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Research Article

Three-Parameter Logarithm and Entropy

Cristina B. Corcino[1,2] **and Roberto B. Corcino**[1,2]

[1]Research Institute for Computational Mathematics and Physics, Cebu Normal University, Cebu City, Philippines
[2]Department of Mathematics, Cebu Normal University, Cebu City, Philippines

Correspondence should be addressed to Roberto B. Corcino; rcorcino@yahoo.com

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A three-parameter logarithmic function is derived using the notion of q-analogue and ansatz technique. The derived three-parameter logarithm is shown to be a generalization of the two-parameter logarithmic function of Schwämmle and Tsallis as the latter is the limiting function of the former as the added parameter goes to 1. The inverse of the three-parameter logarithm and other important properties are also proved. A three-parameter entropic function is then defined and is shown to be analytic and hence Lesche-stable, concave, and convex in some ranges of the parameters.

1. Introduction

The concept of entropy provides deep insight into the direction of spontaneous change for many everyday phenomena. For example, a block of ice placed on a hot stove surely melts, while the stove grows cooler. Such a process is called irreversible because no slight change will cause the melted water to turn back into ice while the stove grows hotter [1]. The concept of entropy was first introduced by German physicist Rudolf Clausius as a precise way of expressing the second law of thermodynamics.

The Boltzmann equation for entropy is

$$S = k_B \ln \omega, \tag{1}$$

where k_B is the Boltzmann constant [2] and ω is the number of different ways or microstates in which the energy of the molecules in a system can be arranged on energy levels [3]. The Boltzmann entropy plays a crucial role in the foundation of statistical mechanics and other branches of science [4].

The Boltzmann-Gibbs-Shannon entropy [5, 6] is given by

$$S_{\mathrm{BGS}} \equiv -k \sum_{i=1}^{\omega} p_i \ln p_i = k \sum_{i=1}^{\omega} p_i \ln \frac{1}{p_i}, \tag{2}$$

where

$$\sum_{i=1}^{\omega} p_i = 1. \tag{3}$$

S_{BGS} is a generalization of the Boltzmann entropy because if $p_i = 1/\omega$, for all i,

$$S_{\mathrm{BGS}} = k \ln \omega. \tag{4}$$

Systems presenting long-range interactions and/or long-duration memory have been shown not well described by the Boltzmann-Gibbs statistics. Some examples may be found in gravitational systems, Levy flights, fractals, turbulence physics, and economics. In an attempt to deal with such systems, Tsallis [7] postulated a nonextensive entropy which generalizes Boltzmann-Gibbs entropy through an entropic index q [8]. Another generalization was also suggested by Renyi [9]. Abe [10] proposed how to generate entropy functionals.

Tsallis q-entropy [7] is given by

$$S_q \equiv k \frac{1 - \sum_{\{i=1\}}^{\omega} p_i^q}{q-1} = k \sum_{i=1}^{\omega} p_i \ln_q \frac{1}{p_i}, \tag{5}$$

where $q \in \mathbb{R}$, $\sum_{i=1}^{\omega} p_i = 1$ and

$$\ln_q x \equiv \frac{x^{1-q}-1}{1-q}, \ (\ln_1 x = \ln x), \tag{6}$$

which is referred to as q-logarithm. If $p_i = 1/\omega$, for all i, then

$$S_q = k \ln_q \omega. \tag{7}$$

The inverse of the q-logarithm is the q-exponential

$$e_q^x \equiv [1+(1-q)x]_+^{1/1-q}, \ \ (e_1^x = e^x), \tag{8}$$

where $[\cdots]_+$ is zero if its argument is nonpositive.

A q-sum and q-product and their calculus studied in [11] were, respectively, defined as follows (these were also mentioned in [5]):

$$\begin{gathered} x \oplus_q y \equiv x + y + (1-q)xy, \ (x \oplus_1 y = x + y), \\ x \otimes_q y \equiv \left(x^{1-q} + y^{1-q} - 1\right)^{\frac{1}{1-q}}, \ (x \otimes_1 y = xy). \end{gathered} \tag{9}$$

The q-logarithm satisfies the following properties:

$$\begin{gathered} \ln_q(xy) = \ln_q x \oplus_q \ln_q y, \\ \ln_q\left(x \otimes_q y\right) = \ln_q x + \ln_q y. \end{gathered} \tag{10}$$

Then, a two-parameter logarithm was defined and presented along with a two-parameter entropy in [5]. It was defined as follows:

$$\ln_{q,q'} x = \frac{1}{1-q'}\left[\exp\left(\frac{1-q'}{1-q}\left(x^{1-q}-1\right)\right)-1\right]. \tag{11}$$

The above doubly deformed logarithm satisfies

$$\ln_{q,q'}\left(x \otimes_q y\right) = \ln_{q,q'} x \oplus_{q'} \ln_{q,q'} y. \tag{12}$$

Properties of the two-parameter logarithm and those of the two-parameter entropy were proved in [5]. Probability distribution in the canonical ensemble of the two-parameter entropy was obtained in [12] while applications were discussed in [13].

