Research Article

Some New Hermite–Hadamard-Type Inequalities Associated with Conformable Fractional Integrals and Their Applications

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In this article, we establish some new Hermite–Hadamard-type inequalities involving the conformable fractional integrals. As applications, several inequalities for the approximation error in the midpoint formula and certain bivariate means are derived.

1. Introduction

Let $I \subseteq \mathbb{R}$ be an interval. Then, a real-valued function $\phi: I \rightarrow \mathbb{R}$ is said to be convex (concave) on I if the inequality

$$\phi(\vartheta d_1 + (1 - \vartheta)d_2) \leq (\geq) \vartheta \phi(d_1) + (1 - \vartheta)\phi(d_2), \quad (1)$$

holds for all $d_1, d_2 \in I$ with $d_1 \neq d_2$ and $\vartheta \in [0, 1]$.

It is well known that the convex functions have wide applications in pure and applied mathematics [1–27]. In particular, many remarkable inequalities and properties for the convex functions can be found in the literature [28–64]. Recently, a great deal of generalizations, extensions, and variants have been made for the convexity [65–72].

Let $d_1, d_2 \in \mathbb{R}$ with $d_1 < d_2$ and $\phi: [d_1, d_2] \rightarrow \mathbb{R}$ be a convex function. Then, the classical Hermite–Hadamard inequality [73–77] states that the double inequality

$$\phi\left(\frac{d_1 + d_2}{2}\right) \leq \frac{1}{d_2 - d_1} \int_{d_1}^{d_2} \phi(z) dz \leq \frac{\phi(d_1) + \phi(d_2)}{2}, \quad (2)$$

holds.

Alomari et al. [78] and Chen and Feng [79] proved the following.

Theorem 1 (see [78]). Let $I \subseteq \mathbb{R}$ be an interval, I° be the interior of I , and $\phi: I^\circ \rightarrow \mathbb{R}$ be a differentiable mapping. Then, the identity

$$\begin{aligned} & \phi\left(\frac{d_1 + d_2}{2}\right) - \frac{1}{d_2 - d_1} \int_{d_1}^{d_2} \phi(z) dz \\ &= \frac{d_2 - d_1}{4} \left[\int_0^1 \phi'\left(\frac{(d_1 + d_2)z}{2} + d_1(1 - z)\right) z dz \right. \\ & \quad \left. + \int_0^1 \phi'\left(\frac{(d_1 + d_2)(1 - z)}{2} + d_2 z\right) (z - 1) dz \right], \end{aligned} \quad (3)$$

holds for $d_1, d_2 \in I^\circ$ with $d_1 < d_2$ if $\phi' \in L[d_1, d_2]$.

Theorem 2 (see [79]). Let $I \subseteq \mathbb{R}$ be an interval, I° be the interior of I , and $\phi: I^\circ \rightarrow \mathbb{R}$ be a differentiable mapping. Then, the identity

$$\begin{aligned} & \phi[\lambda d_1 + (1 - \lambda)d_2] - \frac{1}{d_2 - d_1} \int_{d_1}^{d_2} \phi(z) dz \\ &= \lambda^2 (d_2 - d_1) \left[\int_0^1 \phi'((\lambda d_1 + (1 - \lambda)d_2)(1 - z) + d_2 z) (z - 1) dz \right] \\ & \quad + (1 - \lambda)^2 (d_2 - d_1) \left[\int_0^1 \phi'((\lambda d_1 + (1 - \lambda)d_2)z + d_1(1 - z)) z dz \right], \end{aligned} \quad (4)$$

holds for $d_1, d_2 \in I^\circ$ with $d_1 < d_2$ and $\lambda \in [0, 1]$ if $\phi' \in L[d_1, d_2]$.

Now, we recall the definitions of the conformable fractional derivative and integral.

Definition 1 (see [80]). Let $\phi: [0, \infty) \rightarrow \mathbb{R}$ be a real-valued function and $\alpha \in (0, 1]$. Then, the α -order conformable fractional derivative $D_\alpha(\phi)(z)$ of ϕ at $z \in (0, \infty)$ is defined by

$$D_\alpha(\phi)(z) = \lim_{\nu \rightarrow 0} \frac{\phi(z + \nu z^{1-\alpha}) - \phi(z)}{\nu}, \quad (5)$$

where ϕ is said to be α -differentiable at z if $D_\alpha(\phi)(z)$ exists. In particular, $D_\alpha(\phi)(0)$ is defined by

$$D_\alpha(\phi)(0) = \lim_{z \rightarrow 0^+} D_\alpha(\phi)(z), \quad (6)$$

and we use $\phi^\alpha(z)$ or $(d_\alpha/d_\alpha z)(\phi)$ to denote $D_\alpha(\phi)(z)$.

The following formulas can be derived from Definition 1 immediately.

Theorem 3. Let $\alpha \in (0, 1]$ and ϕ_1 and ϕ_2 be two α -differentiable real-valued functions at $m > 0$. Then, one has

$$\begin{aligned} \frac{d_\alpha}{d_\alpha m} (m^\mu) &= \mu m^{\mu-\alpha}, \\ \frac{d_\alpha}{d_\alpha m} (\mu) &= 0, \\ \frac{d_\alpha}{d_\alpha m} (\mu_1 \phi_1(m) + \mu_2 \phi_2(m)) &= \mu_1 \frac{d_\alpha}{d_\alpha m} (\phi_1(m)) + \mu_2 \frac{d_\alpha}{d_\alpha m} (\phi_2(m)), \\ \frac{d_\alpha}{d_\alpha m} (\phi_1(m) \phi_2(m)) &= \phi_1(m) \frac{d_\alpha}{d_\alpha m} (\phi_2(m)) + \phi_2(m) \frac{d_\alpha}{d_\alpha m} (\phi_1(m)), \\ \frac{d_\alpha}{d_\alpha m} \left(\frac{\phi_1(m)}{\phi_2(m)} \right) &= \frac{\phi_2(m) (d_\alpha/d_\alpha m)(\phi_1(m)) - \phi_1(m) (d_\alpha/d_\alpha m)(\phi_2(m))}{(\phi_2(m))^2}, \end{aligned} \quad (7)$$

for any constants $\mu, \mu_1, \mu_2 \in \mathbb{R}$. If ϕ_1 is differentiable at $\phi_2(m)$, then

$$\frac{d_\alpha}{d_\alpha m} (\phi_1(\phi_2(m))) = \phi_1'(\phi_2(m)) \frac{d_\alpha}{d_\alpha m} (\phi_2(m)). \quad (8)$$

Moreover, if ϕ_1 is differentiable, then

$$\frac{d_\alpha}{d_\alpha m} (\phi_1(m)) = m^{1-\alpha} \frac{d}{dm} (\phi_1(m)). \quad (9)$$

Definition 2. Let $\alpha \in (0, 1]$ and $d_1, d_2 \in [0, \infty)$ with $d_1 < d_2$. Then, the real-valued function $\phi_1: [d_1, d_2] \rightarrow \mathbb{R}$ is said to be α -fractional integrable if the integral

$$\int_{d_1}^{d_2} \phi_1(y) d_\alpha y = \int_{d_1}^{d_2} \phi_1(y) y^{\alpha-1} dy, \quad (10)$$

exists and is finite. All α -fractional integrable functions on $[d_1, d_2]$ are denoted by $L_\alpha^1([d_1, d_2])$.

Remark 1

$$I_\alpha^{d_1}(\phi_1)(m) = I_1^{d_1}(m^{\alpha-1} \phi_1) = \int_{d_1}^m \frac{\phi_1(y)}{y^{1-\alpha}} dy, \quad (11)$$

where the integral is the usual Riemann improper integral and $\alpha \in (0, 1]$.

Anderson [81] established the Hermite–Hadamard inequality for the conformable fractional integral as follows.

Theorem 4. Let $\alpha \in (0, 1]$, $d_1, d_2 \in \mathbb{R}$ with $d_1 < d_2$, and $\phi: [d_1, d_2] \rightarrow \mathbb{R}$ be an α -fractional differentiable function such that $D_\alpha(\phi)$ is increasing. Then, one has

$$\frac{\alpha}{d_2^\alpha - d_1^\alpha} \int_{d_1}^{d_2} \phi(y) d_\alpha y \leq \frac{\phi(d_1) + \phi(d_2)}{2}. \quad (12)$$

Moreover, if the function ϕ is decreasing on $[d_1, d_2]$, then

$$\phi\left(\frac{d_1 + d_2}{2}\right) \leq \frac{\alpha}{d_2^\alpha - d_1^\alpha} \int_{d_1}^{d_2} \phi(y) d_\alpha y. \quad (13)$$

Remark 2. If $\alpha = 1$, then we clearly see that inequalities (12) and (13) reduce to inequality (2).

The aim of this article is to establish some new Hermite–Hadamard-type inequalities for the conformable fractional integrals and present their applications in the error estimations of the midpoint formula and certain bivariate means.

2. Main Results

In order to establish our main results, we need a lemma which we present in this section.

