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Research Article

On Some Types of Multigranulation Covering Based on Binary Relations

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Recently, the notions of right and left covering rough sets were constructed by right and left neighborhoods to propose four types of multigranulation covering rough set (MGCRS) models. These models were constructed using the granulations as equivalence relations. In this paper, we introduce four types of multigranulation covering rough set models under arbitrary relations using the q-minimal and q-maximal descriptors of objects in a given universe. We also study the properties of these new models. Thus, we explore the relationships between these models. Then, we put forward an algorithm to illustrate the method of reduction based on the presented model. Finally, we give an illustrative example to show its efficiency and importance.

1. Introduction

The notion of rough set theory originated by Pawlak in 1982 [1, 2] to deal with uncertain information and knowledge. It is a tool concerned with the approximation of sets described by a single binary relation. In the view of granular computing suggested by Zadeh [3], a general concept described by a set is characterized via the upper and lower approximations under a single granulation (always equivalence relation) on the universe. This tool has been widely used in many subjects including machine learning, data mining, decision support, and analysis. In the past 20 years, many authors have proposed several extensions of the rough set model [4–19]. In some cases, it is important to use multiequivalence relations on the universe to describe precisely a target concept. Recently, more attention is given to multigranulation rough set (MGRS) models and, also, to multigranulation covering rough set (MGCRS) models in which a target concept is approximated by employing the maximal or minimal descriptors of objects in the given universe. In [20, 21], Qian et al. developed a multigranulation rough set (MGRS) model by using equivalence relations. Several scholars worked on MGRS such as the MGRS model through multiple tolerance relations in incomplete information systems, MGRS via the fuzzy approximation space, the hierarchical structures of MGRS, the topological and lattice-theoretic properties of MGRS, and the efficient rough feature selection algorithm with MGRS [22–30]. Moreover, Liu and Miao and Liu and Wang [31, 32] introduced the multigranulation covering via rough set (MGCRS) and fuzzy rough set (MGCFRS). Lin et al. studied two types of the neighborhood via MGRS [33] and three new types of MGCRS [34]. Also, three types of MGRS via the tolerance, ordered, and generalized relations are investigated and developed the multigranulation decision-theoretic rough set [35–38]. In addition, Liu et al. [39] proposed four new types of MGCRS using the minimal and maximal descriptions and discussed relevant characteristics. For more details about MGRS, see, for instance, [40–44].

The notions of left and right covering rough sets proposed by Abd El-Monsef et al. [45] are important tool to make an extension of Liu et al. [39]. The objective of this paper is to develop new models of MGCRS using the notions of left and right covering using the concepts of q-minimal and q-maximal descriptions. Also, we discuss the properties of these models. The relationships between these models are studied. Then, we present the reduction method over our proposed work and establish a numerical example to show its performance. The paper consists of six sections and is organized as follows: Section 1 deals with a brief history to the subject. Section 2 includes the preliminary concepts. Section 3 is the main core of the paper and consists of the new models. In Section 4, the properties and differences between the proposed models are introduced. Section 5 explores new criteria to make a reduction with a test example. We end up with conclusion in the last section.

2. Basic Terminologies and Results

This section provides a short survey of some notions used throughout the article.

Definition 1 [26]. Let $\mathbb{Q}$ be an universal set and $\varnothing \neq \mathbb{E} = \{\tilde{\mathbb{E}}_1, \tilde{\mathbb{E}}_2, \cdots, \tilde{\mathbb{E}}_m\} \subseteq \mathbb{Q}$. We call $\mathbb{E}$ as a covering of $\mathbb{Q}$, if $\bigcup_{i=1}^{m} \tilde{\mathbb{E}}_i(w) = \mathbb{Q}$ for any $w \in \mathbb{Q}$. Also, $(\mathbb{Q}, \mathbb{E})$ is called a covering approximation space (briefly, CAS).

Definition 2 [46]. Let $\mathscr{R}$ be a binary relation on an universe set $\mathbb{Q}$, and for every $w \in \mathbb{Q}$, we have the following two classes. Define the after and fore sets as follows:

$$\begin{aligned} w\mathscr{R} &= \{v \in \mathbb{Q} : w\mathscr{R}v\}, \\ \mathscr{R}w &= \{v \in \mathbb{Q} : v\mathscr{R}w\}. \end{aligned} \tag{1}$$

Definition 3 [45]. Let $\mathscr{R}$ be a binary relation on an universe set $\mathbb{Q}$. For each $w \in \mathbb{Q}$, define the right covering $\mathscr{C}_{\mathbf{r}}$ (resp., the left covering $\mathscr{C}_{\mathbf{l}}$) as follows:

$$\begin{aligned} \mathscr{C}_{\mathbf{r}} &= \left\{ w\mathscr{R} : \mathbb{Q} = \bigcup_{w \in \mathbb{Q}} w\mathscr{R} \right\}, \\ \mathscr{C}_{\mathbf{l}} &= \left\{ \mathscr{R}w : \mathbb{Q} = \bigcup_{w \in \mathbb{Q}} \mathscr{R}w \right\}. \end{aligned} \tag{2}$$

Definition 4 [45]. Let $\mathscr{R}$ be a binary relation on an universe set $\mathbb{Q}$ and $\mathbb{E}_q$ be a q-cover of $\mathbb{Q}$, where $q \in \{\mathbf{r}, \mathbf{l}\}$. Then, $(\mathbb{Q}, \mathscr{R}, \mathbb{E}_q)$ is said to be $\mathbb{E}_q$ covering approximation space (briefly, $\mathbb{E}_q$ -CAS).

Definition 5 [45]. Let $(\mathbb{Q}, \mathscr{R}, \mathbb{E}_q)$ be $\mathbb{E}_q$ -CAS. For every $w \in \mathbb{Q}$, define the right neighborhood $\mathbb{N}_{\mathrm{r}}$, the left neighborhood $\mathbb{N}_{\mathrm{l}}$, the intersection neighborhood $\mathbb{N}_{\mathrm{i}}$, and the union neighborhood $\mathbb{N}_{\mathrm{u}}$, respectively, as follows:

$$\begin{aligned} \mathbb{N}_{\mathrm{r}}(w) &= \{\mathbf{C} \in \mathscr{C}_{\mathrm{r}} : w \in \mathbf{C}\}, \\ \mathbb{N}_{\mathrm{l}}(w) &= \{\mathbf{C} \in \mathscr{C}_{\mathrm{l}} : w \in \mathbf{C}\}, \\ \mathbb{N}_{\mathrm{i}}(w) &= \mathbb{N}_{\mathrm{r}}(w) \cap \mathbb{N}_{\mathrm{l}}(w), \\ \mathbb{N}_{\mathrm{u}}(w) &= \mathbb{N}_{\mathrm{r}}(w) \cup \mathbb{N}_{\mathrm{l}}(w). \end{aligned} \tag{3}$$

Definition 6 [45]. Let $(\mathbb{Q}, \mathscr{R}, \mathbb{E}_q)$ be $\mathbb{E}_q$ -CAS and $\forall p \in \{\mathbf{r}, \mathbf{l}, \mathbf{i}, \mathbf{u}\}$ and $\mathscr{Z} \subseteq \mathbb{Q}$. Define the p-lower approximation, p-upper approximation, p-boundary, p-positive, p-negative, and p-accuracy of $\mathscr{Z}$, respectively, as follows:

$$\begin{aligned} \mathbf{L}_p(\mathscr{Z}) &= \{w \in \mathscr{Z} : \mathbb{N}_p(w) \subseteq \mathscr{Z}\}, \\ \mathbf{U}_p(\mathscr{Z}) &= \{w \in \mathscr{Z} : \mathbb{N}_p(w) \cap \mathscr{Z} \neq \varnothing\}, \\ B_p(\mathscr{Z}) &= \mathbf{U}_p(\mathscr{Z}) - \mathbf{L}_p(\mathscr{Z}), \\ \oplus_p(\mathscr{Z}) &= \mathbf{L}_p(\mathscr{Z}), \\ \ominus_p(\mathscr{Z}) &= \mathbb{Q} - \mathbf{U}_p(\mathscr{Z}) \\ A_p(\mathscr{Z}) &= \frac{|\mathbf{L}_p(\mathscr{Z})|}{|\mathbf{U}_p(\mathscr{Z})|}, \quad \text{where } |\mathbf{U}_p(\mathscr{Z})| \neq 0. \end{aligned} \tag{4}$$

Pawlak's [1, 2] rough set properties are given as follows:
$(L_1)\mathbf{L}(\mathscr{Z}) \subseteq \mathscr{Z}$, $(H_1)\mathscr{Z} \subseteq \mathbf{U}(\mathscr{Z})$.
(L_2) $\mathbf{L}(\mathbb{Q}) = \mathbb{Q}$, $(H_2)\mathbf{U}(\varnothing) = \varnothing$.
(L_3) $\mathbf{L}(\varnothing) = \varnothing$, $(H_3)\mathbf{U}(\mathbb{Q}) = \mathbb{Q}$.
(L_4) If $\mathscr{Z}_1 \subseteq \mathscr{Z}_2$, then $\mathbf{L}(\mathscr{Z}_1) \subseteq \mathbf{L}(\mathscr{Z}_2)$, $(H_4)\mathbf{U}(\mathscr{Z}_1) \subseteq \mathbf{U}(\mathscr{Z}_2)$.
(L_5) $\mathbf{L}(\mathscr{Z}_1 \cap \mathscr{Z}_2) = \mathbf{L}(\mathscr{Z}_1) \cap \mathbf{L}(\mathscr{Z}_2)$.$(H_5)\mathbf{U}(\mathscr{Z}_1 \cup \mathscr{Z}_2) = \mathbf{U}(\mathscr{Z}_1) \cup \mathbf{U}(\mathscr{Z}_2)$.
(L_6) $\mathbf{L}(\mathscr{Z}_1 \cup \mathscr{Z}_2) \supseteq \mathbf{L}(\mathscr{Z}_1) \cup \mathbf{L}(\mathscr{Z}_2)$.$(H_6)\mathbf{U}(\mathscr{Z}_1 \cap \mathscr{Z}_2) \subseteq \mathbf{U}(\mathscr{Z}_1) \cap \mathbf{U}(\mathscr{Z}_2)$.
(L_7) $\mathbf{L}(\mathscr{Z}^c) = (\mathbf{U}(\mathscr{Z}))^c$, $(H_7)\mathbf{U}(\mathscr{Z}^c) = (\mathbf{L}(\mathscr{Z}))^c$.
(L_8) $\mathbf{L}(\mathbf{L}(\mathscr{Z})) = \mathbf{L}(\mathscr{Z})$, $(H_8)\mathbf{U}(\mathbf{U}(\mathscr{Z})) = \mathbf{U}(\mathscr{Z})$.
(L_9) $\mathbf{L}(\mathbf{L}(\mathscr{Z}))^c = (\mathbf{L}(\mathscr{Z}))^c$, $(L_9)\mathbf{U}(\mathbf{U}(\mathscr{Z}))^c = (\mathbf{U}(\mathscr{Z}))^c$.

Definition 7 [47]. Let $(\mathbb{Q}, \mathbb{E})$ be a CAS and $\mathscr{Z} \subseteq \mathbb{Q}$. For any $w \in \mathbb{Q}$, define the minimal and maximal descriptions of w, respectively, as follows:

$$\begin{aligned} \mathcal{H}_{\mathbb{E}} &= \{\mathcal{C} \in \mathbb{E} : w \in \mathcal{C} \wedge (\forall \mathcal{S} \in \mathbb{E} \wedge w \in \mathcal{S} \wedge \mathcal{S} \subseteq \mathcal{C} \Rightarrow \mathcal{S} = \mathcal{C})\}, \\ \mathscr{D}_{\mathbb{E}} &= \{\mathscr{C} \in \mathbb{E} : w \in \mathscr{C} \wedge (\forall \mathscr{S} \in \mathbb{E} \wedge w \in \mathscr{S} \wedge \mathscr{S} \supseteq \mathscr{C} \Rightarrow \mathscr{S} = \mathscr{C})\}. \end{aligned} \tag{5}$$