In Section 2 of the present paper, a three-parameter logarithm $\ln_{q,q',r} x$, where $q, q', r \in \mathbb{R}$, is derived using q-analogues and ansatz technique. In Section 3, the inverse of the three-parameter logarithm is derived and some properties are proved. A three-parameter entropy and its properties are presented in Section 4, and conclusion is given in Section 5.

2. Three-Parameter Logarithm

As $x = e^{\ln x}$, a q-analogue of x will be defined by

$$[x]_q = e^{\ln_q x}, \tag{13}$$

where $\ln_q x$ is defined in (6). Similarly, the q'-analogue of $[x]_q$ is defined by

$$[x]_{q,q}{}' = e^{\ln_{q,q'} x}, \tag{14}$$

where $\ln_{q,q}{}' x$ is as defined in (11), which can be written as

$$\ln_{q,q'} x = \frac{[x]_q^{1-q'}-1}{1-q'} = \frac{\left(e^{\ln_q x}\right)^{1-q'}-1}{1-q'}. \tag{15}$$

The three-parameter logarithm is then defined as

$$\ln_{q,q}{}',_r x = \frac{[x]_{q,q'}^{1-r}-1}{1-r} = \frac{\left(e^{\ln_{q,q'} x}\right)^{1-r}-1}{1-r}, \tag{16}$$

from which

$$\ln_{q,q}{}',_r x \equiv \frac{1}{1-r}\left\{e^{\left(1/1-q'\left\{e^{(1-q')\ln_q x}-1\right\}\right)^{1-r}}-1\right\}. \tag{17}$$

To obtain similar property as that in (12), define $x \otimes_{q,q}{}' y$ as the q'-analogue of $x \otimes_q y$. That is,

$$x \otimes_{q,q}{}' y \equiv \left[x \otimes_q y\right]_q{}' = \left([x]_{q'}^{1-q} + [y]_{q'}^{1-q} - 1\right)^{\frac{1}{1-q}}. \tag{18}$$

Lemma 1. *The following relations hold*

$$\ln_{q,q}{}'\left(x \otimes_{q'} y\right) = \ln_{q,q}{}' x + \ln_{q,q}{}' y, \tag{19}$$

$$\ln_{q,q}{}',_r\left(x \otimes_q{}' y\right) = \ln_{q,q}{}',_r x \oplus_r \ln_{q,q}{}',_r y. \tag{20}$$

Proof. From (16) and (18),

$$\begin{aligned} &\ln_{q,q}{}'\left(x \otimes_q{}' y\right) \\ &= \frac{\left[x \otimes_q{}' y\right]_q^{1-q'}-1}{1-q'} = \frac{\left\{\left([x]_q^{1-q'} + [y]_q^{1-q'} - 1\right)^{1/1-q'}\right\}^{1-q'}-1}{1-q'} \\ &= \frac{[x]_q^{1-q'} + [y]_q^{1-q'} - 1 - 1}{1-q'} = \frac{[x]_q^{1-q'}-1}{1-q'} + \frac{[y]_q^{1-q'}-1}{1-q'} \\ &= \ln_{q,q}{}' x + \ln_{q,q}{}' y. \end{aligned} \tag{21}$$

In similar manner and using (14),

$$\begin{aligned}
&\ln_{q,q',r}\left(x\otimes_{q'}y\right)\\
&=\frac{\left[x\otimes_{q'}y\right]_{q,q'}^{1-r}-1}{1-r}=\frac{\left\{e^{\ln_{q,q'}\left(x\otimes_{q'}y\right)}\right\}^{1-r}-1}{1-r}\\
&=\frac{\left(e^{\ln_{q,q'}x+\ln_{q,q'}y}\right)^{1-r}-1}{1-r}=\frac{\left(e^{\ln_{q,q'}x}\right)^{1-r}\left(e^{\ln_{q,q'}y}\right)^{1-r}-1}{1-r}\\
&=\frac{\left\{\left(e^{\ln_{q,q'}x}\right)^{1-r}-1\right\}+\left\{\left(e^{\ln_{q,q'}y}\right)^{1-r}-1\right\}+\left\{\left(e^{\ln_{q,q'}x}\right)^{1-r}-1\right\}\left\{\left(e^{\ln_{q,q'}y}\right)^{1-r}-1\right\}}{1-r}.
\end{aligned} \tag{22}$$

Thus,

$$\begin{aligned}
\ln_{q,q',r}\left(x\otimes_{q'}y\right)&=\frac{\left(e^{\ln_{q,q'}x}\right)^{1-r}-1}{1-r}+\frac{\left(e^{\ln_{q,q'}y}\right)^{1-r}-1}{1-r}+(1-r)\\
&\quad\cdot\left[\frac{1}{1-r}\left(e^{\ln_{q,q'}x}\right)^{1-r}-1\right]\left[\frac{1}{1-r}\left(e^{\ln_{q,q'}y}\right)^{1-r}-1\right]\\
&=\ln_{q,q',r}x+\ln_{q,q',r}y+(1-r)\left[\ln_{q,q',r}x\right]\left[\ln_{q,q',r}y\right]\\
&=\ln_{q,q',r}x\oplus_r\ln_{q,q',r}y,
\end{aligned} \tag{23}$$

which is the desired relation analogous to (12). ?