Lemma 1. Let $\alpha \in (0, 1]$, $d_1, d_2 \in (0, \infty)$ with $d_1 < d_2$, and $\phi: [d_1, d_2] \rightarrow \mathbb{R}$ be an α -fractional differentiable function. Then, the identity

$$\begin{aligned}
& \phi(\lambda d_1 + (1-\lambda)d_2) - \frac{\alpha}{d_2^\alpha - d_1^\alpha} \int_{d_1}^{d_2} \phi(s) d_\alpha s \\
&= \frac{(d_1 - d_2)(\lambda - 1)}{(d_2^\alpha - d_1^\alpha)} \left[\int_0^1 ((z(\lambda d_1 + (1-\lambda)d_2) + (1-z)d_1)^{2\alpha-1} - d_1^\alpha (z(\lambda d_1 + (1-\lambda)d_2) + (1-z)d_1)^{\alpha-1}) \right. \\
&\quad \times D_\alpha(\phi)(z(\lambda d_1 + (1-\lambda)d_2) + (1-z)d_1) z^{1-\alpha} d_\alpha z \Big] \\
&\quad + \frac{(d_2 - d_1)\lambda}{(d_2^\alpha - d_1^\alpha)} \left[\int_0^1 ((zd_2 + (1-z)(\lambda d_1 + (1-\lambda)d_2))^{2\alpha-1} - d_2^\alpha (zd_2 + (1-z)(\lambda d_1 + (1-\lambda)d_2))^{\alpha-1}) \right. \\
&\quad \times D_\alpha(\phi)(zd_2 + (1-z)(\lambda d_1 + (1-\lambda)d_2)) z^{1-\alpha} d_\alpha z \Big],
\end{aligned} \tag{14}$$

holds for any $\lambda \in [0, 1]$ if $D_\alpha(\phi) \in L_\alpha^1([d_1, d_2])$.

Proof. Integrating by parts, we have

$$\begin{aligned}
I_1 &= \int_0^1 ((z(\lambda d_1 + (1-\lambda)d_2) + (1-z)d_1)^{2\alpha-1} - d_1^\alpha (z(\lambda d_1 + (1-\lambda)d_2) + (1-z)d_1)^{\alpha-1}) \\
&\quad \times D_\alpha(\phi)(z(\lambda d_1 + (1-\lambda)d_2) + (1-z)d_1) z^{1-\alpha} d_\alpha z \\
&= \int_0^1 ((z(\lambda d_1 + (1-\lambda)d_2) + (1-z)d_1)^\alpha - d_1^\alpha) \phi'(z(\lambda d_1 + (1-\lambda)d_2) + (1-z)d_1) dz \\
&= ((z(\lambda d_1 + (1-\lambda)d_2) + (1-z)d_1)^\alpha - d_1^\alpha) \frac{\phi(z(\lambda d_1 + (1-\lambda)d_2) + (1-z)d_1)}{(d_1 - d_2)(\lambda - 1)} \Big|_0^1 \\
&\quad - \alpha \int_0^1 (z(\lambda d_1 + (1-\lambda)d_2) + (1-z)d_1)^{\alpha-1} \phi(z(\lambda d_1 + (1-\lambda)d_2) + (1-z)d_1) dz \\
&= ((\lambda d_1 + (1-\lambda)d_2)^\alpha - d_1^\alpha) \frac{\phi(\lambda d_1 + (1-\lambda)d_2)}{(d_1 - d_2)(\lambda - 1)} - \frac{\alpha}{(d_1 - d_2)(\lambda - 1)} \int_{d_1}^{\lambda d_1 + (1-\lambda)d_2} \phi(s) d_\alpha s, \\
I_2 &= \int_0^1 ((zd_2 + (1-z)(\lambda d_1 + (1-\lambda)d_2))^{2\alpha-1} - d_2^\alpha (zd_2 + (1-z)(\lambda d_1 + (1-\lambda)d_2))^{\alpha-1}) \\
&\quad \times D_\alpha(\phi)(zd_2 + (1-z)(\lambda d_1 + (1-\lambda)d_2)) z^{1-\alpha} d_\alpha z \\
&= \int_0^1 ((zd_2 + (1-z)(\lambda d_1 + (1-\lambda)d_2))^\alpha - d_2^\alpha) \phi'(zd_2 + (1-z)(\lambda d_1 + (1-\lambda)d_2)) dz \\
&= ((zd_2 + (1-z)(\lambda d_1 + (1-\lambda)d_2))^\alpha - d_2^\alpha) \frac{\phi(zd_2 + (1-z)(\lambda d_1 + (1-\lambda)d_2))}{(d_2 - d_1)\lambda} \Big|_0^1 \\
&\quad - \alpha \int_0^1 (zd_2 + (1-z)(\lambda d_1 + (1-\lambda)d_2))^{\alpha-1} \phi(zd_2 + (1-z)(\lambda d_1 + (1-\lambda)d_2)) dz \\
&= (d_2^\alpha - (\lambda d_1 + (1-\lambda)d_2)^\alpha) \frac{\phi(\lambda d_1 + (1-\lambda)d_2)}{(d_2 - d_1)\lambda} - \frac{\alpha}{(d_2 - d_1)\lambda} \int_{\lambda d_1 + (1-\lambda)d_2}^{d_2} \phi(s) d_\alpha s,
\end{aligned} \tag{15}$$

where we use the changes of variables $s = z(\lambda d_1 + (1-\lambda)d_2) + (1-z)d_1$ in I_1 and $s = zd_2 + (1-z)(\lambda d_1 + (1-\lambda)d_2)$ in I_2 . Then, multiply I_1 by $((d_1 - d_2)(\lambda - 1))/(d_2^\alpha - d_1^\alpha)$ and I_2 by $((d_2 - d_1)\lambda)/(d_2^\alpha - d_1^\alpha)$ to get the desired result. $\square$

Remark 3. Let $\alpha = 1$, then identity (14) reduces to (4).

Remark 4. Let $\alpha = 1$ and $\lambda = 1/2$, then identity (14) reduces to (3).

Theorem 5. Let $\alpha \in (0, 1]$, $d_1, d_2 \in (0, \infty)$ with $d_1 < d_2$, and $\phi: [d_1, d_2] \rightarrow \mathbb{R}$ be an α -fractional differentiable function. Then, the Hermite–Hadamard-type inequality

$$\begin{aligned} & \left| \phi(\lambda d_1 + (1-\lambda)d_2) - \frac{\alpha}{d_2^\alpha - d_1^\alpha} \int_{d_1}^{d_2} \phi(s) d_\alpha s \right| \\ & \leq \frac{(d_2 - d_1)(1-\lambda)}{(d_2^\alpha - d_1^\alpha)} \left[A_1(\alpha) |\phi'(d_1)| + B_1(\alpha) |\phi'(\lambda d_1 + (1-\lambda)d_2)| \right] \\ & \quad + \frac{(d_2 - d_1)\lambda}{(d_2^\alpha - d_1^\alpha)} \left[A_2(\alpha) |\phi'(d_2)| + B_2(\alpha) |\phi'(\lambda d_1 + (1-\lambda)d_2)| \right], \end{aligned} \quad (16)$$

holds for any $\lambda \in [0, 1]$ if $D_\alpha(h) \in L_\alpha^1([d_1, d_2])$ and $|\phi'|$ is convex on $[d_1, d_2]$, where

$$\begin{aligned} A_1(\alpha) &= \frac{1}{12} \left[((\lambda+1)(\lambda d_1 + (1-\lambda)d_2)(d_1^{\alpha-1} + d_2^{\alpha-1})) \right. \\ & \quad \left. + (\lambda d_1(d_1^{\alpha-1} - d_2^{\alpha-1}) + d_1(d_2^{\alpha-1} - 5d_1^{\alpha-1})) \right], \\ B_1(\alpha) &= \frac{(\lambda d_1 + (1-\lambda)d_2)}{12} (3d_2^{\alpha-1}(1-\lambda) + d_1^{\alpha-1}(1+3\lambda) \\ & \quad + d_1^\alpha(\lambda-5)), \\ A_2(\alpha) &= \frac{\lambda(d_2^\alpha - d_1^\alpha)}{6}, \\ B_2(\alpha) &= \frac{\lambda(d_2^\alpha - d_1^\alpha)}{3}. \end{aligned} \quad (17)$$

Proof. It follows from Lemma 1 and the convexity of the functions $\psi(y) = y^{\alpha-1}$ and $\theta(y) = -y^\alpha$ for $y > 0$ together with $|\phi'|$ that

$$\begin{aligned} & \left| \phi(\lambda d_1 + (1-\lambda)d_2) - \frac{\alpha}{d_2^\alpha - d_1^\alpha} \int_{d_1}^{d_2} \phi(s) d_\alpha s \right| \\ & \leq \frac{(d_2 - d_1)(1-\lambda)}{(d_2^\alpha - d_1^\alpha)} \left[\int_0^1 \left((z(\lambda d_1 + (1-\lambda)d_2) + (1-z)d_1)^\alpha - d_1^\alpha \right) \times |\phi'(z(\lambda d_1 + (1-\lambda)d_2) + (1-z)d_1)| dz \right] \\ & \quad + \frac{(d_2 - d_1)\lambda}{(d_2^\alpha - d_1^\alpha)} \left[\int_0^1 \left(d_2^\alpha - (zd_2 + (1-z)(\lambda d_1 + (1-\lambda)d_2))^\alpha \right) \times |\phi'(zd_2 + (1-z)(\lambda d_1 + (1-\lambda)d_2))| dz \right] \\ & \leq \frac{(d_2 - d_1)(1-\lambda)}{(d_2^\alpha - d_1^\alpha)} \left[\int_0^1 \left((z(\lambda d_1 + (1-\lambda)d_2) + (1-z)d_1)^{\alpha-1} \times (z(\lambda d_1 + (1-\lambda)d_2) \right. \right. \\ & \quad \left. \left. + (1-z)d_1 - d_1^\alpha) \right) |\phi'(z(\lambda d_1 + (1-\lambda)d_2) + (1-z)d_1)| dz \right] \\ & \quad + \frac{(d_2 - d_1)\lambda}{(d_2^\alpha - d_1^\alpha)} \left[\int_0^1 \left(d_2^\alpha - (zd_2^\alpha + (1-z)(\lambda d_1^\alpha + (1-\lambda)d_2^\alpha)) \right) \times |\phi'(zd_2 + (1-z)(\lambda d_1 + (1-\lambda)d_2))| dz \right] \\ & \leq \frac{(d_2 - d_1)(1-\lambda)}{(d_2^\alpha - d_1^\alpha)} \left[\int_0^1 \left((z(\lambda d_1^{\alpha-1} + (1-\lambda)d_2^{\alpha-1}) + (1-z)d_1^{\alpha-1}) \times (z(\lambda d_1 + (1-\lambda)d_2) \right. \right. \\ & \quad \left. \left. + (1-z)d_1 - d_1^\alpha) \right) (z|\phi'(\lambda d_1 + (1-\lambda)d_2)| + (1-z)|\phi'(d_1)|) dz \right] \\ & \quad + \frac{(d_2 - d_1)\lambda}{(d_2^\alpha - d_1^\alpha)} \left[\int_0^1 \left(d_2^\alpha - (zd_2^\alpha + (1-z)(\lambda d_1^\alpha + (1-\lambda)d_2^\alpha)) \right) \times (z|\phi'(d_2)| + (1-z)|\phi'(\lambda d_1 + (1-\lambda)d_2)|) dz \right] \\ & = \frac{(d_2 - d_1)(1-\lambda)}{(d_2^\alpha - d_1^\alpha)} [A_1(\alpha) |\phi'(d_1)| + B_1(\alpha) |\phi'(\lambda d_1 + (1-\lambda)d_2)|] \\ & \quad + \frac{(d_2 - d_1)\lambda}{(d_2^\alpha - d_1^\alpha)} [A_2(\alpha) |\phi'(d_2)| + B_2(\alpha) |\phi'(\lambda d_1 + (1-\lambda)d_2)|]. \end{aligned} \quad (18)$$