Definition 8 [39]. Let $(\mathbb{Q}, \mathbb{E})$ be MGCAS and $\mathscr{Z} \subseteq \mathbb{Q}$. For any $w \in \mathbb{Q}$, define four types of the lower and upper approximations, respectively, as follows:

$$\mathbf{L}^1_{\sum_{d=1}^{n} \mathbb{E}_d}(\mathcal{Z}) = \{w \in \mathbb{Q} : \cap \mathcal{H}_{\mathbb{E}_1}(w) \subseteq \mathcal{Z} \text{ or } \cap \mathcal{H}_{\mathbb{E}_2}(w) \subseteq \mathcal{Z} \text{ or } \cdots \text{ or } \cap \mathcal{H}_{\mathbb{E}_n}(w) \subseteq \mathcal{Z}\},$$

$$\mathbf{U}^1_{\sum_{d=1}^{n} \mathbb{E}_d}(\mathcal{Z}) = \{w \in \mathbb{Q} : [\cap \mathcal{H}_{\mathbb{E}_1}(w)] \cap \mathcal{Z} \neq \varnothing \text{ and } [\cap \mathcal{H}_{\mathbb{E}_2}(w)] \cap \mathcal{Z} \neq \varnothing \text{ and } \cdots \text{ and } [\cap \mathcal{H}_{\mathbb{E}_n}(w)] \cap \mathcal{Z} \neq \varnothing\},$$

$$\mathbf{L}^2_{\mathbb{E}_d}(\mathcal{Z}) = \{w \in \mathbb{Q} : \cup \mathcal{H}_{\mathbb{E}_1}(w) \subseteq \mathcal{Z} \text{ or } \cup \mathcal{H}_{\mathbb{E}_2}(w) \subseteq \mathcal{Z} \text{ or } \cdots \text{ or } \cup \mathcal{H}_{\mathbb{E}_n}(w) \subseteq \mathcal{Z}\},$$

$$\mathbf{U}^2_{\mathbb{E}_d}(\mathscr{Z}) = \{w \in \mathbb{Q} : [\cup\mathcal{H}_{\mathbb{E}_1}(w)] \cap \mathscr{Z} \neq \varnothing \text{ and } [\cup\mathcal{H}_{\mathbb{E}_2}(w)] \cap \mathscr{Z} \neq \varnothing \text{ and} \cdots \text{and } [\cup\mathcal{H}_{\mathbb{E}_n}(w)] \cap \mathscr{Z} \neq \varnothing\},$$
$$\mathbf{L}^3_{\mathbb{E}_d}(\mathscr{Z}) = \{w \in \mathbb{Q} : \cap\mathcal{D}_{\mathbb{E}_1}(w) \subseteq \mathscr{Z} \text{ or } \cap\mathcal{D}_{\mathbb{E}_2}(w) \subseteq \mathscr{Z} \text{ or} \cdots \text{or } \cap\mathcal{D}_{\mathbb{E}_n}(w) \subseteq \mathscr{Z}\},$$
$$\mathbf{U}^3_{\mathbb{E}_d}(\mathscr{Z}) = \{w \in \mathbb{Q} : [\cap\mathcal{D}_{\mathbb{E}_1}(w)] \cap \mathscr{Z} \neq \varnothing \text{ and } [\cap\mathcal{D}_{\mathbb{E}_2}(w)] \cap \mathscr{Z} \neq \varnothing \text{ and} \cdots \text{and } [\cap\mathcal{D}_{\mathbb{E}_n}(w)] \cap \mathscr{Z} \neq \varnothing\},$$
$$\mathbf{L}^4_{\mathbb{E}_d}(\mathscr{Z}) = \{w \in \mathbb{Q} : \cup\mathcal{D}_{\mathbb{E}_1}(w) \subseteq \mathscr{Z} \text{ or } \cup\mathcal{D}_{\mathbb{E}_2}(w) \subseteq \mathscr{Z} \text{ or} \cdots \text{or } \cup\mathcal{D}_{\mathbb{E}_n}(w) \subseteq \mathscr{Z}\},$$
$$\mathbf{U}^4_{\mathbb{E}_d}(\mathscr{Z}) = \{w \in \mathbb{Q} : [\cup\mathcal{D}_{\mathbb{E}_1}(w)] \cap \mathscr{Z} \neq \varnothing \text{ and } [\cup\mathcal{D}_{\mathbb{E}_2}(w)] \cap \mathscr{Z} \neq \varnothing \text{ and} \cdots \text{and } [\cup\mathcal{D}_{\mathbb{E}_n}(w)] \cap \mathscr{Z} \neq \varnothing\}, \quad (6)$$

If $\mathbf{L}^1_{\mathbb{E}_d}(\mathscr{Z})$ (resp., $\mathbf{L}^2_{\mathbb{E}_d}(\mathscr{Z})$, $\mathbf{L}^3_{\mathbb{E}_d}(\mathscr{Z})$, and $\mathbf{L}^4_{\mathbb{E}_d}(\mathscr{Z})$) $\neq$ $\mathbf{U}^1_{\mathbb{E}_d}(\mathscr{Z})$ (resp., $\mathbf{U}^2_{\mathbb{E}_d}(\mathscr{Z})$, $\mathbf{U}^3_{\mathbb{E}_d}(\mathscr{Z})$, and $\mathbf{U}^4_{\mathbb{E}_d}(\mathscr{Z})$), then $\mathscr{Z}$ is called the first kind of a multigranulation covering rough set (briefly, type 1-MGCRS) (resp., type 2-MGCRS, type 3-MGCRS, and type 4-MGCRS), else it is definable.

Definition 9 [48]. Let $(\mathbb{Q}, \mathbb{E})$ be a covering information system. For any $\mathscr{Z} \subseteq \mathbb{Q}$ and $w \in \mathbb{Q}$, define the first type of optimistic multigranulation covering lower approximation (briefly, 1-OMGCLA) ${}_1\mathbf{L}^O_{\sum_{d=1}^n {}_d\mathbb{E}}(\mathscr{Z})$ and the first type of optimistic multigranulation covering upper approximation (briefly, 1-OMGCUA) ${}_1\mathbf{U}^O_{\sum_{d=1}^n {}_d\mathbb{E}}(\mathscr{Z})$ as follows:

$$ {}_1\mathbf{L}^O_{\sum_{d=1}^n {}_d\mathbb{E}}(\mathscr{Z}) = \{w \in \mathbb{Q} : (w)^1_{\mathbb{E}} \subseteq \mathscr{Z} \vee (w)^2_{\mathbb{E}} \subseteq \mathscr{Z} \vee \cdots \vee (w)^n_{\mathbb{E}} \subseteq \mathscr{Z}\},$$
$$ {}_1\mathbf{L}^O_{\sum_{d=1}^n {}_d\mathbb{E}}(\mathscr{Z}) = \{w \in \mathbb{Q} : (w)^1_{\mathbb{E}} \cap \mathscr{Z} \neq \varnothing \wedge (w)^2_{\mathbb{E}} \cap \mathscr{Z} \neq \varnothing \wedge \cdots \wedge (w)^n_{\mathbb{E}} \cap \mathscr{Z} \neq \varnothing\}. \quad (7)$$

Definition 10 [48]. Let $(\mathbb{Q}, \mathbb{E})$ be a covering information system. For any $\mathscr{Z} \subseteq \mathbb{Q}$ and $w \in \mathbb{Q}$, define the first type of pessimistic multigranulation covering lower approximation (briefly, 1-PMGCLA) ${}_1\mathbf{L}^P_{\sum_{d=1}^n {}_d\mathbb{E}_q}(\mathscr{Z})$ and the first type of pessimistic multigranulation covering upper approximation (briefly, 1-PMGCUA) ${}_1\mathbf{U}^P_{\sum_{d=1}^n {}_d\mathbb{E}_q}(\mathscr{Z})$ as follows:

$$ {}_1\mathbf{L}^P_{\sum_{d=1}^n \mathbb{E}}(\mathscr{Z}) = \{w \in \mathbb{Q} : (w)^1_{\mathbb{E}} \subseteq \mathscr{Z} \wedge (w)^2_{\mathbb{E}} \subseteq \mathscr{Z} \wedge \cdots \wedge (w)^n_{\mathbb{E}} \subseteq \mathscr{Z}\},$$
$$ {}_1\mathbf{U}^P_{\sum_{d=1}^n {}_d\mathbb{E}}(\mathscr{Z}) = \{w \in \mathbb{Q} : (w)^1_{\mathbb{E}} \cap \mathscr{Z} \neq \varnothing \vee (w)^2_{\mathbb{E}} \cap \mathscr{Z} \neq \varnothing \vee \cdots \vee (w)^n_{\mathbb{E}} \cap \mathscr{Z} \neq \varnothing\}. \quad (8)$$

Next, we have the following definitions using the notion of $\mathbb{E}_q$-CAS.

Definition 11. Let $(\mathbb{Q}, \mathscr{R}, \mathbb{E}_q)$ be $\mathbb{E}_q$-CAS and $\mathscr{Z} \subseteq \mathbb{Q}$. For any $w \in \mathbb{Q}$, define the q -minimal and q-maximal descriptions of w, respectively, as follows:

$$\mathcal{H}_{\mathbb{E}_q} = \{\mathcal{C} \in \mathbb{E}_q : w \in \mathcal{C} \wedge (\forall \mathcal{S} \in \mathbb{E}_q \wedge w \in \mathcal{S} \wedge \mathcal{S} \subseteq \mathcal{C} \Rightarrow \mathcal{S} = \mathcal{C})\},$$
$$\mathscr{D}_{\mathbb{E}_q} = \{\mathscr{C} \in \mathbb{E}_q : w \in \mathscr{C} \wedge (\forall \mathcal{S} \in \mathbb{E}_q \wedge w \in \mathcal{S} \wedge \mathcal{S} \supseteq \mathscr{C} \Rightarrow \mathcal{S} = \mathscr{C})\}. \quad (9)$$

We give the following example to illustrate the above definition.

Example 1. Let $(\mathbb{Q}, \mathscr{R}, \mathbb{E}_q)$ be $\mathbb{E}_q$-CAS, $\mathbb{Q} = \{k_1, k_2, k_3, k_4\}$ and $\mathscr{R} = \{(k_1, k_4), (k_2, k_2), (k_2, k_3), (k_3, k_2), (k_4, k_1), (k_4, k_3)\}$. Then, we have the following results:

$$\begin{aligned}
&\mathcal{H}_{\mathbb{E}_r}(k_1) = \{k_1, k_3\}, \mathcal{H}_{\mathbb{E}_r}(k_2) = \{k_2\},\\
&\mathcal{H}_{\mathbb{E}_r}(k_3) = \{\{k_1, k_3\}, \{k_2, k_3\}\}, \mathcal{H}_{\mathbb{E}_r}(k_4) = \{k_4\},\\
&\mathcal{H}_{\mathbb{E}_l}(k_1) = \{k_1\}, \mathcal{H}_{\mathbb{E}_l}(k_2) = \{\{k_2, k_3\}, \{k_2, k_4\}\},\\
&\mathcal{H}_{\mathbb{E}_l}(k_3) = \{\{k_2, k_3\}\}, \mathcal{H}_{\mathbb{E}_l}(k_4) = \{k_4\},\\
&\mathscr{D}_{\mathbb{E}_r}(k_1) = \{k_1, k_3\}, \mathscr{D}_{\mathbb{E}_r}(k_2) = \{k_2, k_3\},\\
&\mathscr{D}_{\mathbb{E}_r}(k_3) = \{\{k_1, k_3\}\{k_2, k_3\}\}, \mathscr{D}_{\mathbb{E}_r}(k_4) = \{k_4\},\\
&\mathscr{D}_{\mathbb{E}_l}(k_1) = \{k_1\}, \mathscr{D}_{\mathbb{E}_l}(k_2) = \{\{k_2, k_3\}, \{k_2, k_4\}\},\\
&\mathscr{D}_{\mathbb{E}_l}(k_3) = \{k_2, k_3\}, \mathscr{D}_{\mathbb{E}_l}(k_4) = \{k_2, k_4\}.
\end{aligned} \quad (10)$$

Definition 12. Let $(\mathbb{Q}, \mathscr{R}, \mathbb{E}_q)$ be $\mathbb{E}_q$ -CAS and $\mathscr{Z} \subseteq \mathbb{Q}$. For any $w \in \mathbb{Q}$, define the lower and upper approximations, respectively, as follows:

$$\begin{aligned}
\mathbf{L}_{\mathbb{E}_q}(\mathscr{Z}) &= \{w \in \mathscr{Z} : (\cap\mathcal{H}_{\mathbb{E}_q}(w)) \subseteq \mathscr{Z}\},\\
\mathbf{U}_{\mathbb{E}_q}(\mathscr{Z}) &= \{w \in \mathbb{Q} : (\cap\mathscr{D}_{\mathbb{E}_q}(w)) \cap \mathscr{Z} \neq \varnothing\}.
\end{aligned} \quad (11)$$

To explain the above definition, we give the following example.

Example 2. Consider Example 1, if $\mathscr{Z} = \{k_1, k_2, k_4\}$, then we have the following results.

$$\begin{aligned}
\mathbf{L}_{\mathbb{E}_r}(\mathscr{Z}) &= \{k_2, k_4\},\\
\mathbf{U}_{\mathbb{E}_r}(\mathscr{Z}) &= \{k_1, k_2, k_4\},\\
\mathbf{L}_{\mathbb{E}_l}(\mathscr{Z}) &= \{k_1, k_2, k_4\},\\
\mathbf{U}_{\mathbb{E}_l}(\mathscr{Z}) &= \mathbb{Q}.
\end{aligned} \quad (12)$$

3. Multi-$\mathbb{E}_q$-Covering Approximation Space

Presume that $\mathbb{Q}$ is an universal set, $\mathfrak{R}$ is a family of binary relations on $\mathbb{Q}$, and $\mathbb{E}_q$ is q-cover of $\mathbb{Q}$ depending on $\mathfrak{R}$, where $q \in \{\mathbf{l}, \mathbf{r}\}$. Thus, $(\mathbb{Q}, \mathfrak{R}, E_q)$ is called a multi-$\mathbb{E}_q$-covering approximation space (briefly, M$\mathbb{E}_q$CAS).

Definition 13. Assume that $(\mathbb{Q}, \mathfrak{R}, \mathbb{E}_q)$ is a M$\mathbb{E}_q$CAS and $\mathfrak{R} = \{\mathcal{R}_1, \mathcal{R}_2, \cdots, \mathcal{R}_S\}, \forall S \in I$, for any $\mathcal{Z} \subseteq \mathbb{Q}$ and $w \in \mathbb{Q}$. Then, we have four novel kinds of lower and upper approximations written as follows:

Style 1

The 1-MCLA ${}_1\mathbf{L}_{\sum_{d=1}^{n} {}_d\mathbb{E}_q}(\mathcal{Z})$ and the 1-MCUA ${}_1\mathbf{L}_{\sum_{d=1}^{n} {}_d\mathbb{E}_q}(\mathcal{Z})$ are shown as follows:

$$\begin{aligned} {}_1\mathbf{L}_{\sum_{d=1}^{n} {}_d\mathbb{E}_q}(\mathcal{Z}) &= \Big\{ w \in \mathbb{Q} : \cap \mathcal{H}_{\mathbb{E}_q}^{\mathcal{R}_1}(w) \subseteq \mathcal{Z} \text{ or } \cap \mathcal{H}_{\mathbb{E}_q}^{\mathcal{R}_2}(w) \subseteq \mathcal{Z} \\ &\quad \text{or} \cdots \text{or} \cap \mathcal{H}_{\mathbb{E}_q}^{\mathcal{R}_n}(w) \subseteq \mathcal{Z} \Big\}, \\ {}_1\mathbf{U}_{\sum_{d=1}^{n} {}_d\mathbb{E}_q}(\mathcal{Z}) &= \Big\{ w \in \mathbb{Q} : \left[\cap \mathcal{H}_{\mathbb{E}_q}^{\mathcal{R}_1}(w)\right] \cap \mathcal{Z} \neq \varnothing \\ &\quad \text{and} \left[\cap \mathcal{H}_{\mathbb{E}_q}^{\mathcal{R}_2}(w)\right] \cap \mathcal{Z} \neq \varnothing \\ &\quad \text{and} \cdots \text{and} \left[\cap \mathcal{H}_{\mathbb{E}_q}^{\mathcal{R}_n}(w)\right] \cap \mathcal{Z} \neq \varnothing \Big\}. \end{aligned} \tag{13}$$

If ${}_1\mathbf{L}_{\sum_{d=1}^{n} {}_d\mathbb{E}_q}(\mathcal{Z}) \neq {}_1\mathbf{U}_{\sum_{d=1}^{n} {}_d\mathbb{E}_q}(\mathcal{Z})$, then $\mathcal{Z}$ is said to be the first kind of q-covering multigranulation rough set (briefly, 1-qMGCRS), else it is definable.

