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Research Article

Ground State Solution for an Autonomous Nonlinear Schrödinger System

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In this paper, we study the following autonomous nonlinear Schrödinger system (discussed in the paper), where λ, μ, and ν are positive parameters; $2^* = 2N/(N-2)$ is the critical Sobolev exponent; and f satisfies general subcritical growth conditions. With the help of the Pohožaev manifold, a ground state solution is obtained.

1. Introduction and Main Result

In this paper, we consider the following autonomous nonlinear Schrödinger system:

$$\begin{cases} -\Delta u + \mu u = \mu f(u) + \lambda v, & x \in \mathbb{R}^N, \\ -\Delta v + \nu v = |v|^{2^*-2} v + \lambda u, & x \in \mathbb{R}^N, \\ u, v \in H^1(\mathbb{R}^N), & N \geq 3, \end{cases} \tag{1}$$

where μ, ν, and λ are positive parameters satisfying $0 < \lambda < \sqrt{\mu\nu}$; $2^* = 2N/(N-2)$ is the critical Sobolev exponent; and f satisfies the following conditions:

(f_1) $f \in C(\mathbb{R}, \mathbb{R})$ is an odd function.

(f_2) $\lim_{s \longrightarrow 0^+} (f(s)/s) = 0$.

(f_3) $\lim_{s \longrightarrow +\infty} (f(s)/s^{2^*-1}) = 0$.

(f_4) There exists $\zeta > 0$ such that $F(\zeta) > (\zeta^2/2)$, where $F(\zeta) = \int_0^\zeta f(t)dt$.

Systems of above type arise in nonlinear optics (cf. [1]). It is well known that a solution $(u, v) \in H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$ of system (1) is called a ground state solution if $(u, v) \neq (0, 0)$ and its energy is minimal among the energy of all the nontrivial solutions.

The following nonlinear Schrödinger system

$$\begin{cases} -\Delta u + \mu u = |u|^{p-1} u + \lambda v, & x \in \mathbb{R}^N, \\ -\Delta v + \nu v = |v|^{q-1} v + \lambda u, & x \in \mathbb{R}^N, \\ u, v \in H^1(\mathbb{R}^N), \end{cases} \tag{2}$$

has been studied by many authors. When $N \leq 3$, $\mu = \nu = 1$, $p = q = 3$, and $\lambda > 0$ small enough, Ambrosetti et al. [2] proved that (2) has multibump solitons. When $N \geq 2$, $\mu = \nu = 1$, $1 < p, q < 2^* - 1$, $0 < \lambda < 1$, and $|u|^{p-1}u$ and $|v|^{q-1}v$ are replaced by $(1 + a(x))|u|^{p-1}u$ and $(1 + b(x))|v|^{q-1}v$, Ambrosetti et al. [3] proved that system (2) has a positive ground state solution. When $u, v \in D^{1,2}(\mathbb{R}^4)$, $\mu = V_1(x)$, and $\nu = V_2(x)$ satisfy the integral conditions and $|u|^{p-1}u, \lambda v, |v|^{q-1}v$ and λu are replaced by $\mu_1 u^3, \beta uv^2, \mu_2 v^3$, and $\beta u^2 v$, respectively, Liu and Liu [4] proved that (2) has a positive solution. When $u, v \in H_0^1(\Omega)$, Ω is a smooth bounded domain in $\mathbb{R}^3$, $p = q = 3$, and $\lambda v, \lambda u$ are replaced by $-\beta uv^2, -\beta vu^2$, respectively, Noris and Ramos [5] proved that (2) admits an unbounded sequence of solutions (u, v) with $u > 0, v > 0$, and $u \neq v$ for sufficiently large $\beta > 0$. When $N \geq 3$, $1 < p < 2^* - 1, q = 2^* - 1$, and $\mu, \nu > 0$, $0 < \lambda < \sqrt{\mu\nu}$, Chen and Zou [6] proved that (2) has a positive ground state solution under λ, μ, ν which satisfied certain conditions. When $N \geq 3$, $1 < p < 2^* - 1, q = 2^* - 1$, and $\mu = a(x), \nu = b(x), \lambda = \lambda(x)$, Li and Tang [7] proved that (2) has a nontrivial solution.

Inspired by the above literatures, especially [6], we investigate the existence of ground state solution of system (1). When $\mu f(u) = |u|^{p-1}u$ with $1 < p < 2^* - 1$, by using the Nehari manifold, Chen and Zou [6] obtained the existence of ground state solution of system (1). But in our paper, without the assumption of the monotonicity of $u \mapsto (f(u))/u$, we have to adopt a new method to replace the Nehari manifold.

The following single Schrödinger equation

$$-\Delta u + u = f(u), \quad u \in H^1(\mathbb{R}^N), N \geq 3, \tag{3}$$

has been widely studied by many researchers, and relevant results can been referred to [8–10] and the references therein. By [9], we know that if f satisfies (f_1)-(f_4); then, equation (3) has a ground state solution. Define

$$a = \inf_{u \in \Gamma} \left[\frac{1}{2} \int_{\mathbb{R}^N} \left(|\nabla u|^2 + u^2 \right) dx - \int_{\mathbb{R}^N} F(u) dx \right], \tag{4}$$

where $\Gamma = \{u \in H^1(\mathbb{R}^N) \colon u \text{ is a nontrivial solution of equation (3)}\}$ and define

$$S = \inf_{u \in D^{1,2}(\mathbb{R}^N) \setminus \{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u|^2 dx}{\left(\int_{\mathbb{R}^N} |u|^{2^*} dx \right)^{2/2^*}}, \tag{5}$$

where S is the optimal constant of the Sobolev embedding $D^{1,2}(\mathbb{R}^N) \longrightarrow L^{2^*}(\mathbb{R}^N)$.