□

Remark 5. Let $\alpha = 1$. Then, inequality (16) leads to

$$\begin{aligned} & \left[\phi(\lambda d_1 + (1-\lambda)d_2) - \frac{\alpha}{d_2 - d_1} \int_{d_1}^{d_2} \phi(s) ds \right] \\ & \leq (1-\lambda) \left[A_1(1) |\phi'(d_1)| + B_1(1) |\phi'(\lambda d_1 + (1-\lambda)d_2)| \right] \\ & \quad + \lambda \left[A_2(1) |\phi'(d_2)| + B_2(1) |\phi'(\lambda d_1 + (1-\lambda)d_2)| \right], \end{aligned} \quad (19)$$

where $A_1(\alpha)$, $B_1(\alpha)$, $A_2(\alpha)$, and $B_2(\alpha)$ are defined as in Theorem 5.

Theorem 6. Let $\alpha \in (0, 1]$, $d_1, d_2 \in (0, \infty)$ with $d_1 < d_2$, and $p, q > 1$ such that $1/p + 1/q = 1$ and $\phi: [d_1, d_2] \rightarrow \mathbb{R}$ be an α -fractional differentiable function. Then, the Hermite-Hadamard-type inequality

$$\begin{aligned} & \left| \phi(\lambda d_1 + (1-\lambda)d_2) - \frac{\alpha}{d_2^\alpha - d_1^\alpha} \int_{d_1}^{d_2} \phi(s) d_\alpha s \right| \\ & \leq \frac{(d_2 - d_1)(1-\lambda)}{(d_2^\alpha - d_1^\alpha)} \left[(A_1(\alpha, p))^{1/p} E_1^{1/q} \right] \\ & \quad + \frac{(d_2 - d_1)\lambda}{(d_2^\alpha - d_1^\alpha)} \left[(A_2(\alpha, p))^{1/p} E_2^{1/q} \right], \end{aligned} \quad (20)$$

is valid for all $\lambda \in [0, 1]$ if $D_\alpha(\phi) \in L_\alpha^1([d_1, d_2])$ and $|\phi'|^q$ is convex on $[d_1, d_2]$, where

$$\begin{aligned} A_1(\alpha, p) &= \int_0^1 ((z(\lambda d_1 + (1-\lambda)d_2) + (1-z)d_1)^\alpha - d_1^\alpha)^p dz \\ &= \frac{1}{(\lambda-1)(d_1-d_2)} \int_{d_1}^{\lambda d_1 + (1-\lambda)d_2} (s^\alpha - d_1^\alpha)^p ds, \\ A_2(\alpha, p) &= \int_0^1 (d_2^\alpha - (z d_2 + (1-z)(\lambda d_1 + (1-\lambda)d_2))^\alpha)^p dz \\ &= \frac{1}{\lambda(d_2-d_1)} \int_{\lambda d_1 + (1-\lambda)d_2}^{d_2} (d_2^\alpha - s^\alpha)^p ds, \\ E_1(\alpha) &= \frac{|\phi'(\lambda d_1 + (1-\lambda)d_2)|^q + |\phi'(d_1)|^q}{2}, \\ E_2(\alpha) &= \frac{|\phi'(d_2)|^q + |\phi'(\lambda d_1 + (1-\lambda)d_2)|^q}{2}. \end{aligned} \quad (21)$$

Proof. Making use of Lemma 1, the convexity of $|\phi'|^q$ on $[d_1, d_2]$, Hölder inequality, and the property of the modulus, we get

$$\begin{aligned} & \left| \phi(\lambda d_1 + (1-\lambda)d_2) - \frac{\alpha}{d_2^\alpha - d_1^\alpha} \int_{d_1}^{d_2} \phi(s) d_\alpha s \right| \\ & \leq \frac{(d_2 - d_1)(1-\lambda)}{(d_2^\alpha - d_1^\alpha)} \left[\int_0^1 ((z(\lambda d_1 + (1-\lambda)d_2) + (1-z)d_1)^\alpha - d_1^\alpha) \times |\phi'((\lambda d_1 + (1-\lambda)d_2) + (1-z)d_1)| dz \right] \\ & \quad + \frac{(d_2 - d_1)\lambda}{(d_2^\alpha - d_1^\alpha)} \left[\int_0^1 (d_2^\alpha - (z d_2 + (1-z)(\lambda d_1 + (1-\lambda)d_2))^\alpha) \times |\phi'(z d_2 + (1-z)(\lambda d_1 + (1-\lambda)d_2))| dz \right] \\ & \leq \frac{(d_2 - d_1)(1-\lambda)}{(d_2^\alpha - d_1^\alpha)} \left[\left(\int_0^1 ((z(\lambda d_1 + (1-\lambda)d_2) + (1-z)d_1)^\alpha - d_1^\alpha)^p dz \right)^{1/p} \times \left(\int_0^1 |\phi'(z(\lambda d_1 + (1-\lambda)d_2) + (1-z)d_1)|^q dz \right)^{1/q} \right] \\ & \quad + \frac{(d_2 - d_1)\lambda}{(d_2^\alpha - d_1^\alpha)} \left[\left(\int_0^1 (d_2^\alpha - (z d_2 + (1-z)(\lambda d_1 + (1-\lambda)d_2))^\alpha)^p dz \right)^{1/p} \times \left(\int_0^1 |\phi'(z d_2 + (1-z)(\lambda d_1 + (1-\lambda)d_2))|^q dz \right)^{1/q} \right] \\ & \leq \frac{(d_2 - d_1)(1-\lambda)}{(d_2^\alpha - d_1^\alpha)} \left[\left(\int_0^1 ((z(\lambda d_1 + (1-\lambda)d_2) + (1-z)d_1)^\alpha - d_1^\alpha)^p dz \right)^{1/p} \times \left(\int_0^1 z |\phi'(\lambda d_1 + (1-\lambda)d_2)|^q + (1-z) |\phi'(d_1)|^q dz \right)^{1/q} \right] \\ & \quad + \frac{(d_2 - d_1)\lambda}{(d_2^\alpha - d_1^\alpha)} \left[\left(\int_0^1 (d_2^\alpha - (z d_2 + (1-z)(\lambda d_1 + (1-\lambda)d_2))^\alpha)^p dz \right)^{1/p} \times \left(z |\phi'(d_2)|^q + (1-z) |\phi'(\lambda d_1 + (1-\lambda)d_2)|^q dz \right)^{1/q} \right] \\ & = \frac{(d_2 - d_1)(1-\lambda)}{(d_2^\alpha - d_1^\alpha)} \left[(A_1(\alpha, p))^{1/p} (E_1)^{1/q} \right] + \frac{(d_2 - d_1)\lambda}{(d_2^\alpha - d_1^\alpha)} \left[(A_2(\alpha, p))^{1/p} (E_2)^{1/q} \right]. \end{aligned}$$

(22)

□

Remark 6. Let $\alpha = 1$. Then, inequality (20) leads to

$$\left| \phi(\lambda d_1 + (1-\lambda)d_2) - \frac{1}{d_2 - d_1} \int_{d_1}^{d_2} \phi(s) ds \right| \leq (1-\lambda) \left[(A_1(1, p))^{1/p} E_1^{1/q} \right] + \lambda \left[(A_2(1, p))^{1/p} E_2^{1/q} \right], \quad (23)$$

where $A_1(\alpha, P)$, $A_2(\alpha, P)$, $E_1(\alpha)$, and $E_2(\alpha)$ are defined as in Theorem 6.