Style 2

The 2-MCLA ${}_2\mathbf{L}_{\sum_{d=1}^{n} {}_d\mathbb{E}_q}(\mathcal{Z})$ and the 2-MCUA ${}_2\mathbf{L}_{\sum_{d=1}^{n} {}_d\mathbb{E}_q}(\mathcal{Z})$ are seen as follows:

$$\begin{aligned} {}_2\mathbf{L}_{\sum_{d=1}^{n} {}_d\mathbb{E}_q}(\mathcal{Z}) &= \Big\{ w \in \mathbb{Q} : \cup \mathcal{H}_{\mathbb{E}_q}^{\mathcal{R}_1}(w) \subseteq \mathcal{Z} \text{ or } \cup \mathcal{H}_{\mathbb{E}_q}^{\mathcal{R}_2}(w) \subseteq \mathcal{Z} \\ &\quad \text{or} \cdots \text{or} \cup \mathcal{H}_{\mathbb{E}_q}^{\mathcal{R}_n}(w) \subseteq \mathcal{Z} \Big\}, \\ {}_2\mathbf{U}_{\sum_{d=1}^{n} {}_d\mathbb{E}_q}(\mathcal{Z}) &= \Big\{ w \in \mathbb{Q} : \left[\cup \mathcal{H}_{\mathbb{E}_q}^{\mathcal{R}_1}(w)\right] \cap \mathcal{Z} \neq \varnothing \\ &\quad \text{and} \left[\cup \mathcal{H}_{\mathbb{E}_q}^{\mathcal{R}_2}(w)\right] \cap \mathcal{Z} \neq \varnothing \\ &\quad \text{and} \cdots \text{and} \left[\cup \mathcal{H}_{\mathbb{E}_q}^{\mathcal{R}_n}(w)\right] \cap \mathcal{Z} \neq \varnothing \Big\}. \end{aligned} \tag{14}$$

If ${}_2\mathbf{L}_{\sum_{d=1}^{n} {}_d\mathbb{E}_q}(\mathcal{Z}) \neq {}_2\mathbf{U}_{\sum_{d=1}^{n} {}_d\mathbb{E}_q}(\mathcal{Z})$, then $\mathcal{Z}$ is said to be the second kind of q-covering multigranulation rough set (briefly, 2-qMGCRS), else it is definable.

Style 3

The 3-MCLA ${}_3\mathbf{L}_{\sum_{d=1}^{n} {}_d\mathbb{E}_q}(\mathcal{Z})$ and the 3-MCUA ${}_3\mathbf{U}_{\sum_{d=1}^{n} {}_d\mathbb{E}_q}(\mathcal{Z})$ are seen as follows:

$$\begin{aligned} {}_3\mathbf{L}_{\sum_{d=1}^{n} {}_d\mathbb{E}_q}(\mathcal{Z}) &= \Big\{ w \in \mathbb{Q} : \cap \mathcal{D}_{\mathbb{E}_q}^{\mathcal{R}_1}(w) \subseteq \mathcal{Z} \text{ or } \cap \mathcal{D}_{\mathbb{E}_q}^{\mathcal{R}_2}(w) \subseteq \mathcal{Z} \\ &\quad \text{or} \cdots \text{or} \cap \mathcal{D}_{\mathbb{E}_q}^{\mathcal{R}_n}(w) \subseteq \mathcal{Z} \Big\}, \\ {}_3\mathbf{U}_{\sum_{d=1}^{n} {}_d\mathbb{E}_q}(\mathcal{Z}) &= \Big\{ w \in \mathbb{Q} : \left[\cap \mathcal{D}_{\mathbb{E}_q}^{\mathcal{R}_1}(w)\right] \cap \mathcal{Z} \neq \varnothing \\ &\quad \text{and} \left[\cap \mathcal{D}_{\mathbb{E}_q}^{\mathcal{R}_2}(w)\right] \cap \mathcal{Z} \neq \varnothing \\ &\quad \text{and} \cdots \text{and} \left[\cap \mathcal{D}_{\mathbb{E}_q}^{\mathcal{R}_n}(w)\right] \cap \mathcal{Z} \neq \varnothing \Big\}. \end{aligned} \tag{15}$$

If ${}_3\mathbf{L}_{\sum_{d=1}^{n} {}_d\mathbb{E}_q}(\mathcal{Z}) \neq {}_3\mathbf{U}_{\sum_{d=1}^{n} {}_d\mathbb{E}_q}(\mathcal{Z})$, then $\mathcal{Z}$ is said to be the third kind of q-covering multigranulation rough set (briefly, 3-qMGCRS), else it is definable.

Style 4

The 4-MCLA ${}_4\mathbf{L}_{\sum_{d=1}^{n} {}_d\mathbb{E}_q}(\mathcal{Z})$ and the 4-MCUA ${}_4\mathbf{U}_{\sum_{d=1}^{n} {}_d\mathbb{E}_q}(\mathcal{Z})$ are seen as follows:

$$\begin{aligned} {}_4\mathbf{L}_{\sum_{d=1}^{n} {}_d\mathbb{E}_q}(\mathcal{Z}) &= \Big\{ w \in \mathbb{Q} : \cup \mathcal{D}_{\mathbb{E}_q}^{\mathcal{R}_1}(w) \subseteq \mathcal{Z} \text{ or } \cup \mathcal{D}_{\mathbb{E}_q}^{\mathcal{R}_2}(w) \subseteq \mathcal{Z} \\ &\quad \text{or} \cdots \text{or} \cup \mathcal{D}_{\mathbb{E}_q}^{\mathcal{R}_n}(w) \subseteq \mathcal{Z} \Big\}, \\ {}_4\mathbf{U}_{\sum_{d=1}^{n} {}_d\mathbb{E}_q}(\mathcal{Z}) &= \Big\{ w \in \mathbb{Q} : \left[\cup \mathcal{D}_{\mathbb{E}_q}^{\mathcal{R}_1}(w)\right] \cap \mathcal{Z} \neq \varnothing \\ &\quad \text{and} \left[\cup \mathcal{D}_{\mathbb{E}_q}^{\mathcal{R}_2}(w)\right] \cap \mathcal{Z} \neq \varnothing \\ &\quad \text{and} \cdots \text{and} \left[\cup \mathcal{D}_{\mathbb{E}_q}^{\mathcal{R}_n}(w)\right] \cap \mathcal{Z} \neq \varnothing \Big\}. \end{aligned} \tag{16}$$

If ${}_4\mathbf{L}_{\sum_{d=1}^{n} {}_d\mathbb{E}_q}(\mathcal{Z}) \neq {}_4\mathbf{U}_{\sum_{d=1}^{n} {}_d\mathbb{E}_q}(\mathcal{Z})$, then $\mathcal{Z}$ is said to be the fourth kind of q-covering multigranulation rough set (briefly, 4-qMGCRS), else it is definable.

Example 3. Consider $(\mathbb{Q}, \mathfrak{R}, \mathbb{E}_q)$ is a M$\mathbb{E}_q$CAS, $\mathbb{Q} = \{k_1, k_2, k_3, k_4\}$ and $\mathfrak{R} = \{\mathcal{R}_1, \mathcal{R}_2\}$, where $\mathcal{R}_1 = \{(k_1, k_4), (k_2, k_2), (k_2, k_3), (k_3, k_2), (k_4, k_1), (k_4, k_3)\}$ and $\mathcal{R}_2 = \{(k_1, k_1), (k_1, k_2), (k_2, k_3), (k_2, k_4), (k_3, k_1), (k_4, k_1)\}$. Take $\mathcal{Z} = \{k_1, k_3\}$; then, we have the presented outcomes:

$(1_\mathbf{r})$ ${}_1\mathbf{L}_{\sum_{d=1}^{2} {}_d\mathbb{E}_\mathbf{r}}(\mathcal{Z}) = \{k_1, k_3\}$, ${}_1\mathbf{U}_{\sum_{d=1}^{2} {}_d\mathbb{E}_\mathbf{r}}(\mathcal{Z}) = \{k_1, k_3\}$

$(1_\mathbf{l})$ ${}_1\mathbf{L}_{\sum_{d=1}^{2} {}_d\mathbb{E}_\mathbf{l}}(\mathcal{Z}) = \{k_1\}$, ${}_1\mathbf{U}_{\sum_{d=1}^{2} {}_d\mathbb{E}_\mathbf{l}}(\mathcal{Z}) = \{k_1, k_3\}$.

$(2_\mathbf{r})$ ${}_2\mathbf{L}_{\sum_{d=1}^{2} {}_d\mathbb{E}_\mathbf{r}}(\mathcal{Z}) = \{k_1\}$, ${}_2\mathbf{U}_{\sum_{d=1}^{2} {}_d\mathbb{E}_\mathbf{r}}(\mathcal{Z}) = \{k_1, k_3\}$

$(2_\mathbf{l})$ ${}_2\mathbf{L}_{\sum_{d=1}^{2} {}_d\mathbb{E}_\mathbf{l}}(\mathcal{Z}) = \{k_1\}$, ${}_2\mathbf{U}_{\sum_{d=1}^{2} {}_d\mathbb{E}_\mathbf{l}}(\mathcal{Z}) = \{k_1, k_3\}$.

$(3_\mathbf{r})$ ${}_3\mathbf{L}_{\sum_{d=1}^{2} {}_d\mathbb{E}_\mathbf{r}}(\mathcal{Z}) = \{k_1, k_3\}$, ${}_3\mathbf{U}_{\sum_{d=1}^{2} {}_d\mathbb{E}_\mathbf{r}}(\mathcal{Z}) = \{k_1, k_2, k_3\}$

$(3_\mathbf{l})$ ${}_3\mathbf{L}_{\sum_{d=1}^{2} {}_d\mathbb{E}_\mathbf{l}}(\mathcal{Z}) = \{k_1\}$, ${}_3\mathbf{U}_{\sum_{d=1}^{2} {}_d\mathbb{E}_\mathbf{l}}(\mathcal{Z}) = \{k_1, k_3\}$.

$(4_{\mathbf{r}})\ {}_4\mathbf{L}_{\sum_{d=1}^{2} {}_d\mathbb{E}_{\mathbf{r}}}(\mathcal{Z})=\{k_1\},\ {}_4\mathbf{U}_{\sum_{d=1}^{2} {}_d\mathbb{E}_{\mathbf{r}}}(\mathcal{Z})=\{k_1,k_2,k_3\}$

$(4_{\mathbf{l}})\ {}_4\mathbf{L}_{\sum_{d=1}^{2} {}_d\mathbb{E}_{\mathbf{l}}}(\mathcal{Z})=\{k_1\},\ {}_4\mathbf{U}_{\sum_{d=1}^{2} {}_d\mathbb{E}_{\mathbf{l}}}(\mathcal{Z})=\{k_1,k_3\}$.

Theorem 14. *Suppose that $(\mathbb{Q},\mathfrak{R},\mathbb{E}_q)$ is a $M\mathbb{E}_q CAS$. For any $\mathcal{Z}\subseteq\mathbb{Q}$, we get the following properties:*

(1) ${}_1\mathbf{L}_{\sum_{d=1}^{n} {}_d\mathbb{E}_q}(\mathcal{Z}^c)=[{}_1\mathbf{U}_{\sum_{d=1}^{n} {}_d\mathbb{E}_q}(\mathcal{Z})]^c$, ${}_1\mathbf{U}_{\sum_{d=1}^{n} {}_d\mathbb{E}_q}(\mathcal{Z}^c)=[{}_1\mathbf{L}_{\sum_{d=1}^{n} {}_d\mathbb{E}_q}(\mathcal{Z})]^c$

(2) ${}_2\mathbf{L}_{\sum_{d=1}^{n} {}_d\mathbb{E}_q}(\mathcal{Z}^c)=[{}_2\mathbf{U}_{\sum_{d=1}^{n} {}_d\mathbb{E}_q}(\mathcal{Z})]^c$, ${}_2\mathbf{U}_{\sum_{d=1}^{n} {}_d\mathbb{E}_q}(\mathcal{Z}^c)=[{}_2\mathbf{L}_{\sum_{d=1}^{n} {}_d\mathbb{E}_q}(\mathcal{Z})]^c$

(3) ${}_3\mathbf{L}_{\sum_{d=1}^{n} {}_d\mathbb{E}_q}(\mathcal{Z}^c)=[{}_3\mathbf{U}_{\sum_{d=1}^{n} {}_d\mathbb{E}_q}(\mathcal{Z})]^c$, ${}_3\mathbf{U}_{\sum_{d=1}^{n} {}_d\mathbb{E}_q}(\mathcal{Z}^c)=[{}_3\mathbf{L}_{\sum_{d=1}^{n} {}_d\mathbb{E}_q}(\mathcal{Z})]^c$

(4) ${}_4\mathbf{L}_{\sum_{d=1}^{n} {}_d\mathbb{E}_q}(\mathcal{Z}^c)=[{}_4\mathbf{U}_{\sum_{d=1}^{n} {}_d\mathbb{E}_q}(\mathcal{Z})]^c$, ${}_4\mathbf{U}_{\sum_{d=1}^{n} {}_d\mathbb{E}_q}(\mathcal{Z}^c)=[{}_4\mathbf{L}_{\sum_{d=1}^{n} {}_d\mathbb{E}_q}(\mathcal{Z})]^c$

Proof. Here, we want to set (1) only.