The main result of this paper is the following.

Theorem 1. *Assume that μ, ν, and λ are positive parameters satisfying $\mu > S^{-N/(N-2)}(aN)^{2/(N-2)}$ and $0 < \lambda < \sqrt{\mu\nu}$. Suppose that f satisfies (f_1)-(f_4). Then, system (1) has a ground state solution.*

Remark 2. There are some examples of functions that satisfy the assumptions (f_1)-(f_4), for example, $f(s) = |s|^{p-2}s$ with $2 < p < 2^*$ and $f(s) = |s|^{p-2}s/(1 + s^2)$ with $4 < p < 2^* + 2$.

Remark 3. It is obvious that system (1) has no semitrivial solutions. Indeed, if $(u, 0)$ is a solution of system (1), then $u = 0$ and if $(0, v)$ is a solution of system (1), then $v = 0$.

Remark 4. There are some recent studies on the ground state solutions for other types of Schrödinger equations or systems, for example, [6, 11]. Moreover, in the bounded domain, the existence and the regularity of solutions to differential problems have been widely investigated by using tools of harmonic and real analysis and variational methods, for example, [12–14].

2. Preliminaries

In order to make a precise explanation of the results in this paper, we will give some notations.

C, C_i denote various positive constants.

$L^p(\mathbb{R}^N)$ is the usual Lebesgue space endowed with the norm

$$|u|_p = \left(\int_{\mathbb{R}^N} |u|^p dx \right)^{1/p}. \tag{6}$$

$D^{1,2}(\mathbb{R}^N) = \{u \in L^{2^*}(\mathbb{R}^N) \mid (\partial u/\partial x_i) \in L^2(\mathbb{R}^N), i = 1, 2, \cdots, N\}$ endowed with the norm

$$\|u\|_{D^{1,2}} = \left(\int_{\mathbb{R}^N} |\nabla u|^2 dx \right)^{1/2}. \tag{7}$$

$H^1(\mathbb{R}^N) = \{u \in L^2(\mathbb{R}^N) \mid (\partial u/\partial x_i) \in L^2(\mathbb{R}^N), i = 1, 2, \cdots, N\}$ endowed with the norm

$$\|u\| = \left(\int_{\mathbb{R}^N} \left(|\nabla u|^2 + u^2 \right) dx \right)^{1/2}. \tag{8}$$

For any $(u, v) \in H \coloneqq H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$, we set

$$\|u, v\|_H = \left(\int_{\mathbb{R}^N} \left(|\nabla u|^2 + \mu u^2 + |\nabla v|^2 + \nu v^2 \right) dx \right)^{1/2}. \tag{9}$$

For any $u \in H^1(\mathbb{R}^N)$, we denote $u_t = u(\cdot/t)$ for all $t > 0$.

The weak solutions of (1) correspond to critical points of the functional

$$I(u, v) = \frac{1}{2} \|u, v\|_H^2 - \mu \int_{\mathbb{R}^N} F(u) dx - \frac{1}{2^*} \int_{\mathbb{R}^N} |v|^{2^*} dx - \lambda \int_{\mathbb{R}^N} uv dx. \tag{10}$$

Obviously, $I \in C^1(H, \mathbb{R})$ and for all $(u, v) \in H$ and $(\varphi, \psi) \in H$, we have

$$\begin{aligned} \left\langle I'(u, v), (\varphi, \psi) \right\rangle = & \int_{\mathbb{R}^N} (\nabla u \cdot \nabla \varphi + \mu u \varphi + \nabla v \cdot \nabla \psi + \nu v \psi) dx \\ & - \mu \int_{\mathbb{R}^N} f(u) \varphi dx - \int_{\mathbb{R}^N} |v|^{2^*-2} v \psi dx \\ & - \lambda \int_{\mathbb{R}^N} (\varphi v + u \psi) dx. \end{aligned} \tag{11}$$

Similar to [15, 16], in order to obtain a ground state solution, we define the Pohožaev manifold

$$\mathscr{P} = \{(u, v) \in H \setminus \{(0,0)\} : J(u, v) = 0\} \tag{12}$$

and consider the constraint minimization problem

$$m = \inf_{(u,v)\in\mathscr{P}} I(u, v), \tag{13}$$

where $J : H \longrightarrow \mathbb{R}$ is defined as

$$J(u, v) = \frac{N-2}{2}\int_{\mathbb{R}^N} \left(|\nabla u|^2 + |\nabla v|^2\right)dx + \frac{N}{2}\int_{\mathbb{R}^N}\left(\mu u^2 + \nu v^2\right)dx - \mu N\int_{\mathbb{R}^N} F(u)dx - \frac{N}{2^*}\int_{\mathbb{R}^N}|v|^{2^*}dx - \lambda N\int_{\mathbb{R}^N} uv dx. \tag{14}$$

We also require the following subcritical system of system (1):

$$\begin{cases} -\Delta u + \mu u = \mu f(u) + \lambda v, & x \in \mathbb{R}^N, \\ -\Delta v + \nu v = |v|^{q-2}v + \lambda u, & x \in \mathbb{R}^N, \\ u, v \in H^1\left(\mathbb{R}^N\right), & N \geq 3, \end{cases} \tag{15}$$

where $2 < q < 2^*$, μ, ν, and λ are positive parameters satisfying $0 < \lambda < \sqrt{\mu\nu}$ and f satisfies (f_1)-(f_4). The energy functional of system (15) is

$$I_q(u, v) = \frac{1}{2}\|u, v\|_H^2 - \mu\int_{\mathbb{R}^N} F(u)dx - \frac{1}{q}\int_{\mathbb{R}^N}|v|^q dx - \lambda\int_{\mathbb{R}^N} uv dx. \tag{16}$$