Theorem 7. Let $\alpha \in (0, 1]$, $d_1, d_2 \in (0, \infty)$ with $d_1 < d_2$, $q \geq 1$, and $\phi: [d_1, d_2] \rightarrow \mathbb{R}$ be an α -fractional differentiable function. Then, the Hermite–Hadamard-type inequality

$$\begin{aligned} & \left| \phi(\lambda d_1 + (1-\lambda)d_2) - \frac{\alpha}{d_2^\alpha - d_1^\alpha} \int_{d_1}^{d_2} \phi(s) d_\alpha s \right| \\ & \leq \frac{(d_2 - d_1)(1-\lambda)}{(d_2^\alpha - d_1^\alpha)} \left[(P_1(\alpha))^{1-(1/q)} \cdot (P_2(\alpha) |\phi'(\lambda d_1 + (1-\lambda)d_2)|^q + P_3(\alpha) |\phi'(d_1)|^q)^{1/q} \right] \\ & \quad + \frac{(d_2 - d_1)\lambda}{(d_2^\alpha - d_1^\alpha)} \left[(Q_1(\alpha))^{1-(1/q)} \cdot (Q_2(\alpha) |\phi'(d_2)|^q + Q_3(\alpha) |\phi'(\lambda d_1 + (1-\lambda)d_2)|^q)^{1/q} \right], \end{aligned} \quad (24)$$

takes place for all $\lambda \in [0, 1]$ if $D_\alpha(\phi) \in L_\alpha^1([d_1, d_2])$ and $|\phi'|^q$ is convex on $[d_1, d_2]$, where

$$\begin{aligned} P_1(\alpha) &= \frac{1}{(\lambda-1)(d_1-d_2)} \left[\frac{((\lambda d_1 + (1-\lambda)d_2)^{\alpha+1} - d_1^{\alpha+1})}{\alpha+1} - d_1^\alpha ((\lambda-1)(d_1-d_2)) \right], \\ Q_1(\alpha) &= \frac{1}{\lambda(d_2-d_1)} \left[\frac{((\lambda d_1 + (1-\lambda)d_2)^{\alpha+1} - d_2^{\alpha+1})}{\alpha+1} - d_2^\alpha (2d_2 - \lambda(d_1+d_2)) \right], \\ P_2(\alpha) &= \frac{1}{(\lambda-1)(d_2-d_1)^2} \left[\frac{((\lambda d_1 + (1-\lambda)d_2)^{\alpha+2} - d_1^{\alpha+2})}{\alpha+2} + d_1 \frac{(d_1^{\alpha+1} - (\lambda d_1 + (1-\lambda)d_2)^{\alpha+1})}{\alpha+1} \right. \\ & \quad \left. + d_1^\alpha \frac{(d_1^2 - (\lambda d_1 + (1-\lambda)d_2)^2)}{2} + d_1^{\alpha+1} ((d_1-d_2)(1-\lambda)) \right], \\ Q_2(\alpha) &= \frac{1}{\lambda^2(d_2-d_1)^2} \left[\frac{d_2(d_2^{\alpha+1} - (\lambda d_1 + (1-\lambda)d_2)^{\alpha+1})}{\alpha+1} + \lambda(d_2-d_1) \frac{((\lambda d_1 + (1-\lambda)d_2)^{\alpha+1} - d_2^{\alpha+1})}{\alpha+1} \right. \\ & \quad \left. + \frac{((\lambda d_1 + (1-\lambda)d_2)^{\alpha+2} - d_2^{\alpha+2})}{\alpha+2} + d_2^{\alpha+1} ((\lambda d_1 + (1-\lambda)d_2) - d_2) + d_2^\alpha \frac{(d_2^2 - (\lambda d_1 + (1-\lambda)d_2)^2)}{2} + \lambda^2 d_2^\alpha (d_2-d_1)^2 \right], \\ P_3(\alpha) &= \frac{1}{(\lambda-1)(d_2-d_1)^2} \left[\frac{((\lambda d_1 + (1-\lambda)d_2)^{\alpha+2} - d_1^{\alpha+2})}{\alpha+2} + d_1 \frac{(d_1^{\alpha+1} - (\lambda d_1 + (1-\lambda)d_2)^{\alpha+1})}{\alpha+1} \right. \\ & \quad \left. + d_1^\alpha \frac{(d_1^2 - (\lambda d_1 + (1-\lambda)d_2)^2)}{2} + d_1^{\alpha+1} ((d_1-d_2)(\lambda-1)) \right], \\ Q_3(\alpha) &= \frac{1}{\lambda^2(d_2-d_1)^2} \left[\frac{d_2((\lambda d_1 + (1-\lambda)d_2)^{\alpha+1} - d_2^{\alpha+1})}{\alpha+1} + \frac{(d_2^{\alpha+2} - (\lambda d_1 + (1-\lambda)d_2)^{\alpha+2})}{\alpha+2} \right. \\ & \quad \left. + d_2^\alpha \frac{((\lambda d_1 + (1-\lambda)d_2)^2 - d_2^2)}{2} + d_2^{\alpha+1} (\lambda(d_2-d_1)) \right]. \end{aligned} \quad (25)$$

Proof. It follows from Lemma 1, convexity of the function $|\phi'|^q$, power mean inequality, and the property of the modulus that

$$\begin{aligned}
& \left| \phi(\lambda d_1 + (1-\lambda)d_2) - \frac{\alpha}{d_2^\alpha - d_1^\alpha} \int_{d_1}^{d_2} \phi(s) d_\alpha s \right| \\
& \leq \frac{(d_2 - d_1)(1-\lambda)}{(d_2^\alpha - d_1^\alpha)} \left[\int_0^1 ((z(\lambda d_1 + (1-\lambda)d_2) + (1-z)d_1)^\alpha - d_1^\alpha) \times |\phi'(z(\lambda d_1 + (1-\lambda)d_2) + (1-z)d_1)| dz \right] \\
& \quad + \frac{(d_2 - d_1)\lambda}{(d_2^\alpha - d_1^\alpha)} \left[\int_0^1 (d_2^\alpha - (zd_2 + (1-z)(\lambda d_1 + (1-\lambda)d_2))^\alpha) \times |\phi'(zd_2 + (1-z)(\lambda d_1 + (1-\lambda)d_2))| dz \right] \\
& \leq \frac{(d_2 - d_1)(1-\lambda)}{(d_2^\alpha - d_1^\alpha)} \left[\left(\int_0^1 ((z(\lambda d_1 + (1-\lambda)d_2) + (1-z)d_1)^\alpha - d_1^\alpha) dz \right)^{1-(1/q)} \right. \\
& \quad \times \left(\int_0^1 ((z(\lambda d_1 + (1-\lambda)d_2) + (1-z)d_1)^\alpha - d_1^\alpha) |\phi'(z(\lambda d_1 + (1-\lambda)d_2) + (1-z)d_1)|^q dz \right)^{1/q} \Big] \\
& \quad + \frac{(d_2 - d_1)\lambda}{(d_2^\alpha - d_1^\alpha)} \left[\left(\int_0^1 (d_2^\alpha - (zd_2 + (1-z)(\lambda d_1 + (1-\lambda)d_2))^\alpha) dz \right)^{1-(1/q)} \right. \\
& \quad \times \left(\int_0^1 (d_2^\alpha - (zd_2 + (1-z)(\lambda d_1 + (1-\lambda)d_2))^\alpha) |\phi'(zd_2 + (1-z)(\lambda d_1 + (1-\lambda)d_2))|^q dz \right)^{1/q} \Big] \\
& \leq \frac{(d_2 - d_1)(1-\lambda)}{(d_2^\alpha - d_1^\alpha)} \left[\left(\int_0^1 ((z(\lambda d_1 + (1-\lambda)d_2) + (1-z)d_1)^\alpha - d_1^\alpha) dz \right)^{1-(1/q)} \right. \\
& \quad \times \left(\int_0^1 ((z(\lambda d_1 + (1-\lambda)d_2) + (1-z)d_1)^\alpha - d_1^\alpha) (z|\phi'(\lambda d_1 + (1-\lambda)d_2)|^q + (1-z)|\phi'(d_1)|^q) dz \right)^{1/q} \Big] \\
& \quad + \frac{(d_2 - d_1)\lambda}{(d_2^\alpha - d_1^\alpha)} \left[\left(\int_0^1 (d_2^\alpha - (zd_2 + (1-z)(\lambda d_1 + (1-\lambda)d_2))^\alpha) dz \right)^{1-(1/q)} \right. \\
& \quad \times \left(\int_0^1 (d_2^\alpha - (zd_2 + (1-z)(\lambda d_1 + (1-\lambda)d_2))^\alpha) (z|\phi'(d_2)|^q + (1-z)|\phi'(\lambda d_1 + (1-\lambda)d_2)|^q) dz \right)^{1/q} \Big] \\
& = \frac{(d_2 - d_1)(1-\lambda)}{(d_2^\alpha - d_1^\alpha)} \left[(P_1(\alpha))^{1-(1/q)} (P_2(\alpha) |\phi'(\lambda d_1 + (1-\lambda)d_2)|^q + P_3(\alpha) |\phi'(d_1)|^q)^{1/q} \right] \\
& \quad + \frac{(d_2 - d_1)\lambda}{(d_2^\alpha - d_1^\alpha)} \left[(Q_1(\alpha))^{1-(1/q)} (Q_2(\alpha) |\phi'(d_2)|^q + Q_3(\alpha) |\phi'(\lambda d_1 + (1-\lambda)d_2)|^q)^{1/q} \right].
\end{aligned} \tag{26}$$

□

Remark 7. Let $\alpha = 1$. Then, from inequality (24), we get

$$\begin{aligned}
& \left| \phi(\lambda d_1 + (1-\lambda)d_2) - \frac{1}{d_2 - d_1} \int_{d_1}^{d_2} \phi(s) ds \right| \\
& \leq (1-\lambda) \left[(P_1(1))^{1-(1/q)} (P_2(1) |\phi'(\lambda d_1 + (1-\lambda)d_2)|^q + P_3(1) |\phi'(d_1)|^q)^{1/q} \right] \\
& \quad + \lambda \left[(Q_1(1))^{1-(1/q)} (Q_2(1) |\phi'(d_2)|^q + Q_3(1) |\phi'(\lambda d_1 + (1-\lambda)d_2)|^q)^{1/q} \right],
\end{aligned} \tag{27}$$

where $P_1(\alpha)$, $P_2(\alpha)$, $P_3(\alpha)$, $Q_1(\alpha)$, $Q_2(\alpha)$, and $Q_3(\alpha)$ are defined as in Theorem 7.