One can also derive (17) using ansatz. To do this, let $x=y$ in (20). Then,

$$\ln_{q,q',r}\left(x\otimes_{q'}x\right)=\ln_{q,q',r}x\oplus_r\ln_{q,q',r}x. \tag{24}$$

Lemma 2. *If* $\ln_{q,q',r}x=G(\ln_{q,q'}x)=G(z)$, *then*

$$G(2z)=2G(z)+(1-r)[G(z)]^2. \tag{25}$$

Moreover, when $z=\ln_{q,q}{'}x$, *the ansatz*

$$G(z)=\frac{1}{1-r}(b^z-1), \tag{26}$$

satisfies equation (25).

Proof. Note that from (21)

$$\begin{aligned}
\ln_{q,q',r}\left(x\otimes_{q'}x\right)&=G\left(\ln_{q,q'}\left(x\otimes_{q'}x\right)\right)\\
&=G\left(\ln_{q,q'}x+\ln_{q,q'}x\right)\\
&=G\left(2\ln_{q,q'}x\right)=G(2z).
\end{aligned} \tag{27}$$

Thus, from (23) and (20),

$$\begin{aligned}
G\left(2\ln_{q,q'}x\right)&=\ln_{q,q',r}x\oplus_r\ln_{q,q',r}x=\ln_{q,q',r}x+\ln_{q,q',r}x\\
&\quad+(1-r)\left(\ln_{q,q',r}x\right)^2=2G\left(\ln_{q,q'}x\right)+(1-r)\\
&\quad\cdot\left[G\left(\ln_{q,q'}x\right)\right]^2 G(2z)=2G(z)+(1-r)[G(z)]^2.
\end{aligned} \tag{28}$$

Then, the ansatz in (26) will give

$$\begin{aligned}
&2G(z)+(1-r)[G(z)]^2\\
&=2\cdot\frac{1}{1-r}(b^z-1)+(1-r)\left[\frac{1}{1-r}(b^z-1)\right]^2\\
&=\frac{2}{1-r}(b^z-1)+\frac{(b^z-1)^2}{1-r}=\frac{2b^z-2+b^{2z}-2b^z+1}{1-r}\\
&=\frac{2b^z-2+b^{2z}-2b^z+1}{1-r}=\frac{b^{2z}-1}{1-r}=G(2z),
\end{aligned} \tag{29}$$

which means that (26) solves equation (25). ?

Lemma 2. implies that

$$G(z)=G\left(\ln_{q,q'}x\right)=\ln_{q,q',r}x=\frac{1}{1-r}\left(b^{\ln_{q,q'}x}-1\right). \tag{30}$$

Using the property that $d/dx\ln_{q,q',r}x|_{x=1}=1$,which is a natural property of a logarithmic function, it is determined that $b=e^{1-r}$. Consequently,

$$\ln_{q,q',r}x=\frac{1}{1-r}\left(e^{(1-r)\ \ln_{q,q'}x}-1\right). \tag{31}$$

Explicitly,

$$\ln_{q,q',r}x=\frac{1}{1-r}\left(e^{1-r/1-q'\ \exp\left(\left[\left(1-q'\right)/(1-q)\right]\left(x^{1-q}-1\right)\right)}-1\right), \tag{32}$$

which is the same as that in (17). The preceding equation can be written as

$$\ln_{q,q',r}x=\ln_r e^{\ln_{q,q'}x}. \tag{33}$$

It can be easily verified that

$$\lim_{r\to1}\ln_{q,q',r}x=\ln_{q,q'}x. \tag{34}$$

Graphs of $\ln_{q,q',r}x$ for $q=q'=r$ are shown in Figure 1 while graphs of $\ln_{q,q',r}x$ with one fixed parameter are shown in Figure 2.