Define

$$\mathscr{P}_q = \left\{(u, v) \in H \setminus \{(0,0)\} : J_q(u, v) = 0\right\} \text{ and } m_q = \inf_{(u,v)\in\mathscr{P}_q} I_q(u, v), \tag{17}$$

where

$$J_q(u, v) = \frac{N-2}{2}\int_{\mathbb{R}^N} \left(|\nabla u|^2 + |\nabla v|^2\right)dx + \frac{N}{2}\int_{\mathbb{R}^N}\left(\mu u^2 + \nu v^2\right)dx - \mu N\int_{\mathbb{R}^N} F(u)dx - \frac{N}{q}\int_{\mathbb{R}^N}|v|^q dx - \lambda N\int_{\mathbb{R}^N} uv dx. \tag{18}$$

3. Proof of Theorem 1

The following two lemmas will be used in proof.

Lemma 5 (compactness lemma of Strauss, see [9, 10]). *Let $P, Q : \mathbb{R} \longrightarrow \mathbb{R}$ be two continuous functions satisfying*

$$\frac{P(s)}{Q(s)} \longrightarrow 0 \quad \text{as } |s| \longrightarrow +\infty. \tag{19}$$

Let u_n be a sequence of measurable functions: $\mathbb{R}^N \longrightarrow \mathbb{R}$ such that

$$\sup_n \int_{\mathbb{R}^N} |Q(u_n(x))|dx < +\infty \tag{20}$$

and $P(u_n(x)) \longrightarrow \upsilon(x)$ a.e. in $\mathbb{R}^N$, as $n \longrightarrow \infty$. Then, for any bounded Borel set B, one has

$$\int_B |P(u_n(x)) - \upsilon(x)|dx \longrightarrow 0 \quad \text{as } n \longrightarrow +\infty. \tag{21}$$

If one further assumes that

$$\frac{P(s)}{Q(s)} \longrightarrow 0 \quad \text{as } |s| \longrightarrow 0 \tag{22}$$

and $u_n(x) \longrightarrow 0$ as $|x| \longrightarrow +\infty$, uniformly with respect to n, then $P(u_n)$ converges to υ in $L^1(\mathbb{R}^N)$ as $n \longrightarrow +\infty$.

Lemma 6 (Strauss inequality, see [17]). *If $N \geq 2$, there exists $C_N > 0$ such that, for every $u(x) = u(|x|) \in H^1(\mathbb{R}^N)$,*

$$|u(x)| \leq C_N |u|_2^{1/2} |\nabla u|_2^{1/2} |x|^{(1-N)/2} \tag{23}$$

a.e. on $\mathbb{R}^N$.

Before proving Theorem 1, we need to prove a series of lemmas.

Lemma 7. *Suppose that (f_1)-(f_4) hold. Then, the Pohožaev manifold $\mathscr{P}$ is not empty.*

Proof. From [17], we know that for any $\varepsilon > 0$,

$$u_\varepsilon = \frac{[N(N-2)]^{(N-2)/4}\varepsilon^{(N-2)/2}}{\left(\varepsilon + |x|^2\right)^{(N-2)/4}} \tag{24}$$

is a positive solution of the following equation:

$$-\Delta u = |u|^{2^*-2}u, \quad x \in \mathbb{R}^N, N \geq 3. \tag{25}$$

Define a cut-off function $\phi \in C_0^\infty(\mathbb{R}^N, [0,1])$ as

$$\phi = \begin{cases} 1, & x \in B_\rho, \\ 0, & x \in \mathbb{R}^N \setminus B_{2\rho}, \end{cases} \tag{26}$$

where $\varrho > 0$ and $B_\varrho = \{x \in \mathbb{R}^N, |x| < \varrho\}$. Let $W_\varepsilon = \phi u_\varepsilon$ and define $V_\varepsilon = W_\varepsilon/(\int_{\mathbb{R}^N}|W_\varepsilon|^{2^*}dx)^{1/2^*}$. By [16], we have

$$\left(\int_{\mathbb{R}^N}|V_\varepsilon|^{2^*}dx\right)^{1/2^*} = 1,$$
$$\int_{\mathbb{R}^N}|V_\varepsilon|^2dx = \begin{cases} o\left(\varepsilon^{1/2}\right), & N=3, \\ o(\varepsilon|\ln\varepsilon|), & N=4, \\ o(\varepsilon), & N=5. \end{cases} \tag{27}$$

Take $\varepsilon>0$ small enough such that $\int_{\mathbb{R}^N}((1/2^*)|V_\varepsilon|^{2^*} - (\nu/2)V_\varepsilon^2)dx>0$. Let $U\in H^1(\mathbb{R}^N)$ be a positive ground state solution of equation (3). Then, we have the following Pohožaev equality:

$$\frac{N-2}{2}\int_{\mathbb{R}^N}|\nabla U|^2dx + \frac{N}{2}\int_{\mathbb{R}^N}U^2dx = N\int_{\mathbb{R}^N}F(U)dx. \tag{28}$$