(1)

$$\begin{aligned}
{}_1\mathbf{L}_{\sum_{d=1}^{n} {}_d\mathbb{E}_q}(\mathcal{Z}^c) &= \Big\{w\in\mathbb{Q}:\ \cap\mathcal{H}_{\mathbb{E}_q}^{\mathcal{R}_1}(w)\subseteq\mathcal{Z}^c \text{ or} \cap\mathcal{H}_{\mathbb{E}_q}^{\mathcal{R}_2}(w)\subseteq\mathcal{Z}^c \\
&\quad \text{or} \cdots \text{or} \cap\mathcal{H}_{\mathbb{E}_q}^{\mathcal{R}_n}(w)\subseteq\mathcal{Z}^c\Big\} \\
&= \Big\{w\in\mathbb{Q}:\ \cap\mathcal{H}_{\mathbb{E}_q}^{\mathcal{R}_1}(w)\cap\mathcal{Z}=\varnothing \text{ or} \cap\mathcal{H}_{\mathbb{E}_q}^{\mathcal{R}_2}(w)\cap\mathcal{Z}=\varnothing \\
&\quad \text{or} \cdots \text{or} \cap\mathcal{H}_{\mathbb{E}_q}^{\mathcal{R}_n}(w)\cap\mathcal{Z}=\varnothing\Big\} \\
&= \Big\{w\in\mathbb{Q}:\ \cap\mathcal{H}_{\mathbb{E}_q}^{\mathscr{R}_1}(w)\cap\mathcal{Z}\neq\varnothing \text{ and} \cap\mathcal{H}_{\mathbb{E}_q}^{\mathcal{R}_2}(w)\cap\mathcal{Z}\neq\varnothing \\
&\quad \text{and} \cdots \text{and} \cap\mathcal{H}_{\mathbb{E}_q}^{\mathcal{R}_n}(w)\cap\varnothing\Big\}^c \\
&= \left[{}_1\mathbf{U}_{\sum_{d=1}^{n} {}_d\mathbb{E}_q}(\mathcal{Z})\right]^c.
\end{aligned} \tag{17}$$

Also, it is easy to see ${}_1\mathbf{U}_{\sum_{d=1}^{n} {}_d\mathbb{E}_q}(\mathcal{Z}^c)=[{}_1\mathbf{L}_{\sum_{d=1}^{n} {}_d\mathbb{E}_q}(\mathcal{Z})]^c$. □

Proposition 15. *Suppose that $(\mathbb{Q},\mathfrak{R},\mathbb{E}_q)$ is a $M\mathbb{E}_q CAS$. For any $\mathcal{Z}\subseteq\mathbb{Q}$, we get the following properties:*

(1) ${}_1\mathbf{L}_{\sum_{d=1}^{n} {}_d\mathbb{E}_q}({}_1\mathbf{L}_{\sum_{d=1}^{n} {}_d\mathbb{E}_q}(\mathcal{Z}))={}_1\mathbf{L}_{\sum_{d=1}^{n} {}_d\mathbb{E}_q}(\mathcal{Z})$, ${}_1\mathbf{U}_{\sum_{d=1}^{n} {}_d\mathbb{E}_q}({}_1\mathbf{U}_{\sum_{d=1}^{n} {}_d\mathbb{E}_q}(\mathcal{Z}))={}_1\mathbf{U}_{\sum_{d=1}^{n} {}_d\mathbb{E}_q}(\mathcal{Z})$

(2) ${}_2\mathbf{L}_{\sum_{d=1}^{n} {}_d\mathbb{E}_q}({}_2\mathbf{L}_{\sum_{d=1}^{n} {}_d\mathbb{E}_q}(\mathcal{Z}))={}_2\mathbf{L}_{\sum_{d=1}^{n} {}_d\mathbb{E}_q}(\mathcal{Z})$, ${}_2\mathbf{U}_{\sum_{d=1}^{n} {}_d\mathbb{E}_q}({}_2\mathbf{U}_{\sum_{d=1}^{n} {}_d\mathbb{E}_q}(\mathcal{Z}))={}_2\mathbf{U}_{\sum_{d=1}^{n} {}_d\mathbb{E}_q}(\mathcal{Z})$

(3) ${}_3\mathbf{L}_{\sum_{d=1}^{n} {}_d\mathbb{E}_q}({}_3\mathbf{L}_{\sum_{d=1}^{n} {}_d\mathbb{E}_q}(\mathcal{Z}))={}_3\mathbf{L}_{\sum_{d=1}^{n} {}_d\mathbb{E}_q}(\mathcal{Z})$, ${}_3\mathbf{U}_{\sum_{d=1}^{n} {}_d\mathbb{E}_q}({}_3\mathbf{U}_{\sum_{d=1}^{n} {}_d\mathbb{E}_q}(\mathcal{Z}))={}_3\mathbf{U}_{\sum_{d=1}^{n} {}_d\mathbb{E}_q}(\mathcal{Z})$

Proof. Here, we want to set (1) only.

(1) It is obvious that ${}_1\mathbf{L}_{\sum_{d=1}^{n} {}_d\mathbb{E}_q}({}_1\mathbf{L}_{\sum_{d=1}^{n} {}_d\mathbb{E}_q}(\mathcal{Z}))\subseteq {}_1\mathbf{L}_{\sum_{d=1}^{n} {}_d\mathbb{E}_q}(\mathcal{Z})$. On the other hand, we have ${}_1\mathbf{L}_{\sum_{d=1}^{n} {}_d\mathbb{E}_q}(\mathcal{Z})={}_1\mathbf{L}_{{}_1\mathbb{E}_q}(\mathcal{Z})\cup{}_1\mathbf{L}_{{}_2\mathbb{E}_q}(\mathcal{Z})\cup\cdots\cup{}_1\mathbf{L}_{{}_n\mathbb{E}_q}(\mathcal{Z})$. Thus, we get that

$$\begin{aligned}
{}_1\mathbf{L}_{\sum_{d=1}^{n} {}_d\mathbb{E}_q}\left({}_1\mathbf{L}_{\sum_{d=1}^{n} {}_d\mathbb{E}_q}(\mathcal{Z})\right)
&= {}_1\mathbf{L}_{{}_1\mathbb{E}_q}\left({}_1\mathbf{L}_{\sum_{d=1}^{n} {}_d\mathbb{E}_q}(\mathcal{Z})\right)\cup{}_1\mathbf{L}_{{}_2\mathbb{E}_q}\left({}_1\mathbf{L}_{\sum_{d=1}^{n} {}_d\mathbb{E}_q}(\mathcal{Z})\right) \\
&\quad \cup\cdots\cup{}_1\mathbf{L}_{{}_n\mathbb{E}_q}\left({}_1\mathbf{L}_{\sum_{d=1}^{n} {}_d\mathbb{E}_q}(\mathcal{Z})\right) \\
&= {}_1\mathbf{L}_{{}_1\mathbb{E}_q}\left({}_1\mathbf{L}_{{}_1\mathbb{E}_q}(\mathcal{Z})\cup{}_1\mathbf{L}_{{}_2\mathbb{E}_q}(\mathcal{Z})\cup\cdots\cup{}_1\mathbf{L}_{{}_n\mathbb{E}_q}(\mathcal{Z})(\mathcal{Z})\right) \\
&\quad \cup{}_1\mathbf{L}_{{}_2\mathbb{E}_q}\left({}_1\mathbf{L}_{{}_1\mathbb{E}_q}(\mathcal{Z})\cup{}_1\mathbf{L}_{{}_2\mathbb{E}_q}(\mathcal{Z})\cup\cdots\cup{}_1\mathbf{L}_{{}_n\mathbb{E}_q}(\mathcal{Z})(\mathcal{Z})\right) \\
&\quad \cup\cdots\cup{}_1\mathbf{L}_{{}_n\mathbb{E}_q}\left({}_1\mathbf{L}_{{}_1\mathbb{E}_q}(\mathcal{Z})\cup{}_1\mathbf{L}_{{}_2\mathbb{E}_q}(\mathcal{Z})\cup\cdots\cup{}_1\mathbf{L}_{{}_n\mathbb{E}_q}(\mathcal{Z})(\mathcal{Z})\right) \\
&\supseteq {}_1\mathbf{L}_{{}_1\mathbb{E}_q}\left({}_1\mathbf{L}_{{}_1\mathbb{E}_q}(\mathcal{Z})\right)\cup{}_1\mathbf{L}_{{}_2\mathbb{E}_q}\left({}_1\mathbf{L}_{{}_2\mathbb{E}_q}(\mathcal{Z})\right) \\
&\quad \cup_{11}\mathbf{L}_{{}_n\mathbb{E}_q}\left({}_1\mathbf{L}_{{}_n\mathbb{E}_q}(\mathcal{Z})\right) \\
&= {}_1\mathbf{L}_{{}_1\mathbb{E}_q}(\mathcal{Z})\cup{}_1\mathbf{L}_{{}_2\mathbb{E}_q}(\mathcal{Z})\cup\cdots\cup{}_1\mathbf{L}_{{}_n\mathbb{E}_q}(\mathcal{Z})={}_1\mathbf{L}_{\sum_{d=1}^{n} {}_d\mathbb{E}_q}(\mathcal{Z})
\end{aligned} \tag{18}$$

Also, it is clear that ${}_1\mathbf{U}_{\sum_{d=1}^{n} {}_d\mathbb{E}_q}({}_1\mathbf{U}_{\sum_{d=1}^{n} {}_d\mathbb{E}_q}(\mathcal{Z}))\subseteq {}_1\mathbf{U}_{\sum_{d=1}^{n} {}_d\mathbb{E}_q}(\mathcal{Z})$. Consequently, we have ${}_1\mathbf{U}_{\sum_{d=1}^{n} {}_d\mathbb{E}_q}(\mathcal{Z})={}_1\mathbf{U}_{{}_1\mathbb{E}_q}(\mathcal{Z})\cap{}_1\mathbf{U}_{{}_2\mathbb{E}_q}(\mathcal{Z})\cap\cdots\cap{}_1\mathbf{U}_{{}_n\mathbb{E}_q}(\mathcal{Z})$. So, we have

$$\begin{aligned}
{}_1\mathbf{U}_{\sum_{d=1}^{n} {}_d\mathbb{E}_q}\left({}_1\mathbf{U}_{\sum_{d=1}^{n} {}_d\mathbb{E}_q}(\mathcal{Z})\right)
&= {}_1\mathbf{U}_{{}_1\mathbb{E}_q}\left({}_1\mathbf{U}_{\sum_{d=1}^{n} {}_d\mathbb{E}_q}(\mathcal{Z})\right)\cap{}_1\mathbf{U}_{{}_2\mathbb{E}_q}\left({}_1\mathbf{U}_{\sum_{d=1}^{n} {}_d\mathbb{E}_q}(\mathcal{Z})\right) \\
&\quad \cap\cdots\cap{}_1\mathbf{U}_{{}_n\mathbb{E}_q}\left({}_1\mathbf{U}_{\sum_{d=1}^{n} {}_d\mathbb{E}_q}(\mathcal{Z})\right) \\
&= {}_1\mathbf{U}_{1\mathbb{E}_q}\left({}_1\mathbf{U}_{{}_1\mathbb{E}_q}(\mathcal{Z})\cap{}_1\mathbf{U}_{{}_2\mathbb{E}_q}(\mathcal{Z})\cap\cdots\cap{}_1\mathbf{U}_{{}_n\mathbb{E}_q}(\mathcal{Z})\right) \\
&\quad \cap{}_1\mathbf{U}_{{}_2\mathbb{E}_q}\left({}_1\mathbf{U}_{{}_1\mathbb{E}_q}(\mathcal{Z})\cap{}_1\mathbf{U}_{{}_2\mathbb{E}_q}(\mathcal{Z})\cap\cdots\cap{}_1\mathbf{U}_{{}_n\mathbb{E}_q}(\mathcal{Z})\right)
\end{aligned}$$

$$\cap \cdots \cap {}_1\mathbf{U}_{{}_n\mathbb{E}_q}\left({}_1\mathbf{U}_{{}_1\mathbb{E}_q}(\mathcal{Z}) \cap {}_1\mathbf{U}_{{}_2\mathbb{E}_q}(\mathcal{Z}) \cap \cdots \cap {}_1\mathbf{U}_{{}_n\mathbb{E}_q}(\mathcal{Z})\right)$$
$$\supseteq {}_1\mathbf{U}_{{}_1\mathbb{E}_q}\left({}_1\mathbf{U}_{{}_1\mathbb{E}_q}(\mathcal{Z})\right) \cap {}_1\mathbf{U}_{{}_2\mathbb{E}_q}\left({}_1\mathbf{U}_{{}_2\mathbb{E}_q}(\mathcal{Z})\right) \cap \cdots$$
$$\cap {}_1\mathbf{U}_{{}_n\mathbb{E}_q}\left({}_1\mathbf{U}_{{}_n\mathbb{E}_q}(\mathcal{Z})\right)$$
$$= {}_1\mathbf{U}_{{}_1\mathbb{E}_q}(\mathcal{Z}) \cap {}_1\mathbf{U}_{{}_2\mathbb{E}_q}(\mathcal{Z}) \cap \cdots \cap {}_1\mathbf{U}_{{}_n\mathbb{E}_q}(\mathcal{Z})$$
$$= {}_1\mathbf{U}_{\sum_{d=1}^{n} {}_d\mathbb{E}_q}(\mathcal{Z}). \quad (19)$$

Hence, ${}_1\mathbf{L}_{\sum_{d=1}^{n} {}_d\mathbb{E}_q}({}_1\mathbf{L}_{\sum_{d=1}^{n} {}_d\mathbb{E}_q}(\mathcal{Z})) = {}_1\mathbf{L}_{\sum_{d=1}^{n} {}_d\mathbb{E}_q}(\mathcal{Z})$ and ${}_1\mathbf{U}_{\sum_{d=1}^{n} {}_d\mathbb{E}_q}({}_1\mathbf{U}_{\sum_{d=1}^{n} {}_d\mathbb{E}_q}(\mathcal{Z})) = {}_1\mathbf{U}_{\sum_{d=1}^{n} {}_d\mathbb{E}_q}(\mathcal{Z})$. □

The above Proposition 15 is not true for 4-qMGCRS as in the following example.