Theorem 8. Let $\alpha \in (0, 1]$, $d_1, d_2 \in (0, \infty)$ with $d_1 < d_2$, $q \geq 1$, and $\phi: [d_1, d_2] \rightarrow \mathbb{R}$ be an α -fractional differentiable function. Then, the Hermite–Hadamard-type inequality

$$\begin{aligned} & \left| \phi(\lambda d_1 + (1-\lambda)d_2) - \frac{\alpha}{d_2^\alpha - d_1^\alpha} \int_{d_1}^{d_2} \phi(s) d_\alpha s \right| \\ & \leq \frac{(d_2 - d_1)(1-\lambda)}{(d_2^\alpha - d_1^\alpha)} \left[\left((M_1(\alpha))^{1-(1/q)} (M_2(\alpha) |\phi'(\lambda d_1 + (1-\lambda)d_2)|^q + M_3(\alpha) |\phi'(d_1)|^q) \right)^{1/q} \right] \\ & \quad + \frac{(d_2 - d_1)\lambda}{(d_2^\alpha - d_1^\alpha)} \left[\left((N_1(\alpha))^{1-(1/q)} (N_2(\alpha) |\phi'(d_2)|^q + N_3(\alpha) |\phi'(\lambda d_1 + (1-\lambda)d_2)|^q) \right)^{1/q} \right], \end{aligned} \quad (28)$$

takes place for all $\lambda \in [0, 1]$ if $D_\alpha(\phi) \in L_\alpha^1([d_1, d_2])$ and $|\phi'|^q$ is convex on $[d_1, d_2]$, where

$$\begin{aligned} M_1(\alpha) &= \left[\frac{d_1^\alpha(\lambda^2 + \lambda - 2) + d_2^\alpha(\lambda - 1)^2}{3} - \frac{d_2 d_1^{\alpha-1}(2\lambda^2 + \lambda - 3) + d_1 d_2^{\alpha-1}(2\lambda^2 - \lambda - 1)}{6} \right], \\ N_1(\alpha) &= \frac{\lambda(d_2^\alpha - d_1^\alpha)}{2} \\ M_2(\alpha) &= \frac{(\lambda - 1)(d_1^\alpha - d_2^\alpha)}{3} \end{aligned} \quad (29)$$

Proof. From Lemma 1 and convexity of $|\phi'|^q$, we get

$$\begin{aligned} & \left| \phi(\lambda d_1 + (1-\lambda)d_2) - \frac{\alpha}{d_2^\alpha - d_1^\alpha} \int_{d_1}^{d_2} \phi(s) d_\alpha s \right| \\ & \leq \frac{(d_2 - d_1)(1-\lambda)}{(d_2^\alpha - d_1^\alpha)} \left[\int_0^1 ((z(\lambda d_1 + (1-\lambda)d_2) + (1-z)d_1)^\alpha - d_1^\alpha) \times |\phi'(z(\lambda d_1 + (1-\lambda)d_2) + (1-z)d_1)| dz \right] \\ & \quad + \frac{(d_2 - d_1)\lambda}{(d_2^\alpha - d_1^\alpha)} \left[\int_0^1 (d_2^\alpha - (zd_2 + (1-z)(\lambda d_1 + (1-\lambda)d_2))^\alpha) \times |\phi'(zd_2 + (1-z)(\lambda d_1 + (1-\lambda)d_2))| dz \right] \\ & \leq \frac{(d_2 - d_1)(1-\lambda)}{(d_2^\alpha - d_1^\alpha)} \left[\left(\int_0^1 ((z(\lambda d_1 + (1-\lambda)d_2) + (1-z)d_1)^\alpha - d_1^\alpha) dz \right)^{1-(1/q)} \right. \\ & \quad \times \left. \left(\int_0^1 ((z(\lambda d_1 + (1-\lambda)d_2) + (1-z)d_1)^\alpha - d_1^\alpha) |\phi'(z(\lambda d_1 + (1-\lambda)d_2) + (1-z)d_1)|^q dz \right)^{1/q} \right] \\ & \quad + \frac{(d_2 - d_1)\lambda}{(d_2^\alpha - d_1^\alpha)} \left[\left(\int_0^1 (d_2^\alpha - (zd_2 + (1-z)(\lambda d_1 + (1-\lambda)d_2))^\alpha) dz \right)^{1-(1/q)} \right. \\ & \quad \times \left. \left(\int_0^1 (d_2^\alpha - (zd_2 + (1-z)(\lambda d_1 + (1-\lambda)d_2))^\alpha) |\phi'(zd_2 + (1-z)(\lambda d_1 + (1-\lambda)d_2))|^q dz \right)^{1/q} \right] \end{aligned}$$

$$\begin{aligned}
&\leq \frac{(d_2 - d_1)(1 - \lambda)}{(d_2^\alpha - d_1^\alpha)} \left[\left(\int_0^1 \left((z(\lambda d_1 + (1 - \lambda)d_2) + (1 - z)d_1)^{\alpha-1} \times (z(\lambda d_1 + (1 - \lambda)d_2) + (1 - z)d_1) - d_1^\alpha \right) dz \right)^{1-(1/q)} \right. \\
&\quad \times \left. \int_0^1 \left((z(\lambda d_1 + (1 - \lambda)d_2) + (1 - z)d_1)^\alpha - d_1^\alpha \right) \left(z|\phi'(\lambda d_1 + (1 - \lambda)d_2)|^q + (1 - z)|\phi'(d_1)|^q \right) dz \right]^{1/q} \\
&\quad + \frac{(d_2 - d_1)\lambda}{(d_2^\alpha - d_1^\alpha)} \left[\left(\int_0^1 \left(d_2^\alpha - (zd_2 + (1 - z)(\lambda d_1 + (1 - \lambda)d_2))^\alpha \right) dz \right)^{1-(1/q)} \right. \\
&\quad \times \left. \left(\int_0^1 \left(d_2^\alpha - (zd_2 + (1 - z)(\lambda d_1 + (1 - \lambda)d_2))^\alpha \right) \left(z|\phi'(d_2)|^q + (1 - z)|\phi'(\lambda d_1 + (1 - \lambda)d_2)|^q \right) dz \right)^{1/q} \right] \\
&\leq \frac{(d_2 - d_1)(1 - \lambda)}{(d_2^\alpha - d_1^\alpha)} \left[\left(\int_0^1 \left((z(\lambda d_1^{\alpha-1} + (1 - \lambda)d_2^{\alpha-1}) + (1 - z)d_1^{\alpha-1}) \right) \times (z(\lambda d_1 + (1 - \lambda)d_2) + (1 - z)d_1) - d_1^\alpha \right) dz \right)^{1-(1/q)} \\
&\quad \times \left(\int_0^1 \left((z(\lambda d_1^\alpha + (1 - \lambda)d_2^\alpha) + (1 - z)d_1^\alpha) - d_1^\alpha \right) \left(z|\phi'(\lambda d_1 + (1 - \lambda)d_2)|^q + (1 - z)|\phi'(d_1)|^q \right) dz \right)^{1/q} \right] \\
&\quad + \frac{(d_2 - d_1)\lambda}{(d_2^\alpha - d_1^\alpha)} \left[\left(\int_0^1 \left(d_2^\alpha - (zd_2^\alpha + (1 - z)(\lambda d_1^\alpha + (1 - \lambda)d_2^\alpha)) \right) dz \right)^{1-(1/q)} \right. \\
&\quad \times \left. \left(\int_0^1 \left(d_2^\alpha - (zd_2^\alpha + (1 - z)(\lambda d_1^\alpha + (1 - \lambda)d_2^\alpha)) \right) \left(z|\phi'(d_2)|^q + (1 - z)|\phi'(\lambda d_1 + (1 - \lambda)d_2)|^q \right) dz \right)^{1/q} \right] \\
&= \frac{(d_2 - d_1)(1 - \lambda)}{(d_2^\alpha - d_1^\alpha)} \left[(M_1(\alpha))^{1-(1/q)} (M_2(\alpha)|\phi'(\lambda d_1 + (1 - \lambda)d_2)|^q + M_3(\alpha)|\phi'(d_1)|^q)^{1/q} \right] \\
&\quad + \frac{(d_2 - d_1)\lambda}{(d_2^\alpha - d_1^\alpha)} \left[(N_1(\alpha))^{1-(1/q)} (N_2(\alpha)|\phi'(d_2)|^q + N_3(\alpha)|\phi'(\lambda d_1 + (1 - \lambda)d_2)|^q)^{1/q} \right].
\end{aligned} \tag{30}$$

Remark 8. Let $\alpha = 1$. Then, inequality (28) becomes

$$\begin{aligned}
&\left| \phi(\lambda d_1 + (1 - \lambda)d_2) - \frac{1}{d_2 - d_1} \int_{d_1}^{d_2} \phi(s) ds \right| \\
&\leq (1 - \lambda) \left[(M_1(1))^{1-(1/q)} (M_2(1)|\phi'(\lambda d_1 + (1 - \lambda)d_2)|^q + M_3(1)|\phi'(d_1)|^q)^{1/q} \right] \\
&\quad + \lambda \left[(N_1(1))^{1-(1/q)} (N_2(1)|\phi'(d_2)|^q + N_3(1)|\phi'(\lambda d_1 + (1 - \lambda)d_2)|^q)^{1/q} \right],
\end{aligned} \tag{31}$$

where $M_1(\alpha)$, $M_2(\alpha)$, $M_3(\alpha)$, $N_1(\alpha)$, $N_2(\alpha)$, and $N_3(\alpha)$ are defined as in Theorem 8.