Then, $\int_{\mathbb{R}^N}(F(U)-(1/2)U^2)dx>0$. Thus, we have

$$\begin{aligned}\tau(t) &:= I\left(U_t,(V_\varepsilon)_t\right) = \frac{t^{N-2}}{2}\int_{\mathbb{R}^N}\left(|\nabla U|^2+|\nabla V_\varepsilon|^2\right)dx \\ &\quad - \mu t^N\int_{\mathbb{R}^N}\left(F(U)-\frac{1}{2}U^2\right)dx \\ &\quad - t^N\int_{\mathbb{R}^N}\left(\frac{1}{2^*}|V_\varepsilon|^{2^*}-\frac{\nu}{2}V_\varepsilon^2\right)dx - \lambda t^N\int_{\mathbb{R}^N}UV_\varepsilon dx.\end{aligned} \tag{29}$$

Define $s=t^N$; we have

$$\begin{aligned}\eta(s) &:= \tau\left(s^{1/N}\right) = \frac{s^{(N-2)/N}}{2}\int_{\mathbb{R}^N}\left(|\nabla U|^2+|\nabla V_\varepsilon|^2\right)dx \\ &\quad - \mu s\int_{\mathbb{R}^N}\left(F(U)-\frac{1}{2}U^2\right)dx \\ &\quad - s\int_{\mathbb{R}^N}\left(\frac{1}{2^*}|V_\varepsilon|^{2^*}-\frac{\nu}{2}V_\varepsilon^2\right)dx - \lambda s\int_{\mathbb{R}^N}UV_\varepsilon dx.\end{aligned} \tag{30}$$

We can easily know that $\eta(s)>0$ for s small enough and $\eta(s)<0$ for large s. Since $(d^2\eta(s))/ds^2<0$, $\eta(s)$ is a concave function. Then, there exists a unique $s_0>0$ such that $\eta'(s_0)=0$. Hence, there exists a unique $t_0=s_0^{1/N}>0$ such that $\tau'(t_0)=0$. Then, we have $t_0\tau'(t_0)=J(U(x/t_0), V_\varepsilon(x/t_0))=0$. Then, $(U_{t_0},(V_\varepsilon)_{t_0})\in\mathscr{P}$. □

Lemma 8. *Suppose that (f_1)-(f_4) hold. Then,* $m=\inf_{(u,v)\in\mathscr{P}} I(u,v)>0$.

Proof. Since $0<\lambda<\sqrt{\mu\nu}$, there exists $0<\theta<1$ such that $0<\lambda<\sqrt{\mu(1-\theta)\nu}$. For any $(u,v)\in\mathscr{P}$, we have $J(u,v)=0$. By using Young's inequality, we have

$$\begin{aligned}&\frac{N-2}{2}\int_{\mathbb{R}^N}\left(|\nabla u|^2+|\nabla v|^2\right)dx + \frac{N}{2}\int_{\mathbb{R}^N}\left(\mu u^2+\nu v^2\right)dx \\ &\quad = \mu N\int_{\mathbb{R}^N}F(u)dx + \frac{N}{2^*}\int_{\mathbb{R}^N}|v|^{2^*}dx + \lambda N\int_{\mathbb{R}^N}uvdx \\ &\quad \le \mu N\frac{\theta}{2}\int_{\mathbb{R}^N}u^2dx + NC\int_{\mathbb{R}^N}|u|^{2^*}dx + \frac{N}{2^*}\int_{\mathbb{R}^N}|v|^{2^*}dx \\ &\qquad + \lambda N\int_{\mathbb{R}^N}uvdx \le \mu N\frac{\theta}{2}\int_{\mathbb{R}^N}u^2dx + NC\int_{\mathbb{R}^N}|u|^{2^*}dx \\ &\qquad + \frac{N}{2^*}\int_{\mathbb{R}^N}|v|^{2^*}dx + N\frac{\mu(1-\theta)}{2}\int_{\mathbb{R}^N}u^2dx + \frac{N\nu}{2}\int_{\mathbb{R}^N}v^2dx.\end{aligned} \tag{31}$$

Therefore, we have

$$\frac{N-2}{2}\int_{\mathbb{R}^N}\left(|\nabla u|^2+|\nabla v|^2\right)dx \le NC\int_{\mathbb{R}^N}|u|^{2^*}dx + \frac{N}{2^*}\int_{\mathbb{R}^N}|v|^{2^*}dx. \tag{32}$$

By using Sobolev's inequality, we have

$$\begin{aligned}&\frac{N-2}{2}\int_{\mathbb{R}^N}\left(|\nabla u|^2+|\nabla v|^2\right)dx \\ &\quad \le C_1\left[\left(\int_{\mathbb{R}^N}|\nabla u|^2dx\right)^{2^*/2} + \left(\int_{\mathbb{R}^N}|\nabla v|^2dx\right)^{2^*/2}\right] \\ &\quad \le C_2\left[\int_{\mathbb{R}^N}\left(|\nabla u|^2+|\nabla v|^2\right)dx\right]^{2^*/2},\end{aligned} \tag{33}$$

which implies $\int_{\mathbb{R}^N}(|\nabla u|^2+|\nabla v|^2)dx \ge ((N-2)/2C_2)^{(N-2)/2}>0$. Therefore, we conclude that for any $(u,v)\in\mathscr{P}$, we have

$$\begin{aligned}I(u,v) &= I(u,v)-\frac{1}{N}J(u,v) = \left(\frac{1}{2}-\frac{1}{2^*}\right)\int_{\mathbb{R}^N}|\nabla u|^2dx \\ &\quad + \left(\frac{1}{2}-\frac{1}{2^*}\right)\int_{\mathbb{R}^N}|\nabla v|^2dx \ge \frac{1}{N}\left(\frac{N-2}{2C_2}\right)^{(N-2)/2}.\end{aligned} \tag{34}$$

Therefore, we have $m>0$. □

Lemma 9. *Suppose that (f_1)-(f_4) hold. Then,* $m<(1/N)S^{N/2}$.