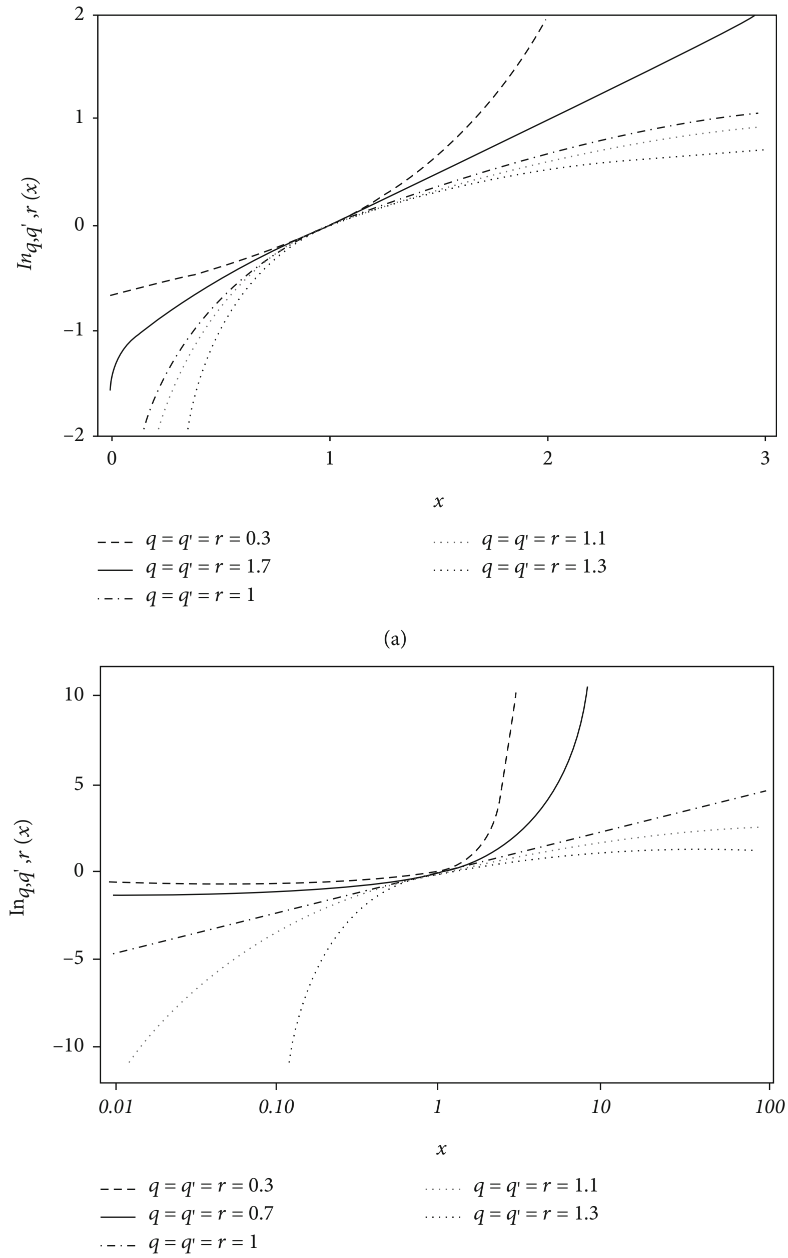


FIGURE 1: Illustration of the three-parameter logarithm in equation (32), setting $q = q' = r$ in (a) linear scales and (b) semilogarithmic scales.

3. Properties

In this section, the inverse of the three-parameter logarithmic function will be derived. It is also verified that the derivative of this logarithm at $x = 1$ is 1 and that the value of the function at $x = 1$ is zero. Moreover, it is shown that the following equality holds

$$\ln_{q,q',r}\frac{1}{x} = -\ln_{2-q,2-q',2-r}x. \tag{35}$$

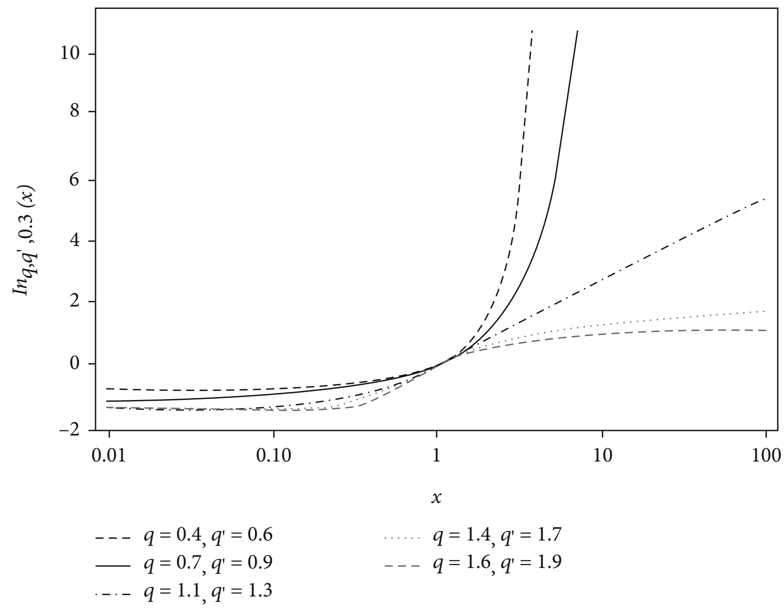
It follows from (16) that the three-parameter logarithmic function is an increasing function of x. Thus, a unique inverse function exists.

Theorem 3. *The inverse of the three-logarithmic function is given by*

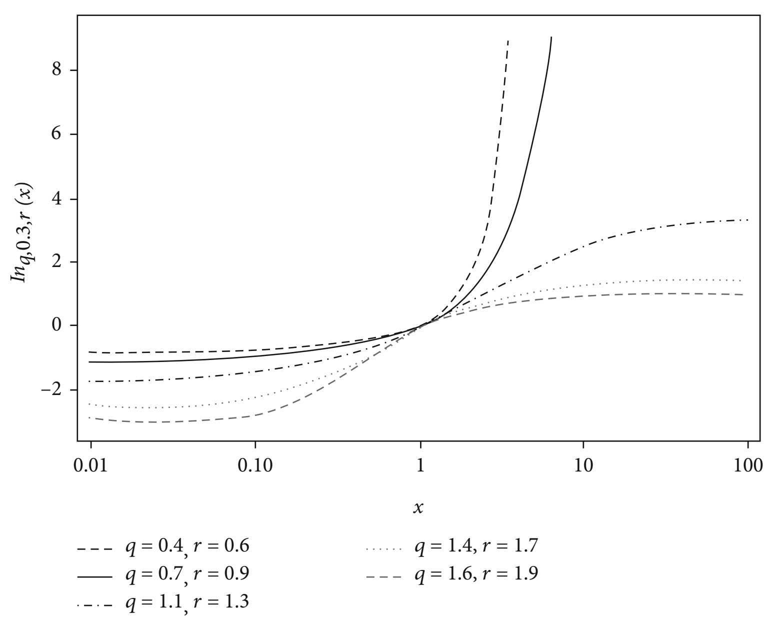
$$e^{y}_{q,q',r} = \exp_q\left\{\ln e_{q'}^{\ln e_r^{y}}\right\}. \tag{36}$$