Theorem 9. Let $\alpha \in (0, 1]$, $d_1, d_2 \in (0, \infty)$ with $d_1 < d_2$, $q > 1$, and $\phi: [d_1, d_2] \rightarrow \mathbb{R}$ be an α -fractional differentiable function. Then, the Hermite–Hadamard-type inequality

$$\begin{aligned}
&\left| \phi(\lambda d_1 + (1 - \lambda)d_2) - \frac{\alpha}{d_2^\alpha - d_1^\alpha} \int_{d_1}^{d_2} \phi(s) d_\alpha s \right| \\
&\leq \frac{(d_2 - d_1)(1 - \lambda)}{(d_2^\alpha - d_1^\alpha)} \left[A_1(\alpha) \phi' \left(\frac{C_1(\alpha)}{A_1(\alpha)} \right) \right] + \frac{(d_2 - d_1)\lambda}{(d_2^\alpha - d_1^\alpha)} \left[B_1(\alpha) \phi' \left(\frac{C_2(\alpha)}{B_1(\alpha)} \right) \right],
\end{aligned} \tag{32}$$

holds for any $\lambda \in [0, 1]$ if $D_\alpha(\phi) \in L_\alpha^1([d_1, d_2])$ and $|\phi'|^q$ is concave on $[d_1, d_2]$, where

$$\begin{aligned} A_1(\alpha) &= \frac{1}{(d_1 - d_2)(\lambda - 1)} \left[\frac{((\lambda d_1 + (1 - \lambda)d_2)^{\alpha+1} - d_1^{\alpha+1})}{\alpha + 1} + d_1^\alpha ((\lambda - 1)(d_2 - d_1)) \right], \\ B_1(\alpha) &= \frac{1}{\lambda(d_2 - d_1)} \left[d_2^\alpha (\lambda(d_2 - d_1)) + \frac{(\lambda d_1 + (1 - \lambda)d_2)^{\alpha+1} - d_2^{\alpha+1}}{\alpha + 1} \right], \\ C_1(\alpha) &= \frac{1}{(d_1 - d_2)(\lambda - 1)} \left[\frac{((\lambda d_1 + (1 - \lambda)d_2)^{\alpha+2} - d_1^{\alpha+2})}{\alpha + 2} + d_1^\alpha \frac{(d_1^2 - (\lambda d_1 + (1 - \lambda)d_2)^2)}{2} \right], \\ C_2(\alpha) &= \frac{1}{\lambda(d_2 - d_1)} \left[\frac{((\lambda d_1 + (1 - \lambda)d_2)^{\alpha+2} - d_2^{\alpha+2})}{\alpha + 2} + d_2^\alpha \frac{(d_2^2 - (\lambda d_1 + (1 - \lambda)d_2)^2)}{2} \right]. \end{aligned} \quad (33)$$

Proof. Let $\gamma \in [0, 1]$. Then, the power mean inequality and the concavity of $|\phi'|^q$ lead to the conclusion that

$$\begin{aligned} & (\gamma|\phi'(d_1)| + (1 - \gamma)|\phi'(d_2)|)^q \\ & \leq \gamma|\phi'(d_1)|^q + (1 - \gamma)|\phi'(d_2)|^q \\ & \leq |\phi'(\gamma d_1 + (1 - \gamma)d_2)|^q, \end{aligned} \quad (34)$$

which shows that $|\phi'|$ is also concave.

Making use of Lemma 1, we get

$$\begin{aligned} & \left| \phi(\lambda d_1 + (1 - \lambda)d_2) - \frac{\alpha}{d_2^\alpha - d_1^\alpha} \int_{d_1}^{d_2} \phi(s) d_\alpha s \right| \\ & \leq \frac{(d_2 - d_1)(1 - \lambda)}{(d_2^\alpha - d_1^\alpha)} \left[\int_0^1 ((z(\lambda d_1 + (1 - \lambda)d_2) + (1 - z)d_1)^\alpha - d_1^\alpha) \times |\phi'((\lambda d_1 + (1 - \lambda)d_2) + (1 - z)d_1)| dz \right] \\ & \quad + \frac{(d_2 - d_1)\lambda}{(d_2^\alpha - d_1^\alpha)} \left[\int_0^1 (d_2^\alpha - (zd_2 + (1 - z)(\lambda d_1 + (1 - \lambda)d_2))^\alpha) \times |\phi'(zd_2 + (1 - z)(\lambda d_1 + (1 - \lambda)d_2))| dz \right]. \end{aligned} \quad (35)$$

Applying Jensen integral inequality, we have

$$\begin{aligned} & \int_0^1 ((z(\lambda d_1 + (1 - \lambda)d_2) + (1 - z)d_1)^\alpha - d_1^\alpha) |\phi'((\lambda d_1 + (1 - \lambda)d_2) + (1 - z)d_1)| dz \\ & \leq \left(\int_0^1 ((z(\lambda d_1 + (1 - \lambda)d_2) + (1 - z)d_1)^\alpha - d_1^\alpha) dz \right) \\ & \quad \times \phi' \left(\frac{\int_0^1 ((z(\lambda d_1 + (1 - \lambda)d_2) + (1 - z)d_1)^\alpha - d_1^\alpha) (z(\lambda d_1 + (1 - \lambda)d_2) + (1 - z)d_1) dz}{\int_0^1 ((z(\lambda d_1 + (1 - \lambda)d_2) + (1 - z)d_1)^\alpha - d_1^\alpha) dz} \right) \\ & = A_1(\alpha) \phi' \left(\frac{C_1(\alpha)}{A_1(\alpha)} \right). \end{aligned} \quad (36)$$

Similarly, one has

$$\begin{aligned}
 & \int_0^1 (d_2^\alpha - (zd_2 + (1-z)(\lambda d_1 + (1-\lambda)d_2))^\alpha) |\phi'(d_2^\alpha - (zd_2 + (1-z)(\lambda d_1 + (1-\lambda)d_2)))| dz \\
 & \leq \left(\int_0^1 (d_2^\alpha - (zd_2 + (1-z)(\lambda d_1 + (1-\lambda)d_2))^\alpha) dz \right) \\
 & \quad \times \phi' \left(\frac{\int_0^1 (d_2^\alpha - (zd_2 + (1-z)(\lambda d_1 + (1-\lambda)d_2))^\alpha) ((zd_2 + (1-z)(\lambda d_1 + (1-\lambda)d_2))) dz}{\int_0^1 (d_2^\alpha - (zd_2 + (1-z)(\lambda d_1 + (1-\lambda)d_2))^\alpha) dz} \right) \\
 & = B_1(\alpha) \phi' \left(\frac{C_2(\alpha)}{B_1(\alpha)} \right).
 \end{aligned} \tag{37}$$

Remark 9. Let $\alpha = 1$. Then, inequality (32) gives

$$\begin{aligned}
 & \left| \phi(\lambda d_1 + (1-\lambda)d_2) - \frac{1}{d_2 - d_1} \int_{d_1}^{d_2} \phi(s) ds \right| \\
 & \leq (1-\lambda) \left[A_1(1) \phi' \left(\frac{C_1(1)}{A_1(1)} \right) \right] + \lambda \left[B_1(1) \phi' \left(\frac{C_2(1)}{B_1(1)} \right) \right],
 \end{aligned} \tag{38}$$

where $A_1(\alpha)$, $C_1(\alpha)$, $B_1(\alpha)$, and $C_2(\alpha)$ are defined as in Theorem 9.

Theorem 10. Let $\alpha \in (0, 1]$, $d_1, d_2 \in (0, \infty)$ with $d_1 < d_2$, and $\phi: [d_1, d_2] \rightarrow \mathbb{R}$ be an α -fractional differentiable function. Then, the Hermite-Hadamard-type inequality

$$\begin{aligned}
 & \left| \phi(\lambda d_1 + (1-\lambda)d_2) - \frac{\alpha}{d_2^\alpha - d_1^\alpha} \int_{d_1}^{d_2} \phi(s) d_\alpha s \right| \\
 & \leq \frac{(d_2 - d_1)(1-\lambda)}{(d_2^\alpha - d_1^\alpha)} [A_1(\alpha) |\phi'(d_1)| + B_1(\alpha) |\phi'(\lambda d_1 + (1-\lambda)d_2)|] \\
 & \quad + \frac{(d_2 - d_1)\lambda}{(d_2^\alpha - d_1^\alpha)} [A_2(\alpha) |\phi'(d_2)| + B_2(\alpha) |\phi'(\lambda d_1 + (1-\lambda)d_2)|],
 \end{aligned} \tag{39}$$

holds for all $\lambda \in [0, 1]$ if $D_\alpha(\phi) \in L_\alpha^1([d_1, d_2])$ and $|\phi'|$ is convex on $[d_1, d_2]$, where $A_1(\alpha)$, $B_1(\alpha)$, $A_2(\alpha)$, and $B_2(\alpha)$ are defined as in Theorem 5.

Proof. It follows from Lemma 1 and the convexity of $|\phi'|$ that

$$\begin{aligned}
 & \left| \phi(\lambda d_1 + (1-\lambda)d_2) - \frac{\alpha}{d_2^\alpha - d_1^\alpha} \int_{d_1}^{d_2} \phi(s) d_\alpha s \right| \\
 & \leq \frac{(d_2 - d_1)(1-\lambda)}{(d_2^\alpha - d_1^\alpha)} \left[\int_0^1 ((z(\lambda d_1 + (1-\lambda)d_2) + (1-z)d_1)^\alpha - d_1^\alpha) \times |\phi'(z(\lambda d_1 + (1-\lambda)d_2) + (1-z)d_1)| dz \right] \\
 & \quad + \frac{(d_2 - d_1)\lambda}{(d_2^\alpha - d_1^\alpha)} \left[\int_0^1 (d_2^\alpha - (zd_2 + (1-z)(\lambda d_1 + (1-\lambda)d_2))^\alpha) \times |\phi'(zd_2 + (1-z)(\lambda d_1 + (1-\lambda)d_2))| dz \right] \\
 & \leq \frac{(d_2 - d_1)(1-\lambda)}{(d_2^\alpha - d_1^\alpha)} \left[\int_0^1 ((z(\lambda d_1 + (1-\lambda)d_2) + (1-z)d_1)^\alpha - d_1^\alpha) \times [z|\phi'(\lambda d_1 + (1-\lambda)d_2)| + (1-z)|\phi'(d_1)|] dz \right] \\
 & \quad + \frac{(d_2 - d_1)\lambda}{(d_2^\alpha - d_1^\alpha)} \left[\int_0^1 (d_2^\alpha - (zd_2 + (1-z)(\lambda d_1 + (1-\lambda)d_2))^\alpha) \times [z|\phi'(d_2)| + (1-z)|\phi'(\lambda d_1 + (1-\lambda)d_2)|] dz \right] \\
 & = \frac{(d_2 - d_1)(1-\lambda)}{(d_2^\alpha - d_1^\alpha)} [A_1(\alpha) |\phi'(d_1)| + B_1(\alpha) |\phi'(\lambda d_1 + (1-\lambda)d_2)|] \\
 & \quad + \frac{(d_2 - d_1)\lambda}{(d_2^\alpha - d_1^\alpha)} [A_2(\alpha) |\phi'(d_2)| + B_2(\alpha) |\phi'(\lambda d_1 + (1-\lambda)d_2)|].
 \end{aligned} \tag{40}$$