Proof. Let $U\in H^1(\mathbb{R}^N)$ be a positive ground state solution of equation (3). Then, (28) holds and

$$\begin{aligned}a &= \frac{1}{2}\int_{\mathbb{R}^N}\left(|\nabla U|^2+U^2\right)dx - \int_{\mathbb{R}^N}F(U)dx \\ &= \frac{1}{2}\int_{\mathbb{R}^N}\left(|\nabla U|^2+U^2\right)dx - \int_{\mathbb{R}^N}F(U)dx \\ &\quad - \frac{1}{N}\left(\frac{N-2}{2}\int_{\mathbb{R}^N}|\nabla U|^2dx + \frac{N}{2}\int_{\mathbb{R}^N}U^2dx - N\int_{\mathbb{R}^N}F(U)dx\right) \\ &= \int_{\mathbb{R}^N}|\nabla U|^2dx.\end{aligned} \tag{35}$$

Moreover, we have also $U(\sqrt{\mu}x)$ which is a solution of equation

$$-\Delta u + \mu u = \mu f(u), \quad u \in H^1(\mathbb{R}^N), N \geq 3. \tag{36}$$

Then, $(U(\sqrt{\mu}x), 0) \in \mathscr{P}$. Since $\mu > S^{-N/(N-2)}(\mathrm{aN})^{2/(N-2)}$, we have

$$\begin{aligned} m &\leq I(U(\sqrt{\mu}x), 0) = I(U(\sqrt{\mu}x), 0) - \frac{1}{N} J(U(\sqrt{\mu}x), 0) \\ &= \frac{1}{N}\int_{\mathbb{R}^N} |\nabla U(\sqrt{\mu}x)|^2 dx = a\mu^{(2-N)/2} < \frac{1}{N} S^{N/2}. \end{aligned} \tag{37}$$

□

Lemma 10. *Suppose that (f1)-(f4) hold. For any $(u_n, v_n) \subset \mathscr{P}$, if $I(u_n, v_n) \leq C$, then (u_n, v_n) is bounded in H.*

Proof. Since $I(u_n, v_n) \leq C$, we have

$$\begin{aligned} C \geq I(u_n, v_n) &= I(u_n, v_n) - \frac{1}{N} J(u_n, v_n) \\ &= \frac{1}{N}\int_{\mathbb{R}^N} \left(|\nabla u_n|^2 + |\nabla v_n|^2\right) dx. \end{aligned} \tag{38}$$

Because $0 < \lambda < \sqrt{\mu\nu}$, there exists $0 < \theta < 1/2$ and $\alpha > 0$ such that $0 < \lambda < \sqrt{\mu(1-2\theta)(\nu-\alpha)}$. Therefore, we have

$$\begin{aligned} &\frac{N-2}{2}\int_{\mathbb{R}^N} \left(|\nabla u_n|^2 + |\nabla v_n|^2\right) dx + \frac{N}{2}\int_{\mathbb{R}^N} \left(\mu u_n^2 + \nu v_n^2\right) dx \\ &\quad = \mu N \int_{\mathbb{R}^N} F(u_n) dx + \frac{N}{2^*}\int_{\mathbb{R}^N} |v_n|^{2^*} dx + \lambda N \int_{\mathbb{R}^N} u_n v_n dx \\ &\quad \leq \mu N \frac{\theta}{2}\int_{\mathbb{R}^N} |u_n|^2 dx + NC\int_{\mathbb{R}^N} |u_n|^{2^*} dx + \frac{N}{2^*}\int_{\mathbb{R}^N} |v_n|^{2^*} dx \\ &\qquad + \lambda N \int_{\mathbb{R}^N} u_n v_n dx \leq \mu N \frac{\theta}{2}\int_{\mathbb{R}^N} |u_n|^2 dx \\ &\qquad + NC\int_{\mathbb{R}^N} |u_n|^{2^*} dx + \frac{N}{2^*}\int_{\mathbb{R}^N} |v_n|^{2^*} dx \\ &\qquad + N\frac{\mu(1-2\theta)}{2}\int_{\mathbb{R}^N} u_n^2 dx + \frac{N(\nu-\alpha)}{2}\int_{\mathbb{R}^N} v_n^2 dx. \end{aligned} \tag{39}$$

Then, we have

$$\begin{aligned} &\frac{N\mu\theta}{2}\int_{\mathbb{R}^N} |u_n|^2 dx + \frac{N\alpha}{2}\int_{\mathbb{R}^N} |v_n|^2 dx \leq CN\int_{\mathbb{R}^N} |u_n|^{2^*} dx \\ &\quad + \frac{N}{2^*}\int_{\mathbb{R}^N} |v_n|^{2^*} dx \leq C_3\left[\int_{\mathbb{R}^N} \left(|\nabla u|^2 + |\nabla v|^2\right) dx\right]^{2^*/2} \leq C_4. \end{aligned} \tag{40}$$

Hence, (u_n, v_n) is bounded in H. □

Lemma 11. *Suppose that (f1)-(f4) hold. Then, $\lim_{q \longrightarrow 2^{*-}} \sup m_q \leq m$.*