Proof. To find the inverse function, let $y = \ln_{q,q',r}(x)$ and solve for x. That is,

$$y = \frac{1}{1-r}\left\{\exp\left(\frac{1-r}{1-q'}\exp\left(\frac{1-q'}{1-q}\left(x^{1-q}-1\right)\right)-1\right)-1\right\}, \tag{37}$$

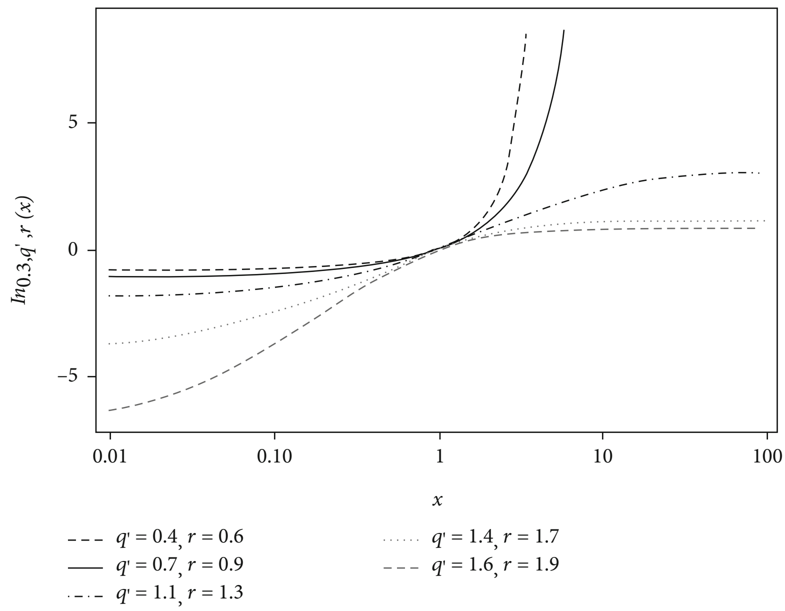


(a)



(b)

Figure 2: Continued.



(c)

FIGURE 2: (a) Illustration of the three-parameter logarithm for fixed value of r. (b) Illustration of the three-parameter logarithm for fixed value of q'. (c) Illustration of the three-parameter logarithm for fixed value of q.

from which

$$x = \left\{1 + \frac{1-q}{1-q'} \ln \left[1 + \frac{1-q'}{1-r} \ln \{1 + (1-r)y\}\right]\right\}^{1/1-q}. \quad (38)$$

Thus, the inverse function is given by

$$\begin{aligned} e^y_{q,q',r} = \exp_{q,q',r} y &= \left\{1 + \frac{1-q}{1-q'} \ln \left[1 + \frac{1-q'}{1-r} \ln \{1 + (1-r)y\}\right]\right\}^{1/1-q} \\ &= \left\{1 + \frac{1-q}{1-q'} \ln \left[1 + \left(1-q'\right) \ln \{1 + (1-r)y\}^{1/1-r}\right]\right\}^{1/1-q} \\ &= \left\{1 + \frac{1-q}{1-q'} \ln \left[1 + \left(1-q'\right) \ln e^y_r\right]\right\}^{1/1-q} \\ &= \left\{1 + (1-q) \ln \left[1 + \left(1-q'\right) \ln e^y_r\right]^{1/1-q'}\right\}^{1/1-q} \\ &= \left\{1 + (1-q) \ln e_{q'}^{\ln e^y_r}\right\}^{1/1-q} = e_q^{\ln e_{q'}^{\ln e^y_r}} = \exp_q\left\{\ln e_q^{\ln e^y_r}\right\}, \end{aligned} \quad (39)$$

where the q-exponential e^x_q is defined in (8).

Theorem 4. *The three-parameter logarithm satisfies the following properties:*

(1) $(d/dx)\ln_{q,q',r} x|_{x=1} = 1$,

(2) $\ln_{q,q',r} 1 = 0$,

(3) The slope of $\ln_{q,q',r} x$ *is positive for all* $x > 0$

(4) $\ln_{q,q',r}(1/x) = -\ln_{2-q,2-q',2-r} x$.

Proof. To find the derivative, use (17) to obtain

$$\frac{d}{dx} \ln_{q,q',r} x = x^{-q} \exp\left\{\frac{1-r}{1-q'}\left(e^{\left(1-q'\right)\ln_q 1} - 1\right) - 1\right\} = 0. \quad (40)$$

From (40), the slope of $\ln_{q,q',r} x$ is positive for all $x > 0$. This is also observed in Figures 1 and 2.