Remark 10. Let $\alpha = 1$. Then, inequality (39) leads to

$$\begin{aligned} & \left| \phi(\lambda d_1 + (1-\lambda)d_2) - \frac{\alpha}{d_2 - d_1} \int_{d_1}^{d_2} \phi(s) ds \right| \\ & \leq (1-\lambda) \left[A_1(1) |\phi'(d_1)| + B_1(1) |\phi'(\lambda d_1 + (1-\lambda)d_2)| \right] \\ & \quad + \lambda \left[A_2(1) |\phi'(d_2)| + B_2(1) |\phi'(\lambda d_1 + (1-\lambda)d_2)| \right]. \end{aligned} \quad (41)$$

3. Applications to Bivariate Means

A real-valued function $M: (0, \infty) \times (0, \infty) \rightarrow (0, \infty)$ is said to be a bivariate mean [82] if $\min\{a, b\} \leq M(a, b) \leq \max\{a, b\}$ for all $a, b \in (0, \infty)$. Recently, the properties and applications for the bivariate means and their related special functions have attracted the attention of many researchers [83–93].

Let $d_1, d_2 > 0$ with $d_1 \neq d_2$, $\alpha \in [-1, 0) \cup (0, 1]$, and $r \neq 0$. Then, the arithmetic mean $A(d_1, d_2)$, logarithmic mean $L(d_1, d_2)$, and (α, r) -th generalized logarithmic mean $L_{(\alpha, r)}(d_1, d_2)$ are defined by

$$\begin{aligned} A(d_1, d_2) &= \frac{d_1 + d_2}{2}, \\ L(d_1, d_2) &= \frac{d_2 - d_1}{\log d_2 - \log d_1}, \\ L_{(\alpha, r)}(d_1, d_2) &= \left[\frac{\alpha(d_2^{r+\alpha} - d_1^{r+\alpha})}{(d_2^\alpha - d_1^\alpha)(r+\alpha)} \right]^{1/r}, \end{aligned} \quad (42)$$

respectively.

In this section, we shall present several inequalities involving the arithmetic, logarithmic, and generalized logarithmic means.

Let $x > 0$ and $\phi(x) = x^r$. Then, Theorems 5 and 6 lead to Theorems 11 and 12 immediately.

Theorem 11. Let $d_1, d_2 > 0$ with $d_1 < d_2$, $r > 1$, and $\alpha \in (0, 1]$. Then, the inequalities

$$\begin{aligned} & \left| A^r(d_1, d_2) - L_{(\alpha, r)}^r(d_1, d_2) \right| \\ & \leq \frac{(d_2 - d_1)(1-\lambda)}{(d_2^\alpha - d_1^\alpha)} \left[A_1(\alpha) d_1^{r-1} + B_1(\alpha) (\lambda d_1 + (1-\lambda)d_2)^{r-1} \right] \\ & \quad + \frac{(d_2 - d_1)\lambda}{(d_2^\alpha - d_1^\alpha)} \left[A_2(\alpha) d_2^{r-1} + B_2(\alpha) (\lambda d_1 + (1-\lambda)d_2)^{r-1} \right], \\ & \left| A^{-1}(d_1, d_2) - L_{(\alpha, -1)}^{-1}(d_1, d_2) \right| \\ & \leq \frac{(d_2 - d_1)(1-\lambda)}{(d_2^\alpha - d_1^\alpha)} \left[A_1(\alpha) d_1^{-2} + B_1(\alpha) (\lambda d_1 + (1-\lambda)d_2)^{-2} \right] \\ & \quad + \frac{(d_2 - d_1)\lambda}{(d_2^\alpha - d_1^\alpha)} \left[A_2(\alpha) d_2^{-2} + B_2(\alpha) (\lambda d_1 + (1-\lambda)d_2)^{-2} \right], \end{aligned} \quad (43)$$

hold for all $\lambda \in [0, 1]$, where $A_1(\alpha)$, $A_2(\alpha)$, $B_1(\alpha)$, and $B_2(\alpha)$ are defined as in Theorem 5.

Theorem 12. Let $d_1, d_2 > 0$ with $d_1 < d_2$, $r > 1$, $\alpha \in (0, 1]$, and $p, q > 1$ with $1/p + 1/q = 1$. Then, the inequalities

$$\begin{aligned} & \left| A^r(d_1, d_2) - L_{(\alpha, r)}^r(d_1, d_2) \right| \\ & \leq \frac{(d_2 - d_1)(1-\lambda)}{(d_2^\alpha - d_1^\alpha)} \left[(A_1(\alpha, p))^{1/p} \left(\frac{(\lambda d_1 + (1-\lambda)d_2)^{(r-1)q} + d_1^{(r-1)q}}{2} \right)^{1/p} \right] \\ & \quad + \frac{(d_2 - d_1)\lambda}{(d_2^\alpha - d_1^\alpha)} \left[(A_2(\alpha, p))^{1/p} \left(\frac{d_2^{(r-1)q} (\lambda d_1 + (1-\lambda)d_2)^{(r-1)q}}{2} \right)^{1/q} \right], \\ & \left| A^{-1}(d_1, d_2) - L_{(\alpha, -1)}^{-1}(d_1, d_2) \right| \\ & \leq \frac{(d_2 - d_1)(1-\lambda)}{(d_2^\alpha - d_1^\alpha)} \left[(A_1(\alpha, p))^{1/p} \left(\frac{(\lambda d_1 + (1-\lambda)d_2)^{-2q} + d_1^{-2q}}{2} \right)^{1/q} \right] \\ & \quad + \frac{(d_2 - d_1)\lambda}{(d_2^\alpha - d_1^\alpha)} \left[(A_2(\alpha, p))^{1/p} \left(\frac{d_2^{-2q} (\lambda d_1 + (1-\lambda)d_2)^{-2q}}{2} \right)^{1/q} \right], \end{aligned} \quad (44)$$

hold for all $\lambda \in [0, 1]$, where $A_1(\alpha, p)$ and $A_2(\alpha, p)$ are defined as in Theorem 6.

Let $x > 0$ and $\phi(x) = 1/x$. Then, Theorem 6 leads to Theorems 13 immediately.

Theorem 13. Let $d_1, d_2 > 0$ with $d_1 < d_2$, $\alpha \in (0, 1]$, and $q > 1$. Then, the inequalities

$$\begin{aligned}
 & \left| A^r(d_1, d_2) - L_{(\alpha, r)}^r(d_1, d_2) \right| \\
 & \leq \frac{(d_2 - d_1)(1 - \lambda)}{(d_2^\alpha - d_1^\alpha)} \left[(M_1(\alpha))^{1-(1/q)} (M_2(\alpha)(\lambda d_1 + (1 - \lambda)d_2)^{(r-1)q} + M_3(\alpha)d_1^{(r-1)q})^{1/q} \right] \\
 & \quad + \frac{(d_2 - d_1)\lambda}{(d_2^\alpha - d_1^\alpha)} \left[(N_1(\alpha))^{1-(1/q)} (N_2(\alpha)d_2^{(r-1)q} + N_3(\alpha)(\lambda d_1 + (1 - \lambda)d_2)^{(r-1)q})^{1/q} \right], \\
 & \left| A^{-1}(d_1, d_2) - L_{(\alpha, -1)}^{-1}(d_1, d_2) \right| \\
 & \leq \frac{(d_2 - d_1)(1 - \lambda)}{(d_2^\alpha - d_1^\alpha)} \left[(M_1(\alpha))^{1-(1/q)} (M_2(\alpha)(\lambda d_1 + (1 - \lambda)d_2)^{-2q} + M_3(\alpha)d_1^{-2q})^{1/q} \right] \\
 & \quad + \frac{(d_2 - d_1)\lambda}{(d_2^\alpha - d_1^\alpha)} \left[(N_1(\alpha))^{1-(1/q)} (N_2(\alpha)d_2^{-2q} + N_3(\alpha)(\lambda d_1 + (1 - \lambda)d_2)^{-2q})^{1/q} \right],
 \end{aligned} \tag{45}$$

hold, where $M_1(\alpha)$, $M_2(\alpha)$, $M_3(\alpha)$, $N_1(\alpha)$, $N_2(\alpha)$, and $N_3(\alpha)$ are defined as in Theorem 8.

4. Applications to Midpoint Formula

Let P be the partition of the points $d_1 = y_0 < y_1 < \dots < y_{n-1} < y_n = d_2$ of the interval $[d_1, d_2]$ and consider the quadrature formula

$$\int_{d_1}^{d_2} \phi(y) d_\alpha y = T_\alpha(\phi, P) + E_\alpha(\phi, P), \tag{46}$$

where

$$T_\alpha(\phi, P) = \sum_{i=0}^{n-1} \phi\left(\frac{y_i + y_{i+1}}{2}\right) \frac{(y_{i+1}^\alpha - y_i^\alpha)}{\alpha}, \tag{47}$$

is the midpoint version and $E_\alpha(\phi, P)$ denotes the associated approximation error. In this section, we are going to derive some new estimates for the midpoint formula.

