(y, Tx_2) \quad (54)$$

implies

$$d(u, v) \leq kM(x, y, u, v), \quad (55)$$

where $M(x, y, u, v) = \max \{d(x, y), d(x, u), d(y, v), ((d(x, v) + d(y, u))/2)\}$, then there is a unique common best proximity coincidence point $x^* \in A$ of the pair (S, T) .

Corollary 7. Let (A, B) be a pair of nonempty subsets of a complete metric space (X, d) . Assume that A_0 and B_0 are nonempty such that A_0 is closed. If $S, T : A \rightarrow B$ are mappings such that the following assertions hold:

- (1) $S(A_0) \subseteq B_0$ and $S(A_0) \subseteq T(A_0)$
- (2) S and T are continuous
- (3) S and T commute proximally
- (4) There exists $k > 0$ such that for each $x_1, x_2, u, v, x, y \in A$

$$d(u, Sx_1) = d(v, Sx_2) = d(A, B) = d(x, Tx_1) = d(y, Tx_2) \quad (56)$$

implies

$$d(u, v) \leq e^{-kM(x, y, u, v)} M(x, y, u, v), \quad (57)$$

where $M(x, y, u, v) = \max \{d(x, y), d(x, u), d(y, v), ((d(x, v) + d(y, u))/2)\}$, then there is a unique common best proximity coincidence point $x^* \in A$ of the pair (S, T) .

4. Conclusion

In this work, we give an idea of dominating proximal generalized Geraghty for a pair of mappings and give the existence and uniqueness theorem for a common best proximity coincidence point of these pairs in a complete metric space with some extra assumptions. Further, we present an example of this result. Now, we pose the following open problem.

Open Problem 1. Can Theorem 4 be extended to the framework of complete metric spaces endowed with graphs?

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors have no conflict of interests regarding the publication of this paper.

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