To prove part (4) of the theorem, let $q \longrightarrow 2-q$, $q' \longrightarrow 2-q'$, and $r \longrightarrow 2-r$. From [5],

$$\ln_{q,q'} \frac{1}{x} = -\ln_{2-q,2-q'} x, \quad (41)$$

then

$$\begin{aligned} \ln_{q,q'}(1/x) &= \frac{\left(e^{\ln_{\{q,q'\}}(1/x)}\right)^{1-r} - 1}{1-r} = \frac{\left(e^{-\ln_{2-q,2-q'} x}\right)^{1-r} - 1}{1-r} \\ &= \frac{\left(e^{\ln_{2-q,2-q'} x}\right)^{r-1} - 1}{-(r-1)} = \frac{-\left\{\left(e^{\ln_{2-q,2-q'} x}\right)^{1-(2-r)} - 1\right\}}{1-(2-r)} \\ &= -\ln_{2-q,2-q',2-r} x.? \end{aligned} \quad (42)$$

4. Three-Parameter Entropy

A three-parameter generalization of the Boltzmann-Gibbs-Shannon entropy is constructed here, and its properties are proved. Based on the three-parameter logarithm the entropic function is defined as follows:

$$S_{q,q',r} \equiv k \sum_{i=1}^{\omega} p_i \ln_{q,q',r} \frac{1}{p_i}. \quad (43)$$

If $p_i = 1/\omega, \forall i$,

$$S_{q,q',r} = k \ln_{q,q',r} \omega, \quad (44)$$

where ω is the number of states.

4.1. Lesche-Stability (or Experimental Robustness). The functional form of $\ln_{q,q',r} x$ given in the previous section is analytic in x as $\ln_{q,q',r} x$ is analytic in x. Consequently, $S_{q,q',r}$ is Lesche-stable.

4.2. Expansibility. An entropic function S satisfies this condition if a zero probability $(p_i = 0)$ state does not contribute to the entropy. That is, $S(p_1, p_2, \cdots, p_w, 0) = S(p_1, p_2, \cdots, p_w)$ for any distribution $\{p_i\}$. Observe that in the limit $p_i = 0$, $\ln_{q,q',r} 1/p_i$ is finite if one of q, q', r is greater than 1. Consequently,

$$S_{q,q',r}(p_1, p_2, \cdots, p_w, 0) = S_{q,q',r}(p_1, p_2, \cdots, p_w) \quad (45)$$

provided that one of q, q', r is greater than 1.

4.3. Concavity. Concavity of the entropic function $S_{q,q',r}$ is assured if

$$\frac{d^2}{dp_i^2}\left(p_i \ln_{q,q',r} \frac{1}{p_i}\right) < 0. \quad (46)$$

Theorem 5. *The three-parameter entropic function $S_{q,q',r}$ is concave provided $q + q' + r > 2$.*

Proof. By manual calculation (which is a bit tedious),

$$\frac{d^2}{dp_i^2}\left(p_i \ln_{q,q',r} \frac{1}{p_i}\right) = \exp\left\{\frac{1-r}{1-q'}\left(e^{(1-q')\ln_q 1/p_i} - 1\right)\right\} \cdot e^{(1-q')\ln_q 1/p_i} \times \left\{-q p_i^{q-2} + \left(1-q'\right) p_i^{2q-3} + (1-r) p_i^{2q-3} e^{(1-q')\ln_q 1/p_i}\right\}. \quad (47)$$

In the limit $p_i \longrightarrow 1$, the second derivative given in (47) is less than zero if $q + q' + r > 2$. Thus, concavity of $S_{q,q',r}$ is guaranteed if $q + q' + r > 2$. In the limit $p_i \longrightarrow 0$, concavity is guaranteed if $r > 1$. If $r < 1$, concavity holds if $q > 1$.

4.4. Convexity. A twice-differentiable function of a single variable is convex if and only if its second derivative is nonnegative on its entire domain. The analysis on the convexity of $S_{q,q',r}$ is analogous to that of its concavity. In the limit $p_i \longrightarrow 1$, convexity is guaranteed if $q + q' + r \leq 2$. In the limit $p_i \longrightarrow 0$, convexity is assured if $q, r < 1$. Thus, we have the following theorem.

Theorem 6. *The three-parameter entropic function $S_{q,q}{}',r$ is convex provided $q + q' + r \leq 2$.*

Concavity of $S_{q,q',r}$ is illustrated in Figure 3(a) while convexity is illustrated in Figure 3(b).

4.5. Composability. An entropic function S is said to be composable if for events A and B,

$$S(A+B) = \Phi\left(S(A), S(B), \text{indices}\right), \quad (48)$$

where Φ is some single-valued function [5]. The Boltzmann-Gibbs-Shannon entropy satisfies

$$S_{\mathrm{BGS}}(A+B) = S_{\mathrm{BGS}}(A) + S_{\mathrm{BGS}}(B). \quad (49)$$

Hence, it is composable and additive. The one-parameter entropy S_q,for $q \neq 1$ is also composable as it satisfies

$$\frac{S_q^{A+B}}{k} = \frac{S_q^A}{k} \oplus_q \frac{S_q^B}{k} = \frac{S_q(A)}{k} + \frac{S_q(B)}{k} + (1-q)\frac{S_q(A)}{k}\frac{S_q(B)}{k}. \quad (50)$$

The two-parameter entropy $S_{q,q'}$ [5] satisfies, in the microcanonical ensemble (i.e., equal probabilities), that

$$Y\left(S^{A+B}\right) = Y\left(S^A\right) + Y\left(S^B\right) + \frac{1-q'}{1-q} Y\left(S^A\right) Y\left(S^B\right), \quad (51)$$

where

$$Y(S) \equiv 1 + \frac{1-q}{1-q'} \ln\left[1 + \left(1-q'\right)\frac{S}{k}\right]. \quad (52)$$

However, this does not hold true for arbitrary distributions $\{p_i\}$, which means $S_{q,q}{}'$ is not composable in general. For the 3-parameter entropy $S_{q,q',r}$, a similar property as that of (51) is obtained as shown in the following theorem.