Theorem 14. Let $d_1, d_2 > 0$ with $d_1 < d_2$, $\alpha \in (0, 1]$, and $\phi: [d_1, d_2] \rightarrow \mathbb{R}$ be an α -differentiable function. Then, one has

$$\begin{aligned}
 & |E_\alpha(\phi, P)| \\
 & \leq \sum_{i=0}^{n-1} \frac{(y_{i+1} - y_i)(1 - \lambda)}{(y_{i+1}^\alpha - y_i^\alpha)} \left[A_i(\alpha) |\phi'(y_i)| \right. \\
 & \quad \left. + B_i(\alpha) |\phi'(\lambda y_i + (1 - \lambda)y_{i+1})| \right] \\
 & \quad + \sum_{i=0}^{n-1} \frac{(y_{i+1} - y_i)\lambda}{(y_{i+1}^\alpha - y_i^\alpha)} \left[\bar{A}_i(\alpha) |\phi'(y_{i+1})| \right. \\
 & \quad \left. + \bar{B}_i(\alpha) |\phi'(\lambda y_i + (1 - \lambda)y_{i+1})| \right],
 \end{aligned} \tag{48}$$

if $D_\alpha(\phi) \in L_\alpha^1([d_1, d_2])$ and $|\phi'|$ is convex on $[d_1, d_2]$, where

$$\begin{aligned}
 A_i(\alpha) &= \frac{1}{12} \left[(\lambda y_i + (1 - \lambda)y_{i+1})(\lambda + 1)(y_i^{\alpha-1} + y_{i+1}^{\alpha-1}) + \lambda y_i(y_i^{\alpha-1} - y_{i+1}^{\alpha-1}) + y_i(y_{i+1}^{\alpha-1} - 5y_i^{\alpha-1}) \right], \\
 B_i(\alpha) &= \frac{(\lambda y_i + (1 - \lambda)y_{i+1})}{12} \left[3\lambda(y_i^{\alpha-1} - y_{i+1}^{\alpha-1}) + (3y_{i+1}^{\alpha-1} + y_i^{\alpha-1}) + y_i^\alpha(\lambda - 5) \right], \\
 \bar{A}_i(\alpha) &= \frac{\lambda(y_{i+1}^\alpha - y_i^\alpha)}{6}, \\
 \bar{B}_i(\alpha) &:= \frac{\lambda(y_{i+1}^\alpha - y_i^\alpha)}{3}.
 \end{aligned} \tag{49}$$

Proof. Applying Theorem 5 on the subintervals $[y_i, y_{i+1}]$ ($i = 0, 1, \dots, n-1$) of the partition P , we have

$$\begin{aligned}
 & \left| \phi(\lambda y_i + (1-\lambda)y_{i+1}) \frac{(y_{i+1}^\alpha - y_i^\alpha)}{\alpha} - \int_{y_i}^{y_{i+1}} \phi(y) d_\alpha y \right| \\
 & \leq \frac{(y_{i+1} - y_i)(1-\lambda)}{(y_{i+1}^\alpha - y_i^\alpha)} \left[A_i(\alpha) |\phi'(y_i)| + B_i(\alpha) |\phi'(\lambda y_i + (1-\lambda)y_{i+1})| \right] \\
 & \quad + \frac{(y_{i+1} - y_i)\lambda}{(y_{i+1}^\alpha - y_i^\alpha)} \left[\bar{A}_i(\alpha) |\phi'(y_{i+1})| + \bar{B}_i(\alpha) |\phi'(\lambda y_i + (1-\lambda)y_{i+1})| \right], \\
 & \left| \int_{d_1}^{d_2} \phi(y) d_\alpha y - T_\alpha(\phi, P) \right| \\
 & = \left| \sum_{i=0}^{n-1} \left\{ \int_{y_i}^{y_{i+1}} \phi(y) d_\alpha y - \phi(\lambda y_i + (1-\lambda)y_{i+1}) \frac{(y_{i+1}^\alpha - y_i^\alpha)}{\alpha} \right\} \right| \\
 & \leq \sum_{i=0}^{n-1} \left| \left\{ \int_{y_i}^{y_{i+1}} \phi(y) d_\alpha y - \phi(\lambda y_i + (1-\lambda)y_{i+1}) \frac{(y_{i+1}^\alpha - y_i^\alpha)}{\alpha} \right\} \right| \\
 & \leq \sum_{i=0}^{n-1} \frac{(y_{i+1} - y_i)(1-\lambda)}{(y_{i+1}^\alpha - y_i^\alpha)} \left[A_i(\alpha) |\phi'(y_i)| + B_i(\alpha) |\phi'(\lambda y_i + (1-\lambda)y_{i+1})| \right] \\
 & \quad + \sum_{i=0}^{n-1} \frac{(y_{i+1} - y_i)\lambda}{(y_{i+1}^\alpha - y_i^\alpha)} \left[\bar{A}_i(\alpha) |\phi'(y_{i+1})| + \bar{B}_i(\alpha) |\phi'(\lambda y_i + (1-\lambda)y_{i+1})| \right].
 \end{aligned} \tag{50}$$

Making use of Theorems 6–8 and similar arguments as in the proof of Theorem 14, we get Theorems 15 and 16 immediately. $\square$

Theorem 15. Let $d_1, d_2 > 0$ with $d_1 < d_2$, $\alpha \in (0, 1]$, and $\phi: [d_1, d_2] \rightarrow \mathbb{R}$ be an α -differentiable function and $p, q > 1$ with $1/p + 1/q = 1$. Then, the inequality

$$\begin{aligned}
 |E_\alpha(\phi, P)| & \leq \sum_{i=0}^{n-1} \frac{(y_{i+1} - y_i)(1-\lambda)}{(y_{i+1}^\alpha - y_i^\alpha)} \left[(A_i(\alpha, p))^{1/p} E_i^{1/q} \right] \\
 & \quad + \sum_{i=0}^{n-1} \frac{(y_{i+1} - y_i)\lambda}{(y_{i+1}^\alpha - y_i^\alpha)} \left[(\bar{A}_i(\alpha, p))^{1/p} \bar{E}_i^{1/q} \right],
 \end{aligned} \tag{51}$$

holds for all $\lambda \in [0, 1]$ if $D_\alpha(\phi) \in L_\alpha^1([d_1, d_2])$ and $|\phi'|^q$ is convex on $[d_1, d_2]$, where

$$E_i(\alpha) = \frac{|\phi'(\lambda y_i + (1-\lambda)y_{i+1})|^q + |\phi'(y_i)|^q}{2},$$

$$\bar{E}_i(\alpha) = \frac{|\phi'(y_{i+1})|^q + |\phi'(\lambda y_i + (1-\lambda)y_{i+1})|^q}{2},$$

$$A_i(\alpha, p) = \frac{1}{(\lambda - 1)(y_i - y_{i+1})} \int_{y_i}^{\lambda y_i + (1-\lambda)y_{i+1}} (s^\alpha - y_i^\alpha)^p ds,$$

$$\bar{A}_i(\alpha, p) = \frac{1}{\lambda(y_{i+1} - y_i)} \int_{\lambda y_i + (1-\lambda)y_{i+1}}^{y_{i+1}} (y_{i+1}^\alpha - s^\alpha)^p ds. \tag{52}$$

Theorem 16. Let $d_1, d_2 > 0$ with $d_1 < d_2$, $\alpha \in (0, 1]$, and $\phi: [d_1, d_2] \rightarrow \mathbb{R}$ be an α -differentiable function and $q \geq 1$. Then, the inequality

$$\begin{aligned}
 & |E_\alpha(\phi, P)| \\
 & \leq \sum_{i=0}^{n-1} \frac{(y_{i+1} - y_i)(1-\lambda)}{(y_{i+1}^\alpha - y_i^\alpha)} \left[(M_i(\alpha))^{1-(1/q)} (\bar{M}_i(\alpha) |\phi'(\lambda y_i + (1-\lambda)y_{i+1})|^q + \bar{M}_i(\alpha) |\phi'(y_i)|^q)^{1/q} \right] \\
 & \quad + \sum_{i=0}^{n-1} \frac{(y_{i+1} - y_i)\lambda}{(y_{i+1}^\alpha - y_i^\alpha)} \left[(N_i(\alpha))^{1-(1/q)} (\bar{N}_i(\alpha) |\phi'(y_{i+1})|^q + \bar{N}_i(\alpha) |\phi'(\lambda y_i + (1-\lambda)y_{i+1})|^q)^{1/q} \right],
 \end{aligned} \tag{53}$$

holds for all $\lambda \in [0, 1]$ if $D_\alpha(\phi) \in L^1_\alpha([d_1, d_2])$ and $|\phi'|^q$ is convex on $[d_1, d_2]$, where

$$\begin{aligned} M_i(\alpha) &= \left[\frac{y_i^\alpha(\lambda^2 + \lambda - 2) + y_{i+1}^\alpha(\lambda - 1)^2}{3} - \frac{y_{i+1}y_i^{\alpha-1}(2\lambda^2 + \lambda - 3) + y_iy_{i+1}^{\alpha-1}(2\lambda^2 - \lambda - 1)}{6} \right], \\ N_i(\alpha) &= \frac{\lambda(y_{i+1}^\alpha - y_i^\alpha)}{2} = 3\overline{N}_i(\alpha) = \frac{3}{2}\overline{N}_i(\alpha), \\ \overline{M}_i(\alpha) &= \frac{(\lambda - 1)(y_i^\alpha - y_{i+1}^\alpha)}{3} = 2\overline{M}_i(\alpha). \end{aligned} \quad (54)$$

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Authors' Contributions

All authors contributed equally to writing of this paper. All authors read and approved the final manuscript.

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