Example 4. Consider that $(\mathbb{Q}, \mathfrak{R}, \mathbb{E}_q)$ is a M$\mathbb{E}_q$CAS, $\mathbb{Q} = \{k_1, k_2, k_3, k_4, k_5\}$ and $\mathfrak{R} = \{\mathcal{R}_1, \mathcal{R}_2\}$, where $\mathcal{R}_1 = \{(k_1, k_1), (k_1, k_3), (k_1, k_5), (k_2, k_2), (k_3, k_3), (k_4, k_2), (k_4, k_4), (k_4, k_5), (k_5, k_5)\}$ and $\mathcal{R}_2 = \{(k_1, k_1), (k_1, k_3), (k_2, k_2), (k_2, k_5), (k_3, k_3), (k_4, k_3), (k_4, k_4), (k_5, k_2), (k_5, k_5)\}$.

Take $\mathcal{Z}_1 = \{k_1, k_3\}$ and $\mathcal{Z}_2 = \{k_4, k_5\}$; then, we have the presented outcomes.

($1_\mathbf{r}$) ${}_4\mathbf{L}_{\sum_{d=1}^{2} \mathrm{d}\mathbb{E}_\mathrm{r}}(\mathcal{Z}_1) = \{k_1\}$ and ${}_4\mathbf{L}_{\sum_{d=1}^{2} {}_d\mathbb{E}_\mathrm{r}}({}_4\mathbf{U}_{\sum_{d=1}^{2} \mathrm{d}\mathbb{E}_\mathrm{r}}(\mathcal{Z}_1)) = \varnothing$. Then, ${}_4\mathbf{L}_{\sum_{d=1}^{2} \mathrm{d}\mathbb{E}_\mathrm{r}}({}_4\mathbf{L}_{\sum_{d=1}^{2} {}_d\mathbb{E}_\mathrm{r}}(\mathcal{Z}_1)) \neq {}_4\mathbf{L}_{\sum_{d=1}^{2} {}_d\mathbb{E}_\mathrm{r}}(\mathcal{Z}_1)$.

($2_\mathbf{r}$) ${}_4\mathbf{U}_{\sum_{d=1}^{2} {}_d\mathbb{E}_\mathrm{r}}(\mathcal{Z}_2) = \{k_2, k_3, k_4, k_5\}$ and ${}_4\mathbf{U}_{\sum_{d=1}^{2} \mathrm{d}\mathbb{E}_\mathrm{r}}({}_4\mathbf{U}_{\sum_{d=1}^{2} {}_d\mathbb{E}_\mathrm{r}}(\mathcal{Z}_2)) = \mathbb{Q}$. Then, ${}_4\mathbf{U}_{\sum_{d=1}^{2} {}_d\mathbb{E}_\mathrm{r}}({}_4\mathbf{U}_{\sum_{d=1}^{2} \mathrm{d}\mathbb{E}_\mathrm{r}}(\mathcal{Z}_2)) \neq {}_4\mathbf{U}_{\sum_{d=1}^{2} {}_d\mathbb{E}_\mathrm{r}}(\mathcal{Z}_2)$.

($1_\mathbf{l}$) ${}_4\mathbf{L}_{\sum_{d=1}^{2} {}_d\mathbb{E}_\mathrm{l}}(\mathcal{Z}_1) = \{k_3\}$ and ${}_4\mathbf{L}_{\sum_{d=1}^{2} {}_d\mathbb{E}_\mathrm{l}}({}_4\mathbf{L}_{\sum_{d=1}^{2} \mathrm{d}\mathbb{E}_\mathrm{l}}(\mathcal{Z}_1)) = \varnothing$. Then, ${}_4\mathbf{L}_{\sum_{d=1}^{2} \mathrm{d}\mathbb{E}_\mathrm{l}}({}_4\mathbf{L}_{\sum_{d=1}^{2} \mathrm{d}\mathbb{E}_\mathrm{l}}(\mathcal{Z}_1)) \neq {}_4\mathbf{L}_{\sum_{d=1}^{2} {}_d\mathbb{E}_\mathrm{l}}(\mathcal{Z}_1)$.

($2_\mathbf{l}$) ${}_4\mathbf{U}_{\sum_{d=1}^{2} {}_d\mathbb{E}_\mathrm{l}}(\mathcal{Z}_2) = \{k_1, k_2, k_4, k_5\}$ and ${}_4\mathbf{U}_{\sum_{d=1}^{2} {}_d\mathbb{E}_\mathrm{l}}({}_4\mathbf{U}_{\sum_{d=1}^{2} {}_d\mathbb{E}_\mathrm{l}}(\mathcal{Z}_2)) = \mathbb{Q}$. Then, ${}_4\mathbf{U}_{\sum_{d=1}^{2} {}_d\mathbb{E}_\mathrm{l}}({}_4\mathbf{U}_{\sum_{d=1}^{2} {}_d\mathbb{E}_\mathrm{l}}(\mathcal{Z}_2)) \neq {}_4\mathbf{U}_{\sum_{d=1}^{2} {}_d\mathbb{E}_\mathrm{l}}(\mathcal{Z}_2)$.

Next, we will establish new properties in Proposition 16. These characteristics are done for 1-qMGCRS, 2-qMGCRS, 3-qMGCRS, and 4-qMGCRS, though we demonstrate it in the case of 1-qMGCRS and others are similar.

Proposition 16. *Suppose that $(\mathbb{Q}, \mathfrak{R}, \mathbb{E}_q)$ is a M$\mathbb{E}_q$CAS. For any $\mathcal{Z}_1, \mathcal{Z}_2 \subseteq \mathbb{Q}$, we get the following properties:*

(1) If $\mathcal{Z}_1 \subseteq \mathcal{Z}_2$, then ${}_1\mathbf{L}_{\sum_{d=1}^{n} {}_d\mathbb{E}_q}(\mathcal{Z}_1) \subseteq {}_1\mathbf{L}_{\sum_{d=1}^{n} {}_d\mathbb{E}_q}(\mathcal{Z}_2)$

(2) If $\mathcal{Z}_1 \subseteq \mathcal{Z}_2$, then ${}_1\mathbf{U}_{\sum_{d=1}^{n} {}_d\mathbb{E}_q}(\mathcal{Z}_1) \subseteq {}_1\mathbf{U}_{\sum_{d=1}^{n} {}_d\mathbb{E}_q}(\mathcal{Z}_2)$

(3) ${}_1\mathbf{L}_{\sum_{d=1}^{n} {}_d\mathbb{E}_q}(\mathcal{Z}_1 \cap \mathcal{Z}_2) \subseteq {}_1\mathbf{L}_{\sum_{d=1}^{n} {}_d\mathbb{E}_q}(\mathcal{Z}_1) \cap {}_1\mathbf{L}_{\sum_{d=1}^{n} {}_d\mathbb{E}_q}(\mathcal{Z}_2)$

(4) ${}_1\mathbf{L}_{\sum_{d=1}^{n} {}_d\mathbb{E}_q}(\mathcal{Z}_1 \cup \mathcal{Z}_2) \supseteq {}_1\mathbf{L}_{\sum_{d=1}^{n} {}_d\mathbb{E}_q}(\mathcal{Z}_1) \cup {}_1\mathbf{L}_{\sum_{d=1}^{n} {}_d\mathbb{E}_q}(\mathcal{Z}_2)$

(5) ${}_1\mathbf{U}_{\sum_{d=1}^{n} {}_d\mathbb{E}_q}(\mathcal{Z}_1 \cup \mathcal{Z}_2) \supseteq {}_1\mathbf{U}_{\sum_{d=1}^{n} {}_d\mathbb{E}_q}(\mathcal{Z}_1) \cup {}_1\mathbf{U}_{\sum_{d=1}^{n} {}_d\mathbb{E}_q}(\mathcal{Z}_2)$

(6) ${}_1\mathbf{U}_{\sum_{d=1}^{n} {}_d\mathbb{E}_q}(\mathcal{Z}_1 \cap \mathcal{Z}_2) \subseteq {}_1\mathbf{U}_{\sum_{d=1}^{n} {}_d\mathbb{E}_q}(\mathcal{Z}_1) \cap {}_1\mathbf{U}_{\sum_{d=1}^{n} {}_d\mathbb{E}_q}(\mathcal{Z}_2)$

Proof. Now, we just need to show (1) and (2).

(1) From Definition 13 and since $\mathcal{Z}_1 \subseteq \mathcal{Z}_2$, then, we obtain the following:

$$\begin{aligned}
{}_1\mathbf{L}_{\sum_{d=1}^{n} {}_d\mathbb{E}_q}(\mathcal{Z}_1) &= \left\{w \in \mathbb{Q} : \cap \mathcal{H}_{\mathbb{E}_q}^{\mathcal{R}_1}(w) \subseteq \mathcal{Z}_1 \text{ or} \cap \mathcal{H}_{\mathbb{E}_q}^{\mathcal{R}_2}(w) \subseteq \mathcal{Z}_1 \right. \\
&\quad \left. \text{or} \cdots \text{or} \cap \mathcal{H}_{\mathbb{E}_q}^{\mathcal{R}_n}(w) \subseteq \mathcal{Z}_1 \right\} \\
&\subseteq \left\{w \in \mathbb{Q} : \cap \mathcal{H}_{\mathbb{E}_q}^{\mathcal{R}_1}(w) \subseteq \mathcal{Z}_2 \text{ or} \cap \mathcal{H}_{\mathbb{E}_q}^{\mathcal{R}_2}(w) \right. \\
&\quad \left. \subseteq \mathcal{Z}_2 \text{ or} \cdots \text{or} \cap \mathcal{H}_{\mathbb{E}_q}^{\mathcal{R}_n}(w) \subseteq \mathcal{Z}_2 \right\} \\
&= {}_1\mathbf{L}_{\sum_{d=1}^{n} {}_d\mathbb{E}_q}(\mathcal{Z}_2)
\end{aligned} \quad (20)$$

(2) From Definition 13 and since $\mathcal{Z}_1 \subseteq \mathcal{Z}_2$, then, we have the following:

$$\begin{aligned}
{}_1\mathbf{U}_{\sum_{d=1}^{n} {}_d\mathbb{E}_q}(\mathcal{Z}_1) &= \left\{w \in \mathbb{Q} : \left[\cap \mathcal{H}_{\mathbb{E}_q}^{\mathcal{R}_1}(w)\right] \cap \mathcal{Z}_1 \right. \\
&\quad \neq \varnothing \text{ and} \left[\cap \mathcal{H}_{\mathbb{E}_q}^{\mathcal{R}_2}(w)\right] \cap \mathcal{Z}_1 \\
&\quad \left. \neq \varnothing \text{ and} \cdots \text{and} \left[\cap \mathcal{H}_{\mathbb{E}_q}^{\mathcal{R}_n}(w)\right] \cap \mathcal{Z}_1 \neq \varnothing \right\} \\
&\subseteq \left\{w \in \mathbb{Q} : \left[\cap \mathcal{H}_{\mathbb{E}_q}^{\mathcal{R}_1}(w)\right] \cap \mathcal{Z}_2 \right. \\
&\quad \neq \varnothing \text{ and} \left[\cap \mathcal{H}_{\mathbb{E}_q}^{\mathcal{R}_2}(w)\right] \cap \mathcal{Z}_2 \\
&\quad \left. \neq \varnothing \text{ and} \cdots \text{and} \left[\cap \mathcal{H}_{\mathbb{E}_q}^{\mathcal{R}_n}(w)\right] \cap \mathcal{Z}_2 \neq \varnothing \right\} \\
&= {}_1\mathbf{U}_{\sum_{d=1}^{n} {}_d\mathbb{E}_q}(\mathcal{Z}_2)
\end{aligned} \quad (21)$$

□

Example 5. Consider Example 4. Then, we have the following:

(1_r) Take $\mathcal{Z}_1 = \{k_2, k_3, k_4\}$ and $\mathcal{Z}_2 = \{k_2, k_4, k_5\}$, then we have ${}_1\mathbf{L}_{\sum_{d=1}^{2} {}_d\mathbb{E}_\mathrm{r}}(\mathcal{Z}_1) = \{k_2, k_3, k_4\}$, ${}_1\mathbf{L}_{\sum_{d=1}^{2} {}_d\mathbb{E}_\mathrm{r}}(\mathcal{Z}_2) = \{k_2, k_4, k_5\}$ and ${}_1\mathbf{L}_{\sum_{d=1}^{2} {}_d\mathbb{E}_\mathrm{r}}(\mathcal{Z}_1 \cap \mathcal{Z}_2) = \{k_2\}$. Thus, ${}_1\mathbf{L}_{\sum_{d=1}^{2} {}_d\mathbb{E}_\mathrm{r}}(\mathcal{Z}_1 \cap \mathcal{Z}_2) \neq {}_1\mathbf{L}_{\sum_{d=1}^{2} {}_d\mathbb{E}_\mathrm{r}}(\mathcal{Z}_1) \cap {}_1\mathbf{L}_{\sum_{d=1}^{2} {}_d\mathbb{E}_\mathrm{r}}(\mathcal{Z}_2)$

(1_l) Take $\mathcal{Z}_1 = \{k_2, k_4\}$ and $\mathcal{Z}_2 = \{k_2, k_5\}$, then we have ${}_1\mathbf{L}_{\sum_{d=1}^{2} {}_d\mathbb{E}_\mathrm{l}}(\mathcal{Z}_1) = \{k_2, k_4\}$, ${}_1\mathbf{L}_{\sum_{d=1}^{2} {}_d\mathbb{E}_\mathrm{l}}(\mathcal{Z}_2) = \{k_2, k_5\}$ and ${}_1\mathbf{L}_{\sum_{d=1}^{2} {}_d\mathbb{E}_\mathrm{l}}(\mathcal{Z}_1 \cap \mathcal{Z}_2) = \varnothing$. Thus, ${}_1\mathbf{L}_{\sum_{d=1}^{2} {}_d\mathbb{E}_\mathrm{l}}(\mathcal{Z}_1 \cap \mathcal{Z}_2) \neq {}_1\mathbf{L}_{\sum_{d=1}^{2} {}_d\mathbb{E}_\mathrm{l}}(\mathcal{Z}_1) \cap {}_1\mathbf{L}_{\sum_{d=1}^{2} {}_d\mathbb{E}_\mathrm{l}}(\mathcal{Z}_2)$