Proof. For any $\varepsilon \in (0, 1/2)$, there exists $(u, v) \in \mathscr{P}$ such that $I(u, v) < m + \varepsilon$. Since $J(u, v) = 0$, for any $t > 0$, we have

$$\begin{aligned} I(u_t, v_t) &= I(u_t, v_t) - \frac{t^N}{N} J(u, v) \\ &= \left(\frac{t^{N-2}}{2} - \frac{N-2}{2N} t^N\right)\int_{\mathbb{R}^N} \left(|\nabla u|^2 + |\nabla v|^2\right) dx. \end{aligned} \tag{41}$$

Define $h(t) = t^{N-2}/2 - ((N-2)/2N)t^N$. Through simple calculations, we have $h'(t) = (N-2)/2(t^{N-3} - t^{N-1})$. We can easily see that h is increasing for $t \in (0, 1)$ and h is decreasing for $t > 1$. Then, we have $\max_{t>0} I(u_t, v_t) = I(u, v)$ and $I(u_t, v_t) < I(u, v)$ for any $t \neq 1$. By calculation, we have $I(u_t, v_t) < 0$ for $t > \sqrt{N/(N-2)}$. Take large T such that

$$\begin{aligned} I(u_T, v_T) &= \frac{T^{N-2}}{2}\int_{\mathbb{R}^N} \left(|\nabla u|^2 + |\nabla v|^2\right) dx \\ &\quad + \frac{T^N}{2}\int_{\mathbb{R}^N} \left(\mu u^2 + \nu v^2\right) dx - \mu T^N \int_{\mathbb{R}^N} F(u) dx \\ &\quad - \frac{T^N}{2^*}\int_{\mathbb{R}^N} |v|^{2^*} dx - \lambda T^N \int_{\mathbb{R}^N} uv dx \leq -1. \end{aligned} \tag{42}$$

Then, there exists $\sigma \in (0, 2^*)$ such that

$$\left|I_q(u_t, v_t) - I(u_t, v_t)\right| = \left|\frac{t^N}{2^*}\int_{\mathbb{R}^N} |v|^{2^*} dx - \frac{t^N}{q}\int_{\mathbb{R}^N} |v|^q dx\right| < \varepsilon, \tag{43}$$

for all $2^* - \sigma < q < 2^*$ and $0 \leq t \leq T$. Then, we have $I_q(u_T, v_T) \leq -(1/2)$ for all $2^* - \sigma < q < 2^*$. Since

$$\begin{aligned} I_q(u_t, v_t) &= \frac{t^{N-2}}{2}\int_{\mathbb{R}^N} \left(|\nabla u|^2 + |\nabla v|^2\right) dx \\ &\quad + \frac{t^N}{2}\int_{\mathbb{R}^N} \left(\mu u^2 + \nu v^2\right) dx - \mu t^N \int_{\mathbb{R}^N} F(u) dx \\ &\quad - \frac{t^N}{q}\int_{\mathbb{R}^N} |v|^q dx - \lambda t^N \int_{\mathbb{R}^N} uv dx, \end{aligned} \tag{44}$$

$I_q(u_t, v_t) > 0$ for t small enough. Then, there exists $t_q \in (0, T)$ such that $(d/dt)I_q(u_t, v_t)|_{t=t_q} = 0$. So, $(u_{t_q}, v_{t_q}) \in \mathscr{P}_q$. Hence, we have

$$m_q \leq I_q\left(u_{t_q}, v_{t_q}\right) \leq I\left(u_{t_q}, v_{t_q}\right) + \varepsilon \leq I(u, v) + \varepsilon < m + 2\varepsilon, \tag{45}$$

for all $2^* - \sigma < q < 2^*$.

From [18, 19], we know that system (15) has a positive and radial ground state solution. Then, for any $q_n \in (2, 2^*)$ and $q_n \longrightarrow 2^{*-}$, there exists a positive and radial sequence $\{(u_n, v_n)\} \subset H$ such that

$$\begin{aligned} &I_{q_n}(u_n, v_n) = m_{q_n}, \\ &I'_{q_n}(u_n, v_n) = 0, \\ &J_{q_n}(u_n, v_n) = 0. \end{aligned} \tag{46}$$

By Lemmas 10 and 11, we know that $\{(u_n, v_n)\}$ is bounded in H. □

Lemma 12. *Suppose that (f1)-(f4) and (46) hold. Then,* $\liminf_{n\longrightarrow\infty} m_{q_n} > 0$.

Proof. Similar to the proof of Lemma 8, we have

$$\begin{aligned} \frac{N-2}{2}\int_{\mathbb{R}^N}\left(|\nabla u_n|^2 + |\nabla v_n|^2\right)dx \le NC\int_{\mathbb{R}^N}|u_n|^{2^*}dx \\ + \frac{N}{q_n}\int_{\mathbb{R}^N}|v_n|^{q_n}dx. \end{aligned} \tag{47}$$

Using Young's inequality implies

$$\begin{aligned} \frac{N}{q_n}\int_{\mathbb{R}^N}|v_n|^{q_n}dx &= \frac{N}{q_n}\int_{\mathbb{R}^N}|v_n|^{(2(2^*-q_n))/(2^*-2)}|v_n|^{(2^*(q_n-2))/(2^*-2)}dx \\ &\le \frac{N}{q_n}\frac{2^*-q_n}{2^*-2}\int_{\mathbb{R}^N}|v_n|^2dx + \frac{N}{q_n}\frac{q_n-2}{2^*-2}\int_{\mathbb{R}^N}|v_n|^{2^*}dx \\ &= \frac{N}{2^*}\int_{\mathbb{R}^N}|v_n|^{2^*}dx + o(1). \end{aligned} \tag{48}$$