Theorem 7. *The three-parameter entropy $S_{q,q',r}$ satisfies*

$$U\left(S^{A+B}\right) = U\left(S^A\right) + U\left(S^B\right) + \frac{1-q'}{1-q} U\left(S^A\right) U\left(S^B\right), \quad (53)$$

where

$$U(S) = \ln\left[1 + \frac{1-q'}{1-r} \ln\left[1 + (1-r)\frac{S}{k}\right]\right]. \quad (54)$$

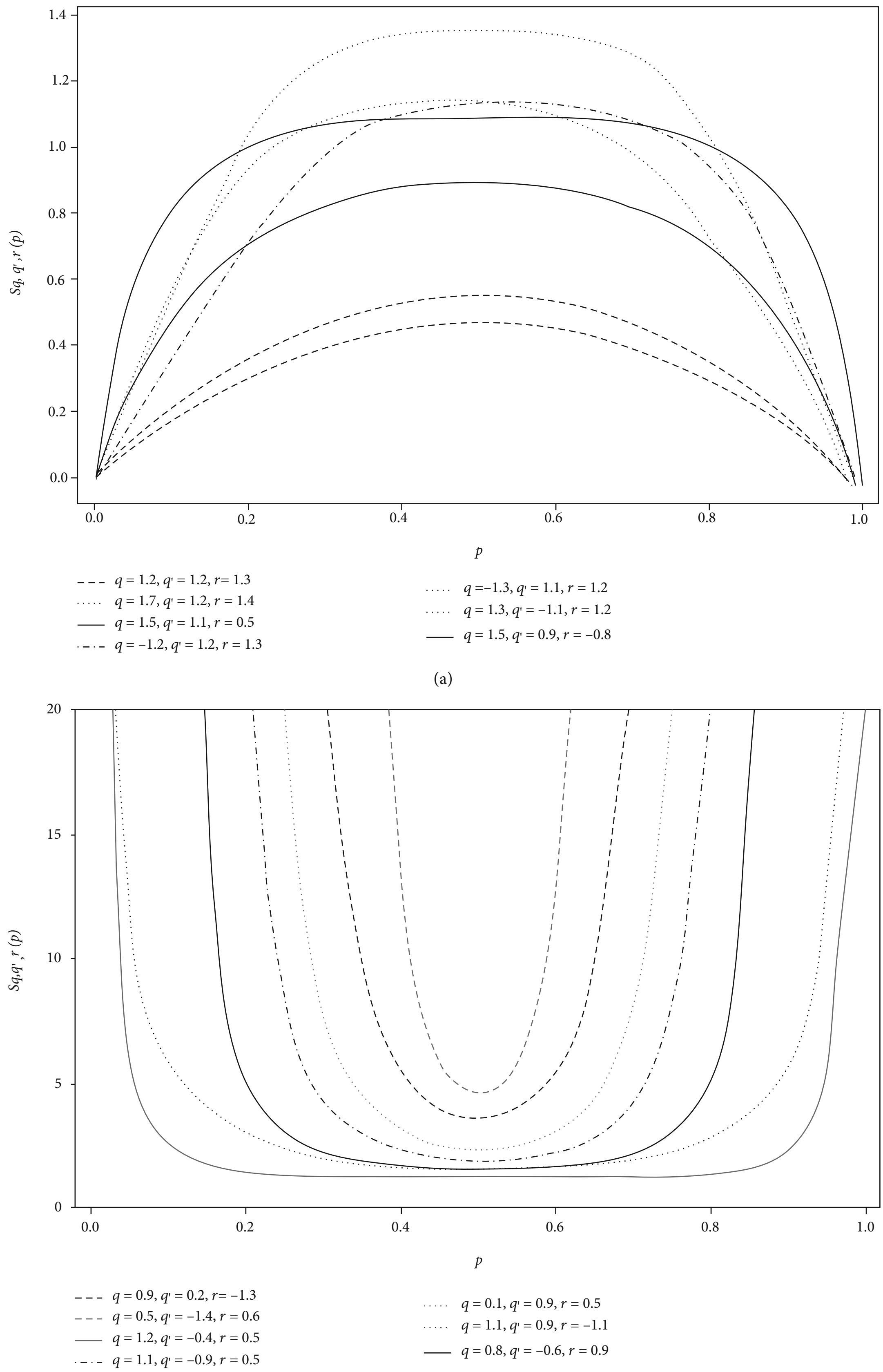


Figure 3: Illustration of the three-parameter entropic function: (a) concavity and (b) convexity.