(2_r) Take $\mathcal{Z}_1 = \{k_1\}$ and $\mathcal{Z}_2 = \{k_3\}$, then we have ${}_1\mathbf{L}_{\sum_{d=1}^{2} {}_d\mathbb{E}_\mathrm{r}}(\mathcal{Z}_1) = \varnothing$, ${}_1\mathbf{L}_{\sum_{d=1}^{2} {}_d\mathbb{E}_\mathrm{r}}(\mathcal{Z}_2) = \{k_3\}$ and ${}_1\mathbf{L}_{\sum_{d=1}^{2} {}_d\mathbb{E}_\mathrm{r}}$

$(\mathscr{Z}_1 \cup \mathscr{Z}_2) = \{k_1, k_3\}$. Thus, ${}_1\mathbf{L}_{\sum_{d=1}^{2} {}_d\mathbb{E}_r}(\mathscr{Z}_1 \cup \mathscr{Z}_2) \neq {}_1\mathbf{L}_{\sum_{d=1}^{2} {}_d\mathbb{E}_r}(\mathscr{Z}_1) \cup {}_1\mathbf{L}_{\sum_{d=1}^{2} {}_d\mathbb{E}_r}(\mathscr{Z}_2)$

(2_l) Take $\mathscr{Z}_1 = \{k_2\}$ and $\mathscr{Z}_2 = \{k_4\}$, then we have ${}_1\mathbf{L}_{\sum_{d=1}^{2} {}_d\mathbb{E}_l}(\mathscr{Z}_1) = \varnothing$, ${}_1\mathbf{L}_{\sum_{d=1}^{2} {}_d\mathbb{E}_l}(\mathscr{Z}_2) = \{k_4\}$ and ${}_1\mathbf{L}_{\sum_{d=1}^{2} {}_d\mathbb{E}_l}(\mathscr{Z}_1 \cup \mathscr{Z}_2) = \{k_2, k_4\}$. Thus, ${}_1\mathbf{L}_{\sum_{d=1}^{2} {}_d\mathbb{E}_l}(\mathscr{Z}_1 \cup \mathscr{Z}_2) \neq {}_1\mathbf{L}_{\sum_{d=1}^{2} {}_d\mathbb{E}_l}(\mathscr{Z}_1) \cup {}_1\mathbf{L}_{\sum_{d=1}^{2} {}_d\mathbb{E}_l}(\mathscr{Z}_2)$

(3_r) Take $\mathscr{Z}_1 = \{k_2\}$ and $\mathscr{Z}_2 = \{k_3\}$, then we have ${}_1\mathbf{U}_{\sum_{d=1}^{2} {}_d\mathbb{E}_r}(\mathscr{Z}_1) = \{k_2\}$, ${}_1\mathbf{U}_{\sum_{d=1}^{2} {}_d\mathbb{E}_r}(\mathscr{Z}_2) = \{k_1, k_3\}$ and ${}_1\mathbf{U}_{\sum_{d=1}^{2} {}_d\mathbb{E}_r}(\mathscr{Z}_1 \cup \mathscr{Z}_2) = \{k_1, k_2, k_3, k_4\}$. Thus, ${}_1\mathbf{U}_{\sum_{d=1}^{2} {}_d\mathbb{E}_r}(\mathscr{Z}_1 \cup \mathscr{Z}_2) \neq {}_1\mathbf{U}_{\sum_{d=1}^{2} {}_d\mathbb{E}_r}(\mathscr{Z}_1) \cup {}_1\mathbf{U}_{\sum_{d=1}^{2} {}_d\mathbb{E}_r}(\mathscr{Z}_2)$

(3_l) Take $\mathscr{Z}_1 = \{k_1\}$ and $\mathscr{Z}_2 = \{k_2\}$, then we have ${}_1\mathbf{U}_{\sum_{d=1}^{2} {}_d\mathbb{E}_l}(\mathscr{Z}_1) = \{k_1, k_3\}$, ${}_1\mathbf{U}_{\sum_{d=1}^{2} {}_d\mathbb{E}_l}(\mathscr{Z}_2) = \{k_2\}$ and ${}_1\mathbf{U}_{\sum_{d=1}^{2} {}_d\mathbb{E}_l}(\mathscr{Z}_1 \cup \mathscr{Z}_2) = \{k_1, k_2, k_3, k_5\}$. Thus, ${}_1\mathbf{U}_{\sum_{d=1}^{2} {}_d\mathbb{E}_l}(\mathscr{Z}_1 \cup \mathscr{Z}_2) \neq {}_1\mathbf{U}_{\sum_{d=1}^{2} {}_d\mathbb{E}_l}(\mathscr{Z}_1) \cup {}_1\mathbf{U}_{\sum_{d=1}^{2} {}_d\mathbb{E}_l}(\mathscr{Z}_2)$

(4_r) Take $\mathscr{Z}_1 = \{k_2, k_3\}$ and $\mathscr{Z}_2 = \{k_2, k_4\}$, then we have ${}_1\mathbf{U}_{\sum_{d=1}^{2} {}_d\mathbb{E}_r}(\mathscr{Z}_1) = \{k_1, k_2, k_3, k_4\}$, ${}_1\mathbf{U}_{\sum_{d=1}^{2} {}_d\mathbb{E}_r}(\mathscr{Z}_2) = \{k_2, k_4\}$ and ${}_1\mathbf{U}_{\sum_{d=1}^{2} {}_d\mathbb{E}_r}(\mathscr{Z}_1 \cap \mathscr{Z}_2) = \{k_2\}$. Thus, ${}_1\mathbf{U}_{\sum_{d=1}^{2} {}_d\mathbb{E}_r}(\mathscr{Z}_1 \cap \mathscr{Z}_2) \neq {}_1\mathbf{U}_{\sum_{d=1}^{2} {}_d\mathbb{E}_r}(\mathscr{Z}_1) \cap {}_1\mathbf{U}_{\sum_{d=1}^{2} {}_d\mathbb{E}_r}(\mathscr{Z}_2)$

(4_l) Take $\mathscr{Z}_1 = \{k_2, k_4\}$ and $\mathscr{Z}_2 = \{k_2, k_5\}$, then we have ${}_1\mathbf{U}_{\sum_{d=1}^{2} {}_d\mathbb{E}_l}(\mathscr{Z}_1) = \{k_2, k_4, k_5\}$, ${}_1\mathbf{U}_{\sum_{d=1}^{2} {}_d\mathbb{E}_l}(\mathscr{Z}_2) = \{k_2, k_5\}$ and ${}_1\mathbf{U}_{\sum_{d=1}^{2} {}_d\mathbb{E}_l}(\mathscr{Z}_1 \cap \mathscr{Z}_2) = \{k_2\}$. Thus, ${}_1\mathbf{U}_{\sum_{d=1}^{2} {}_d\mathbb{E}_l}(\mathscr{Z}_1 \cap \mathscr{Z}_2) \neq {}_1\mathbf{U}_{\sum_{d=1}^{2} {}_d\mathbb{E}_l}(\mathscr{Z}_1) \cap {}_1\mathbf{U}_{\sum_{d=1}^{2} {}_d\mathbb{E}_l}$

4. Relationships among Different Proposed Models

Next, we present the relationships between the proposed $M\mathbb{E}_q$CAS models.

By using Definition 13, we obtain the following properties.

Proposition 17. *Let $(\mathbb{Q}, \mathfrak{R}, \mathbb{E}_q)$ be a $M\mathbb{E}_q$CAS and $\mathcal{Z} \subseteq \mathbb{Q}$. Then, we have the following results:*

(1) ${}_4\mathbf{L}_{\sum_{d=1}^{n} {}_d\mathbb{E}_q}(\mathcal{Z}) \subseteq {}_2\mathbf{L}_{\sum_{d=1}^{n} {}_d\mathbb{E}_q}(\mathcal{Z}) \subseteq {}_1\mathbf{L}_{\sum_{d=1}^{n} {}_d\mathbb{E}_q}(\mathcal{Z}) \subseteq \mathcal{Z}$

(2) ${}_4\mathbf{L}_{\sum_{d=1}^{n} {}_d\mathbb{E}_q}(\mathcal{Z}) \subseteq {}_3\mathbf{U}_{\sum_{d=1}^{n} {}_d\mathbb{E}_q}(\mathcal{Z}) \subseteq {}_1\mathbf{L}_{\sum_{d=1}^{n} {}_d\mathbb{E}_q}(\mathcal{Z}) \subseteq \mathcal{Z}$

(3) $\mathcal{Z} \subseteq {}_1\mathbf{U}_{\sum_{d=1}^{n} {}_d\mathbb{E}_q}(\mathcal{Z}) \subseteq {}_2\mathbf{U}_{\sum_{d=1}^{n} {}_d\mathbb{E}_q}(\mathcal{Z}) \subseteq {}_4\mathbf{U}_{\sum_{d=1}^{n} {}_d\mathbb{E}_q}(\mathcal{Z})$

(4) $\mathcal{Z} \subseteq {}_1\mathbf{U}_{\sum_{d=1}^{n} {}_d\mathbb{E}_q}(\mathcal{Z}) \subseteq {}_3\mathbf{U}_{\sum_{d=1}^{n} {}_d\mathbb{E}_q}(\mathcal{Z}) \subseteq {}_4\mathbf{U}_{\sum_{d=1}^{n} {}_d\mathbb{E}_q}(\mathcal{Z})$

Remark 18. Let $(\mathbb{Q}, \mathfrak{Re}, \mathbb{E}_q)$ be a $M\mathbb{E}_q$CAS and $\mathcal{Z} \subseteq \mathbb{Q}$. Then, we have the following results:

(1) ${}_2\mathbf{L}_{\sum_{d=1}^{n} {}_d\mathbb{E}_q}(\mathcal{Z}) \nsubseteq {}_3\mathbf{L}_{\sum_{d=1}^{n} {}_d\mathbb{E}_q}(\mathcal{Z})$ and ${}_3\mathbf{L}_{\sum_{d=1}^{n} {}_d\mathbb{E}_q}(\mathcal{Z}) \nsubseteq {}_2\mathbf{L}_{\sum_{d=1}^{n} {}_d\mathbb{E}_q}(\mathcal{Z})$

(2) ${}_2\mathbf{U}_{\sum_{d=1}^{n} {}_d\mathbb{E}_q}(\mathcal{Z}) \nsubseteq {}_3\mathbf{U}_{\sum_{d=1}^{n} {}_d\mathbb{E}_q}(\mathcal{Z})$ and ${}_3\mathbf{U}_{\sum_{d=1}^{n} {}_d\mathbb{E}_q}(\mathcal{Z}) \nsubseteq {}_2\mathbf{U}_{\sum_{d=1}^{n} {}_d\mathbb{E}_q}(\mathcal{Z})$

This means that 2-qMGCRS and 3-qMGCRS are independent.

Proposition 19. *Let $(\mathbb{Q}, \mathfrak{Re}, \mathbb{E}_q)$ be a $M\mathbb{E}_q$CAS and $\mathcal{Z} \subseteq \mathbb{Q}$. Then, we have the following results:*

(1) ${}_1\mathbf{L}_{\sum_{d=1}^{n} {}_d\mathbb{E}_q}(\mathcal{Z}) = {}_2\mathbf{L}_{\sum_{d=1}^{n} {}_d\mathbb{E}_q}(\mathcal{Z}) \cup {}_3\mathbf{L}_{\sum_{d=1}^{n} {}_d\mathbb{E}_q}(\mathcal{Z})$

(2) ${}_1\mathbf{U}_{\sum_{d=1}^{n} {}_d\mathbb{E}_q}(\mathcal{Z}) = {}_2\mathbf{U}_{\sum_{d=1}^{n} {}_d\mathbb{E}_q}(\mathcal{Z}) \cap {}_3\mathbf{U}_{\sum_{d=1}^{n} {}_d\mathbb{E}_q}(\mathcal{Z})$

To illustrate the above characteristic, we give the following example.