Then,

$$\begin{aligned} \frac{N-2}{2}\int_{\mathbb{R}^N}\left(|\nabla u_n|^2 + |\nabla v_n|^2\right)dx \le NC\int_{\mathbb{R}^N}|u_n|^{2^*}dx \\ + \frac{N}{2^*}\int_{\mathbb{R}^N}|v_n|^{2^*}dx + o(1) \le C_5\left[\int_{\mathbb{R}^N}\left(|\nabla u_n|^2 + |\nabla v_n|^2\right)dx\right]^{2^*/2} + o(1). \end{aligned} \tag{49}$$

So there exists $\varpi > 0$ such that up to a subsequence, $\int_{\mathbb{R}^N}(|\nabla u_n|^2 + |\nabla v_n|^2)dx + o(1) \ge \varpi$. On the other hand,

$$\begin{aligned} m_{q_n} &= I_{q_n}(u_n, v_n) = I_{q_n}(u_n, v_n) - \frac{1}{N}J_{q_n}(u_n, v_n) \\ &= \frac{1}{N}\int_{\mathbb{R}^N}\left(|\nabla u_n|^2 + |\nabla v_n|^2\right)dx. \end{aligned} \tag{50}$$

Then, $\liminf_{n\longrightarrow\infty} m_{q_n} > 0$. □

Proof of Theorem 1. Because (46) holds, there exists $(u, v) \in H$ such that $u_n u, v_n v$ in $H^1(\mathbb{R}^N)$, $u_n \longrightarrow u, v_n \longrightarrow v$ in $L^p(\mathbb{R}^N), 2 < p < 2^*$, and $u_n(x) \longrightarrow u(x), v_n(x) \longrightarrow v(x)$ a.e. in $\mathbb{R}^N$. For any $(\varphi, \psi) \in H$, we have

$$0 = \left\langle I'_{q_n}(u_n, v_n), (\varphi, \psi)\right\rangle \longrightarrow \left\langle I'(u, v), (\varphi, \psi)\right\rangle, \tag{51}$$

i.e., (u, v) is a solution of system (1). Suppose that $u = 0$. Set $P(s) = f(s)s$ and $Q(s) = |s|^2 + |s|^{2^*}$. Through Lemma 5 and Lemma 6, we have $\int_{\mathbb{R}^N}P(u_n)dx \longrightarrow 0$ as $n \longrightarrow +\infty$. Since $\langle I'_{q_n}(u_n, v_n), (u_n, v_n)\rangle = 0$, by using Young's inequality, we have

$$\begin{aligned} \|u_n, v_n\|_H &= \mu\int_{\mathbb{R}^N}f(u_n)u_n dx + \int_{\mathbb{R}^N}|v_n|^{q_n}dx + 2\lambda\int_{\mathbb{R}^N}u_n v_n dx \\ &\le \mu\int_{\mathbb{R}^N}P(u_n)dx + \int_{\mathbb{R}^N}|v_n|^{(2(2^*-q_n))/(2^*-2)}|v_n|^{(2^*(q_n-2))/(2^*-2)}dx \\ &\quad + \int_{\mathbb{R}^N}\left(\mu u_n^2 + \nu v_n^2\right)dx \le \mu\int_{\mathbb{R}^N}P(u_n)dx + \frac{2^*-q_n}{2^*-2}\int_{\mathbb{R}^N}|v_n|^2dx \\ &\quad + \frac{q_n-2}{2^*-2}\int_{\mathbb{R}^N}|v_n|^{2^*}dx + \int_{\mathbb{R}^N}\left(\mu u_n^2 + \nu v_n^2\right)dx = \int_{\mathbb{R}^N}|v_n|^{2^*}dx \\ &\quad + \int_{\mathbb{R}^N}\left(\mu u_n^2 + \nu v_n^2\right)dx + o(1). \end{aligned} \tag{52}$$

One has

$$\begin{aligned} \int_{\mathbb{R}^N}|\nabla v_n|^2dx \le \int_{\mathbb{R}^N}\left(|\nabla u_n|^2 + |\nabla v_n|^2\right)dx \le \int_{\mathbb{R}^N}|v_n|^{2^*}dx + o(1) \\ \le \left(\frac{\int_{\mathbb{R}^N}|\nabla v_n|^2dx}{S}\right)^{2^*/2} + o(1). \end{aligned} \tag{53}$$

So we have (i) $\int_{\mathbb{R}^N}|\nabla v_n|^2dx \longrightarrow 0$ or (ii) $\limsup_{n\longrightarrow\infty}\int_{\mathbb{R}^N}|\nabla v_n|^2dx \ge S^{N/2}$. If (i) holds, then we have

$$\begin{aligned} m_{q_n} &= I_{q_n}(u_n, v_n) - \frac{1}{N}J_{q_n}(u_n, v_n) \\ &= \frac{1}{N}\int_{\mathbb{R}^N}\left(|\nabla u_n|^2 + |\nabla v_n|^2\right)dx \longrightarrow 0, \end{aligned} \tag{54}$$

which contradicts with Lemma 12. If (ii) holds, then we have

$$\begin{aligned} m &\ge \limsup_{n\longrightarrow\infty} m_{q_n} = \limsup_{n\longrightarrow\infty}\left[I_{q_n}(u_n, v_n) - \frac{1}{N}J_{q_n}(u_n, v_n)\right] \\ &= \limsup_{n\longrightarrow\infty}\left[\frac{1}{N}\int_{\mathbb{R}^N}\left(|\nabla u_n|^2 + |\nabla v_n|^2\right)dx\right] \\ &\ge \limsup_{n\longrightarrow\infty}\frac{1}{N}\int_{\mathbb{R}^N}|\nabla v_n|^2dx \ge \frac{1}{N}S^{N/2}. \end{aligned} \tag{55}$$