Proof. Note that

$$\ln_{q,q}{}'(W_A W_B) = \frac{1}{1-q'}\left[e^{(1-q')\ln_q(W_A W_B)} - 1\right] = \frac{S^{A+B}_{q,q'}}{k}, \quad (55)$$

from which

$$\begin{aligned}\frac{S^{A}_{q,q',r}}{k} &= \ln_{q,q',r} W_A = \frac{1}{1-r}\left[e^{(1-r)\ln_{q,q'} W_A} - 1\right] \\ &= \frac{1}{1-r}\left[e^{(1-r)S^{A}_{q,q'}/k} - 1\right].\end{aligned} \quad (56)$$

Similarly,

$$\begin{aligned}\frac{S^{B}_{q,q',r}}{k} &= \ln_{q,q}{}' r W_B = \frac{1}{1-r}\left[e^{(1-r)S^{B}_{q,q'}/k} - 1\right], \\ \frac{S^{A+B}_{q,q',r}}{k} &= \ln_{\{q,q',r\}} W_A W_B = \frac{1}{1-r}\left[e^{(1-r)S^{A+B}_{q,q'}/k} - 1\right] \\ &= \frac{1}{1-r}e^{(1-r)S^{A+b}_{q,q'}/k} - \frac{1}{1-r}.\end{aligned} \quad (57)$$

From (57),

$$\ln\left[(1-r)\frac{S^{A+B}_{q,q',r}}{k} + 1\right] = (1-r)\frac{S^{A+B}_{q,q'}}{k}. \quad (58)$$

Using the following result in [5],

$$\frac{S^{A+B}_{q,q'}}{k} = \frac{1}{1-q'}\left\{e^{1-q'/1-q\ \ln\left[1+(1-q')S^{A}_{q,q'}/k\right]\cdot\ln\left[1+(1-q')S^{B}_{q,q'}/k\right]}\left[1+(1-q')S^{A}_{q,q'}/k\right]\left[1+(1-q')S^{B}_{q,q'}/k\right] - 1\right\}. \quad (59)$$

Equation (58) becomes

$$\begin{aligned}\ln\left[1+(1-r)\frac{S^{A+B}_{q,q',r}}{k}\right] = \frac{1-r}{1-q'}&\left\{e^{[(1-q')/(1-q)]\ln\left[1+[(1-q')/(1-r)]\ln\left[1+(1-r)S^{A}_{q,q',r}/k\right]\right]\cdot\ln\left[1+[(1-q')/(1-r)]\ln\left[1+(1-r)S^{B}_{q,q',r}/k\right]\right]}\right. \\ &\left.\times\left[1+\frac{1-q'}{1-r}\ln\left[1+(1-r)\frac{S^{A}_{q,q',r}}{k}\right]\right]\times\left[1+\frac{1-q'}{1-r}\ln\left[1+(1-r)\frac{S^{B}_{q,q',r}}{k}\right]\right] - 1\right\}.\end{aligned} \quad (60)$$

Now, with

$$U(S) = \ln\left[1+\frac{1-q'}{1-r}\ln\left[1+(1-r)\frac{S}{k}\right]\right], \quad (61)$$

we have

$$\begin{aligned}&1+\frac{1-q'}{1-r}\ln\left[1+(1-r)\frac{S^{A+B}_{q,q',r}}{k}\right] \\ &= e^{[(1-q')/(1-q)]U(S^A)\cdot U(S^B)} \\ &\quad\times\left[1+\frac{1-q'}{1-r}\ln\left[1+(1-r)\frac{S^{A}_{q,q',r}}{k}\right]\right] \\ &\quad\times\left[1+\frac{1-q'}{1-r}\ln\left[1+(1-r)\frac{S^{B}_{q,q',r}}{k}\right]\right].\end{aligned} \quad (62)$$

Consequently,

$$\begin{aligned}&\ln\left[1+\frac{1-q'}{1-r}\ln\left[1+(1-r)\frac{S^{A+B}_{q,q',r}}{k}\right]\right] \\ &= \frac{1-q'}{1-q}U(S^A)\cdot U(S^B) \\ &\quad+\ln\left[1+\frac{1-q'}{1-r}\ln\left[1+(1-r)\frac{S^{A}_{q,q',r}}{k}\right]\right] \\ &\quad+\ln\left[1+\frac{1-q'}{1-r}\ln\left[1+(1-r)\frac{S^{B}_{q,q',r}}{k}\right]\right],\end{aligned} \quad (63)$$

which can be written as

$$U(S^{A+B}) = U(S^A) + U(S^B) + \frac{1-q'}{1-q}U(S^A)U(S^B). \quad (64)$$

In view of the noncomposability of the 2-parameter entropy, $S_{q,q',r}$ is also noncomposable.

5. Conclusion

It is shown that the two-parameter logarithm of Schwämmle and Tsallis [5] can be generalized to three-parameter logarithm using q-analogues. Consequently, a three-parameter entropic function is defined, and its properties are proved. It will be interesting to study the applicability of the three-parameter entropy to adiabatic ensembles [13] and other ensembles [14] and how these applications relate to generalized Lambert W function.

Data Availability

The computer programs and articles used to generate the graphs and support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The author's declare that they have no conflicts of interest.

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