Example 6. Consider Example 4 and let $\mathcal{Z} = \{k_1, k_2\}$. Then, we have the following outcomes:

(1) For $q = \mathbf{r}$, we have

$$\begin{aligned} {}_1\mathbf{L}_{\sum_{d=1}^{2} {}_d\mathbb{E}_q}(\mathcal{Z}) &= {}_2\mathbf{L}_{\sum_{d=1}^{2} {}_d\mathbb{E}_q}(\mathcal{Z}) \\ &= \{k_2\},\ {}_3\mathbf{L}_{\sum_{d=1}^{2} {}_d\mathbb{E}_q}(\mathcal{Z}) \\ &= {}_4\mathbf{L}_{\sum_{d=1}^{2} {}_d\mathbb{E}_q}(\mathcal{Z}) = \varnothing, \\ {}_1\mathbf{U}_{\sum_{d=1}^{2} {}_d\mathbb{E}_q}(\mathcal{Z}) &= {}_2\mathbf{U}_{\sum_{d=1}^{2} {}_d\mathbb{E}_q}(\mathcal{Z}) = {}_3\mathbf{U}_{\sum_{d=1}^{2} {}_d\mathbb{E}_q}(\mathcal{Z}) \\ &= \{k_1, k_2\},\ {}_4\mathbf{U}_{\sum_{d=1}^{2} {}_d\mathbb{E}_q}(\mathcal{Z}) = \{k_1, k_2, k_3, k_5\} \end{aligned} \tag{22}$$

(2) For $q = \mathbf{l}$, we have

$$\begin{aligned} {}_1\mathbf{L}_{\sum_{d=1}^{2} = {}_d\mathbb{E}_q}(\mathcal{Z}) &= {}_2\mathbf{L}_{\sum_{d=1}^{2} {}_d\mathbb{E}_q}(\mathcal{Z}) = {}_3\mathbf{L}_{\sum_{d=1}^{2} {}_d\mathbb{E}_q}(\mathcal{Z}) \\ &= \{k_1\},\ {}_4\mathbf{L}_{\sum_{d=1}^{2} {}_d\mathbb{E}_q}(\mathcal{Z}) = \varnothing, \\ {}_1\mathbf{U}_{\sum_{d=1}^{2} {}_d\mathbb{E}_q}(\mathcal{Z}) &= {}_2\mathbf{U}_{\sum_{d=1}^{2} {}_d\mathbb{E}_q}(\mathcal{Z}) = {}_3\mathbf{U}_{\sum_{d=1}^{2} {}_d\mathbb{E}_q}(\mathcal{Z}) \\ &= \{k_1, k_2, k_3, k_5\},\ {}_4\mathbf{U}_{\sum_{d=1}^{2} {}_d\mathbb{E}_q}(\mathcal{Z}) = \mathbb{Q} \end{aligned} \tag{23}$$

So, you can find the following:

(1) ${}_4\mathbf{L}_{\sum_{d=1}^{n} {}_d\mathbb{E}_\mathbf{r}}(\mathcal{Z}) \subseteq {}_2\mathbf{L}_{\sum_{d=1}^{n} {}_d\mathbb{E}_\mathbf{r}}(\mathcal{Z}) \subseteq {}_1\mathbf{L}_{\sum_{d=1}^{n} {}_d\mathbb{E}_\mathbf{r}}(\mathcal{Z}) \subseteq \mathcal{Z}$

(2) ${}_4\mathbf{L}_{\sum_{d=1}^{n} {}_d\mathbb{E}_\mathbf{r}}(\mathcal{Z}) \subseteq {}_3\mathbf{L}_{\sum_{d=1}^{n} {}_d\mathbb{E}_\mathbf{r}}(\mathcal{Z}) \subseteq {}_1\mathbf{L}_{\sum_{d=1}^{n} {}_d\mathbb{E}_\mathbf{r}}(\mathcal{Z}) \subseteq \mathcal{Z}$

(3) $\mathcal{Z} \subseteq {}_1\mathbf{U}_{\sum_{d=1}^{n} {}_d\mathbb{E}_\mathbf{r}}(\mathcal{Z}) \subseteq {}_2\mathbf{U}_{\sum_{d=1}^{n} {}_d\mathbb{E}_\mathbf{r}}(\mathcal{Z}) \subseteq {}_4\mathbf{U}_{\sum_{d=1}^{n} {}_d\mathbb{E}_\mathbf{r}}(\mathcal{Z})$

Table 1: Table for the lower approximations.

	${}_1\mathbf{L}_{\sum_{d=1}^{n} {}_d\mathbb{E}_{\mathbf{r}}}(\mathscr{Z})$	${}_2\mathbf{L}_{\sum_{d=1}^{n} {}_d\mathbb{E}_{\mathbf{r}}}(\mathscr{Z})$	${}_3\mathbf{L}_{\sum_{d=1}^{n} {}_d\mathbb{E}_{\mathbf{r}}}(\mathscr{Z})$	${}_4\mathbf{L}_{\sum_{d=1}^{n} {}_d\mathbb{E}_{\mathbf{r}}}(\mathscr{Z})$	${}_1\mathbf{L}_{\sum_{d=1}^{n} {}_d\mathbb{E}_{\mathbf{l}}}(\mathscr{Z})$	${}_2\mathbf{L}_{\sum_{d=1}^{n} {}_d\mathbb{E}_{\mathbf{l}}}(\mathscr{Z})$	${}_3\mathbf{L}_{\sum_{d=1}^{n} {}_d\mathbb{E}_{\mathbf{l}}}(\mathscr{Z})$	${}_4\mathbf{L}_{\sum_{d=1}^{n} {}_d\mathbb{E}_{\mathbf{l}}}(\mathscr{Z})$
L_1	√	√	√	√	√	√	√	√
L_2	√	√	√	√	√	√	√	√
L_3	√	√	√	√	√	√	√	√
L_4	×	×	×	×	×	×	×	×
L_5	√	√	√	√	√	√	√	√
L_6	√	√	√	√	√	√	√	√
L_7	√	√	√	√	√	√	√	√
L_8	√	√	√	×	√	√	√	×
L_9	×	×	×	×	×	×	×	×

Table 2: Table for the upper approximations.

	${}_1\mathbf{U}_{\sum_{d=1}^{n} {}_d\mathbb{E}_{\mathbf{r}}}(\mathscr{Z})$	${}_2\mathbf{U}_{\sum_{d=1}^{n} {}_d\mathbb{E}_{\mathbf{r}}}(\mathscr{Z})$	${}_3\mathbf{U}_{\sum_{d=1}^{n} {}_d\mathbb{E}_{\mathbf{r}}}(\mathscr{Z})$	${}_4\mathbf{U}_{\sum_{d=1}^{n} {}_d\mathbb{E}_{\mathbf{r}}}(\mathscr{Z})$	${}_1\mathbf{U}_{\sum_{d=1}^{n} {}_d\mathbb{E}_{\mathbf{l}}}(\mathscr{Z})$	${}_2\mathbf{U}_{\sum_{d=1}^{n} {}_d\mathbb{E}_{\mathbf{l}}}(\mathscr{Z})$	${}_3\mathbf{U}_{\sum_{d=1}^{n} {}_d\mathbb{E}_{\mathbf{l}}}(\mathscr{Z})$	${}_4\mathbf{U}_{\sum_{d=1}^{n} {}_d\mathbb{E}_{\mathbf{l}}}(\mathscr{Z})$
H_1	√	√	√	√	√	√	√	√
H_2	√	√	√	√	√	√	√	√
H_3	√	√	√	√	√	√	√	√
H_4	×	×	×	×	×	×	×	×
H_5	√	√	√	√	√	√	√	√
H_6	√	√	√	√	√	√	√	√
H_7	√	√	√	√	√	√	√	√
H_8	√	√	√	×	√	√	√	×
H_9	×	×	×	×	×	×	×	×

(4) $\mathscr{Z} \subseteq {}_1\mathbf{U}_{\sum_{d=1}^{n} {}_d\mathbb{E}_{\mathbf{r}}}(\mathscr{Z}) \subseteq {}_3\mathbf{U}_{\sum_{d=1}^{n} {}_d\mathbb{E}_{\mathbf{r}}}(\mathscr{Z}) \subseteq {}_4\mathbf{U}_{\sum_{d=1}^{n} {}_d\mathbb{E}_{\mathbf{r}}}(\mathscr{Z})$

(5) ${}_4\mathbf{L}_{\sum_{d=1}^{n} {}_d\mathbb{E}_{\mathbf{l}}}(\mathscr{Z}) \subseteq {}_2\mathbf{L}_{\sum_{d=1}^{n} {}_d\mathbb{E}_{\mathbf{l}}}(\mathscr{Z}) \subseteq {}_1\mathbf{L}_{\sum_{d=1}^{n} {}_d\mathbb{E}_{\mathbf{l}}}(\mathscr{Z}) \subseteq \mathscr{Z}$

(6) ${}_4\mathbf{L}_{\sum_{d=1}^{n} {}_d\mathbb{E}_{\mathbf{l}}}(\mathscr{Z}) \subseteq {}_3\mathbf{L}_{\sum_{d=1}^{n} {}_d\mathbb{E}_{\mathbf{l}}}(\mathscr{Z}) \subseteq {}_1\mathbf{L}_{\sum_{d=1}^{n} {}_d\mathbb{E}_{\mathbf{l}}}(\mathscr{Z}) \subseteq \mathscr{Z}$

(7) $\mathscr{Z} \subseteq {}_1\mathbf{U}_{\sum_{d=1}^{n} {}_d\mathbb{E}_{\mathbf{l}}}(\mathscr{Z}) \subseteq {}_2\mathbf{U}_{\sum_{d=1}^{n} {}_d\mathbb{E}_{\mathbf{l}}}(\mathscr{Z}) \subseteq {}_4\mathbf{U}_{\sum_{d=1}^{n} {}_d\mathbb{E}_{\mathbf{l}}}(\mathscr{Z})(\mathscr{Z})$

(8) $\mathscr{Z} \subseteq {}_1\mathbf{U}_{\sum_{d=1}^{n} {}_d\mathbb{E}_{\mathbf{l}}}(\mathscr{Z}) \subseteq {}_3\mathbf{U}_{\sum_{d=1}^{n} {}_d\mathbb{E}_{\mathbf{l}}}(\mathscr{Z}) \subseteq {}_4\mathbf{U}_{\sum_{d=1}^{n} {}_d\mathbb{E}_{\mathbf{l}}}(\mathscr{Z})$

Tables 1 and 2 show the Pawlak characteristics for the lower and upper approximations which are given in Definition 13.

5. Relative Reduction of a $\mathrm{M}\mathbb{E}_q$CAS

This section is aimed at discussing a relative reduction of a pessimistic multigranulation q-covering rough sets (briefly, $\mathrm{PM}\mathbb{E}_q$CRS). First, we give the following couple of definitions.

Definition 20. Let $(\mathbb{Q}, \mathfrak{R}, \mathbb{E}_q)$ be a $\mathrm{M}\mathbb{E}_q$CAS and $\mathfrak{R} = \{\mathscr{R}_1, \mathscr{R}_2, \cdots, \mathscr{R}_S\}, \forall S \in I$. For any $\mathscr{Z} \subseteq \mathbb{Q}$ and $w \in \mathbb{Q}$, define the pessimistic multigranulation q-covering lower approximation (briefly, $\mathrm{PMG}\mathbb{E}_q$CLA) $\boldsymbol{L}^P_{\sum_{d=1}^{n} {}_d\mathbb{E}_q}(\mathscr{Z})$ and pessimistic multigranulation q-covering lower approximation (briefly, $\mathrm{PMG}\mathbb{E}_q$CLA) $\boldsymbol{U}^P_{\sum_{d=1}^{n} {}_d\mathbb{E}_q}(\mathscr{Z})$ as follows:

$$
\begin{aligned}
\mathbf{L}^P_{\sum_{d=1}^{n} {}_d\mathbb{E}_q}(\mathscr{Z}) &= \Big\{ w \in \mathbb{Q} : (w)^{\mathscr{R}_1}_{\mathbb{E}_q} \subseteq \mathscr{Z} \text{ and } (w)^{\mathscr{R}_2}_{\mathbb{E}_q} \subseteq \mathscr{Z} \\
&\quad \text{and} \cdots \text{and } (w)^{\mathscr{R}_n}_{\mathbb{E}_q} \subseteq \mathscr{Z} \Big\}, \\
\mathbf{U}^P_{\sum_{d=1}^{n} {}_d\mathbb{E}_q}(\mathscr{Z}) &= \Big\{ w \in \mathbb{Q} : (w)^{\mathscr{R}_1}_{\mathbb{E}_q} \cap \mathscr{Z} \neq \varnothing \text{or } (w)^{\mathscr{R}_2}_{\mathbb{E}_q} \cap \mathscr{Z} \neq \varnothing \\
&\quad \text{or} \cdots \text{or } (w)^{\mathscr{R}_n}_{\mathbb{E}_q} \cap \mathscr{Z} \neq \varnothing \Big\}.
\end{aligned} \tag{24}
$$

Definition 21. Let $(\mathbb{Q}, \mathfrak{R}, \mathbb{E}_q)$ be a $\mathrm{M}\mathbb{E}_q$CAS and $\mathfrak{R} = \{\mathscr{R}_1, \mathscr{R}_2, \cdots, \mathscr{R}_S\}, \forall S \in I$. Suppose that $\mathscr{D} = \{\mathscr{D}_1, \mathscr{D}_2, \cdots, \mathscr{D}_t\}$ is a decision partition of $\mathbb{Q}$. Then,

$$
{}^P\mathbf{L}^{\mathscr{R}_k}_{\mathbb{E}_q}(\mathscr{D}) = \left[\mathbf{L}^P_{\sum_{d=1}^{n} {}_d\mathbb{E}_q}(\mathscr{D}_1), \mathbf{L}^P_{\sum_{d=1}^{n} {}_d\mathbb{E}_q}(\mathscr{D}_2), \cdots, \mathbf{L}^P_{\sum_{d=1}^{n} {}_d\mathbb{E}_q}(\mathscr{D}_t) \right],
$$

Input: $(\mathbb{Q}, \mathfrak{R}, \mathbb{E}_q)$ with information system.
Output: Reduction of PM$\mathbb{E}_q$CLA.
1: Calculate ${}^{P}\mathbf{L}_{\mathbb{E}_q}^{\mathscr{R}_k}(\mathscr{D})$.
2: Remove $\mathbb{E}_q^{\mathscr{R}_k}$, $\mathscr{B}_q^{\mathscr{R}_k} = \mathbb{E}_q^{\mathscr{R}_i} - \mathbb{E}_q^{\mathscr{R}_k}$ and ${}^{P}\mathbf{L}_{\mathscr{B}_q}^{\mathscr{R}_{i-k}}(\mathscr{D}) = {}^{P}\mathbf{L}_{\mathscr{B}_q}^{\mathscr{R}_i}(\mathscr{D})$.
3: Remove a covering in $\mathscr{B}_q^{\mathscr{R}_k}$ again and get $\mathscr{B}\wedge_q^{\mathscr{R}_{i-k}}$. If ${}^{P}\mathbf{L}_{\mathscr{B}\wedge_q}^{\mathscr{R}_{i-k}}(\mathscr{D}) \neq {}^{P}\mathbf{L}_{\mathscr{B}_q}^{\mathscr{R}_i}(\mathscr{D})$, return $\mathscr{B}_q^{\mathscr{R}_k}$; else, go to **Step 2**.
4: : Repeat the Steps 2 and 3 for each covering in $\mathbb{E}_q^{\mathscr{R}_i}$ to get all the relative reduce of the covering family.

Algorithm 1: Algorithm for reduction of PM$\mathbb{E}_q$CLA.

Table 3: Table for house assessment problem.