This is a contradiction. So $u \ne 0$ and through Remark 3, we know that $v \ne 0$. Applying the weak lower-semicontinuity of the norm, we have

$$m \leq I(u, v) = I(u, v) - \frac{1}{N} J(u, v) = \frac{1}{N} \int_{\mathbb{R}^N} \left(|\nabla u|^2 + |\nabla v|^2 \right) dx$$
$$\leq \liminf_{n \longrightarrow \infty} \frac{1}{N} \int_{\mathbb{R}^N} \left(|\nabla u_n|^2 + |\nabla v_n|^2 \right) dx$$
$$= \liminf_{n \longrightarrow \infty} \left[I_{q_n}(u_n, v_n) - \frac{1}{N} J_{q_n}(u_n, v_n) \right] = \liminf_{n \longrightarrow \infty} m_{q_n} \leq m. \quad (56)$$

This implies $I(u, v) = m$. We complete the proof. □

Data Availability

The findings in this research do not make use of data.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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References

[1] N. Akhmediev and A. Ankiewicz, "Novel soliton states and bifurcation phenomena in nonlinear fiber couplers," *Physical Review Letters*, vol. 70, no. 16, pp. 2395–2398, 1993.

[2] A. Ambrosetti, E. Colorado, and D. Ruiz, "Multi-bump solitons to linearly coupled systems of nonlinear Schrödinger equations," *Calculus of Variations and Partial Differential Equations*, vol. 30, no. 1, pp. 85–112, 2007.

[3] A. Ambrosetti, G. Cerami, and D. Ruiz, "Solitons of linearly coupled systems of semilinear non-autonomous equations on $\mathbb{R}^N$," *Journal of Functional Analysis*, vol. 254, no. 11, pp. 2816–2845, 2008.

[4] H. D. Liu and Z. L. Liu, "A coupled Schrödinger system with critical exponent," *Calculus of Variations and Partial Differential Equations*, vol. 59, no. 5, p. 28, 2020.

[5] B. Noris and M. Ramos, "Existence and bounds of positive solutions for a nonlinear Schrödinger system," *Proceedings of the American Mathematical Society*, vol. 138, no. 5, pp. 1681–1692, 2010.

[6] Z. Chen and W. Zou, "Ground states for a system of Schrodinger equations with critical exponent," *Journal of Functional Analysis*, vol. 262, no. 7, pp. 3091–3107, 2012.

[7] G. Li and X. H. Tang, "Nehari-type ground state solutions for Schrodinger equations including critical exponent," *Applied Mathematics Letters*, vol. 37, pp. 101–106, 2014.

[8] A. Bahri and P. L. Lions, "On the existence of a positive solution of semilinear elliptic equations in unbounded domains," *Annales de l'Institut Henri Poincaré C, Analyse non linéaire*, vol. 14, no. 3, pp. 365–413, 1997.

[9] H. Berestycki and P. L. Lions, "Nonlinear scalar field equations. I existence of a ground state," *Archive for Rational Mechanics and Analysis*, vol. 82, no. 4, pp. 313–345, 1983.

[10] W. A. Strauss, "Existence of solitary waves in higher dimensions," *Communications in Mathematical Physics*, vol. 55, no. 2, pp. 149–162, 1977.

[11] W. Xie, H. Chen, and T.-F. Wu, "Ground state solutions for a class of Schrödinger-Poisson systems with Hartree-type nonlinearity," *Applicable Analysis*, vol. 100, no. 13, pp. 2777–2803, 2021.

[12] V. S. Guliyev, M. N. Omarova, M. A. Ragusa, and A. Scapellato, "Regularity of solutions of elliptic equations in divergence form in modified local generalized Morrey spaces," *Analysis and Mathematical Physics*, vol. 11, no. 1, p. 20, 2021.

[13] N. S. Papageorgiou and A. Scapellato, "Multiple solutions for Robin (p, q)-equations plus an indefinite potential and a reaction concave near the origin," *Analysis and Mathematical Physics*, vol. 11, no. 2, p. 27, 2021.

[14] N. S. Papageorgiou and A. Scapellato, "Nonlinear singular problems with convection," *Journal of Differential Equations*, vol. 296, pp. 493–511, 2021.

[15] C. Keller, "Large-time asymptotic behavior of solutions of nonlinear wave equations perturbed from a stationary ground state," *Communications in Partial Differential Equations*, vol. 8, no. 10, pp. 1073–1099, 1983.

[16] J. Liu, J.-F. Liao, and C.-L. Tang, "Ground state solution for a class of Schrödinger equations involving general critical growth term," *Nonlinearity*, vol. 30, no. 3, pp. 899–911, 2017.

[17] M. Willem, "Minimax theorems," in *Progress in Nonlinear Differential Equations and their Applications, vol. 24*, Birkhäuser Boston, Inc., Boston, MA, USA, 1996.

[18] H. Brezis and E. H. Lieb, "Minimum action solutions of some vector field equations," *Communications in Mathematical Physics*, vol. 96, no. 1, pp. 97–113, 1984.

[19] Z. J. Chen and W. M. Zou, "On coupled systems of Schrödinger equations," *Advances in Differential Equations*, vol. 16, pp. 775–800, 2011.