	Equally shared area	Color	Price	Surrounding	Purchase options
k_1	{Large}	{Good}	{High}	{Very noisy}	Oppose
k_2	{Small, large}	{Excellent}	{Middle, low}	{Quiet, noisy}	Support
k_3	{Small, large}	{Excellent, good}	{Middle, low}	{Noisy}	Support
k_4	{Small, ordinary}	{Bad}	{High, middle}	{Noisy, very noisy}	Oppose
k_5	{Small, ordinary}	{Bad}	{High, middle}	{Very noisy}	Oppose
k_6	{Ordinary, large}	{Excellent, good}	{High, low}	{Quiet, noisy}	Support

$$ {}^{P}\mathbf{U}_{\mathbb{E}_q}^{\mathscr{R}_k}(\mathscr{D}) = \left[\mathbf{U}^{P}_{n \atop \sum_{d=1} {}_d\mathbb{E}_q}(\mathscr{D}_1), \mathbf{U}^{P}_{n \atop \sum_{d=1} {}_d\mathbb{E}_q}(\mathscr{D}_2), \cdots, \mathbf{U}^{P}_{n \atop \sum_{d=1} {}_d\mathbb{E}_q}(\mathscr{D}_t) \right]. \tag{25} $$

(i) $\mathscr{B}_q^{\mathscr{R}_k} \subseteq \mathbb{E}_q^{\mathscr{R}_k}$ and ${}^{P}\mathbf{L}_{\mathscr{B}_q}^{\mathscr{R}_k}(\mathscr{D}) = {}^{P}\mathbf{L}_{\mathbb{E}_q}^{\mathscr{R}_k}(\mathscr{D})$, but ${}^{P}\mathbf{L}_{\mathscr{B}\wedge_q}^{\mathscr{R}_k}(\mathscr{D}) \neq {}^{P}\mathbf{L}_{\mathbb{E}_q}^{\mathscr{R}_k}(\mathscr{D})$, for $\mathscr{B}\wedge_q^{\mathscr{R}_k} \subseteq \mathscr{B}_q^{\mathscr{R}_k}$; then, $\mathscr{B}_q^{\mathscr{R}_k}$ is a $\mathscr{D}$ reduction of PM$\mathbb{E}_q$CLA

(ii) $\mathscr{B}_q^{\mathscr{R}_k} \subseteq \mathbb{E}_q^{\mathscr{R}_k}$ and ${}^{P}\mathbf{U}_{\mathscr{B}_q}^{\mathscr{R}_k}(\mathscr{D}) = {}^{P}\mathbf{U}_{\mathbb{E}_q}^{\mathscr{R}_k}(\mathscr{D})$, but ${}^{P}\mathbf{U}_{\mathscr{B}\wedge_q}^{\mathscr{R}_k}(\mathscr{D}) \neq {}^{P}\mathbf{U}_{\mathbb{E}_q}^{\mathscr{R}_k}(\mathscr{D})$, for $\mathscr{B}\wedge_q^{\mathscr{R}_k} \subseteq \mathscr{B}_q^{\mathscr{R}_k}$, then, $\mathscr{B}_q^{\mathscr{R}_k}$ is a $\mathscr{D}$ reduction of PM$\mathbb{E}_q$CUA

We can illustrate the method of reduction as the following Algorithm 1.

Example 7. Presume that $\mathbb{Q} = \{k_1, k_2, \cdots, k_6\}$ is a set of six houses, $\mathscr{Z} = \{\text{equally shared area, color, price, surroundings}\}$ is a set of attributes, and $D = \{\text{purchase opinions}\}$ is a set of decisions. The values of equally shared area could be {large, ordinary, small}. The values of color could be {excellent, good, bad}. The values of price could be {high, middle, low}. The values of surroundings could be {quiet, noisy, very noisy}. The decision values of purchase opinions could be {support, oppose}, which is randomly chosen from experts. The evaluation results are shown in Table 3.

As for the attribute set $\mathscr{Z}$, the binary relation is obtained as follows $\forall k \in \mathscr{Z}$:

$$ \mathscr{R}_k = \{(v, w): \mathscr{F}_k(v) \subseteq \mathscr{F}_k(w)\}. \tag{26} $$

It is easy to see that the $\mathscr{R}_k$ is reflexive and transitive but not symmetric.

If $\mathscr{D}$ is the decision set, then the nonequivalence relation is defined as follows:

$$ \mathscr{R}_{\mathscr{D}} = \{(v, w): \mathscr{F}_{\mathscr{D}}(v) \subseteq \mathscr{F}_{\mathscr{D}}(w)\}. \tag{27} $$

Then, we can construct the following two covers:

(i) Right covering (**r**-cover for short)

$$ \mathscr{C}_{\mathrm{r}} = \left\{ w\mathscr{R}_{\mathscr{K}} : \forall \mathscr{K} \in \{\mathscr{Z}, \mathscr{D}\}, \quad w \in \mathbb{Q}, \mathbb{Q} = \bigcup_{w \in \mathbb{Q}} w\mathscr{R}_{\mathscr{K}} \right\} \tag{28} $$

(ii) Left covering (**l**-cover for short)

$$ \mathscr{C}_{\mathrm{l}} = \left\{ \mathscr{R}_{\mathscr{K}}w : \forall \mathscr{K} \in \{\mathscr{Z}, \mathscr{D}\}, \quad w \in \mathbb{Q}, \mathbb{Q} = \bigcup_{w \in \mathbb{Q}} \mathscr{R}_{\mathscr{K}}w \right\} \tag{29} $$

So, we have the following results:

$$ \mathscr{C}_{\mathrm{r}}^{\mathscr{R}_1} = \{\{k_1, k_2, k_3\}, \{k_2, k_3\}, \{k_4, k_5\}, \{k_6\}\}, $$

$$ \mathscr{C}_{\mathrm{r}}^{\mathscr{R}_2} = \{\{k_1, k_3, k_6\}, \{k_2, k_3, k_6\}, \{k_3, k_6\}, \{k_4, k_5\}\}, $$

Table 4: Table for house assessment problem.

	k_1	k_2	k_3	k_4	k_5	k_6
$k\mathscr{C}_r^{\mathfrak{R}_1}$	$\{k_1, k_2, k_3\}$	$\{k_2, k_3\}$	$\{k_2, k_3\}$	$\{k_4, k_5\}$	$\{k_4, k_5\}$	$\{k_6\}$
$k\mathscr{C}_r^{\mathfrak{R}_2}$	$\{k_1, k_3, k_6\}$	$\{k_2, k_3, k_6\}$	$\{k_3, k_6\}$	$\{k_4, k_5\}$	$\{k_4, k_5\}$	$\{k_3, k_6\}$
$k\mathscr{C}_r^{\mathfrak{R}_3}$	$\{k_1, k_4, k_5, k_6\}$	$\{k_2, k_3\}$	$\{k_2, k_3\}$	$\{k_4, k_5\}$	$\{k_4, k_5\}$	$\{k_6\}$
$k\mathscr{C}_r^{\mathfrak{R}_4}$	$\{k_1, k_4, k_5\}$	$\{k_2, k_6\}$	$\{k_2, k_3, k_6\}$	$\{k_4\}$	$\{k_1, k_4, k_5\}$	$\{k_2, k_6\}$

Table 5: Table for house assessment problem.

	k_1	k_2	k_3	k_4	k_5	k_6
$k\mathscr{C}_1^{\mathfrak{R}_1}$	$\{k_1\}$	$\{k_1, k_2, k_3\}$	$\{k_1, k_2, k_3\}$	$\{k_4, k_5\}$	$\{k_4, k_5\}$	$\{k_6\}$
$k\mathscr{C}_1^{\mathfrak{R}_2}$	$\{k_1\}$	$\{k_2\}$	$\{k_1, k_2, k_3, k_6\}$	$\{k_4, k_5\}$	$\{k_4, k_5\}$	$\{k_1, k_2, k_3, k_6\}$
$k\mathscr{C}_1^{\mathfrak{R}_3}$	$\{k_1\}$	$\{k_2, k_3\}$	$\{k_2, k_3\}$	$\{k_1, k_4, k_5\}$	$\{k_1, k_4, k_5\}$	$\{k_1, k_6\}$
$k\mathscr{C}_1^{\mathfrak{R}_4}$	$\{k_1, k_5\}$	$\{k_2, k_3, k_6\}$	$\{k_3\}$	$\{k_1, k_3, k_4, k_5\}$	$\{k_1, k_5\}$	$\{k_2, k_3, k_6\}$

$$\mathscr{C}_r^{\mathscr{R}_3} = \{\{k_1, k_4, k_5, k_6\}, \{k_2, k_3\}, \{k_4, k_5\}, \{k_6\}\},$$

$$\mathscr{C}_r^{\mathscr{R}_4} = \{\{k_1, k_4, k_5\}, \{k_2, k_6\}, \{k_2, k_3, k_4, k_6\}, \{k_4\}\},$$

$$\mathscr{C}_r^{\mathscr{R}_\mathscr{D}} = \{\{k_1, k_4, k_5\}, \{k_2, k_3, k_6\}\},$$

$$\mathscr{C}_1^{\mathscr{R}_1} = \{\{k_1\}, \{k_1, k_2, k_3\}, \{k_4, k_5\}, \{k_6\}\},$$

$$\mathscr{C}_1^{\mathscr{R}_2} = \{\{k_1\}, \{k_2\}, \{k_1, k_2, k_3, k_6\}, \{k_4, k_5\}\},$$

$$\mathscr{C}_1^{\mathscr{R}_3} = \{\{k_1\}, \{k_2, k_3\}, \{k_1, k_4, k_5\}, \{k_1, k_6\}\},$$

$$\mathscr{C}_\mathbf{1}^{\mathcal{R}_4} = \{\{k_1, k_5\}, \{k_2, k_3, k_6\}, \{k_3\}, \{k_1, k_3, k_4, k_5\}\},$$

$$\mathscr{C}_1^{\mathscr{R}_\mathscr{D}} = \{\{k_1, k_4, k_5\}, \{k_2, k_3, k_6\}\}. \quad (30)$$

Thus, we can establish Tables 4 and 5 for the neighborhood of k as follows.

Now, we can apply Algorithm 1 as follows.

Step 1. ${}^P\mathbf{L}_{\mathbb{E}_r}^{\mathscr{R}_k}(\mathscr{D}) = [\mathbf{L}^P_{\sum_{d=1}^4 {}_d\mathbb{E}_r}(\mathscr{D}_1), \mathbf{L}^P_{\sum_{d=1}^4 {}_d\mathbb{E}_\mathbf{r}}(\mathscr{D}_2)] = [\{k_4, k_5\}, \{k_2, k_3, k_6\}]$.

Step 2. ${}^P\mathbf{L}_{\mathscr{B}_r}^{\mathcal{R}_1}(\mathscr{D}) = (\{k_4, k_5\}, \{k_2, k_3, k_6\})$, ${}^P\mathbf{L}_{\mathscr{B}_r}^{\mathscr{R}_2}(\mathscr{D}) = (\{k_4, k_5\}, \{k_2, k_3, k_6\})$. Therefore, $\mathscr{B}_r^{\mathscr{R}_k} = \{\mathscr{C}_r^{\mathscr{R}_3}, \mathscr{C}_r^{\mathscr{R}_4}\}$ is a reduction of the $\mathrm{PM}\mathbb{E}_r\mathrm{CRS}$.

Also, we can get the following outcomes of the left covering:

$$
\begin{aligned}
{}^P\mathbf{L}_{\mathbb{E}_1}^{\mathscr{R}_k}(\mathscr{D}) &= \left[\mathbf{L}^P_{\sum_{d=1}^{4} {}_d\mathbb{E}_1}(\mathscr{D}_1), \mathbf{L}^P_{\sum_{d=1}^{4} {}_d\mathbb{E}_1}(\mathscr{D}_2)\right] = [\{k_1, k_5\}, \varnothing],\\
{}^P\mathbf{L}_{\mathscr{B}_1}^{\mathcal{R}_1}(\mathscr{D}) &= (\{k_1, k_5\}, \{k_2\}),\\
{}^P\mathbf{L}_{\mathscr{B}_1}^{\mathcal{R}_2}(\mathscr{D}) &= (\{k_1, k_5\}, \phi),\\
{}^P\mathbf{L}_{\mathscr{B}_1}^{\mathcal{R}_3}(\mathscr{D}) &= (\{k_1, k_5\}, \{k_6\}),\\
{}^P\mathbf{L}_{\mathscr{B}_1}^{\mathcal{R}_4}(\mathscr{D}) &= (\{k_1, k_4, k_5\}, \varnothing).
\end{aligned}
\quad (31)
$$

Therefore, $\mathscr{B}_1^{\mathscr{R}_k} = \{\mathscr{C}_1^{\mathscr{R}_1}, \mathscr{C}_1^{\mathscr{R}_3}, \mathscr{C}_1^{\mathscr{R}_4}\}$ is a reduction of the $\mathrm{PM}\mathbb{E}_1\mathrm{CRS}$.

6. Conclusion

In this article, we present a notion called multi-$\mathbb{E}_q$-covering approximation space ($\mathrm{M}\mathbb{E}_q\mathrm{CAS}$) by using the concept of q-minimal and q-maximal descriptions. Based on these notions, we establish four new types of multigranulation covering rough sets, denoted $\mathrm{M}\mathbb{E}_q\mathrm{CAS}$. We also study the properties of these new models. Further, we put forward a new methodology to make a reduction by the presented work. Then, we demonstrate the reduction method with the help of an illustrative example which shows its effectiveness and reliability. The main differences between our proposed work and the previous one in [39] are that the authors in [39] introduced four types of MGCRSs using the minimal and maximal description based on equivalence relations and here we used the notions of right (resp., left) covering rough sets to investigate four kinds of multigranulation right (resp., left) covering rough sets using the right (resp., left) minimal and right (resp., left) maximal description induced by binary relations. In further research, we hope to use this approach in fuzzy rough covering-based fuzzy neighborhoods [49], fuzzy soft covering-based rough sets [50], and soft fuzzy covering-based rough sets [51].

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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