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Research Article

A Comparison Study of Numerical Techniques for Solving Ordinary Differential Equations Defined on a Semi-Infinite Domain Using Rational Chebyshev Functions

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A rational Chebyshev (RC) spectral collocation technique is considered in this paper to solve high-order linear ordinary differential equations (ODEs) defined on a semi-infinite domain. Two definitions of the derivative of the RC functions are introduced as operational matrices. Also, a theoretical study carried on the RC functions shows that the RC approximation has an exponential convergence. Due to the two definitions, two schemes are presented for solving the proposed linear ODEs on the semi-infinite interval with the collocation approach. According to the convergence of the RC functions at the infinity, the proposed technique deals with the boundary value problem which is defined on semi-infinite domains easily. The main goal of this paper is to present a comparison study for differential equations defined on semi-infinite intervals using the proposed two schemes. To demonstrate the validity of the comparisons, three numerical examples are provided. The obtained numerical results are compared with the exact solutions of the proposed problems.

1. Introduction

In the spectral methods, the most common basis functions are Chebyshev polynomials (CPs), which play an important role in the interpolation problems. Many researchers considered CPs to solve differential equations in the finite domain $[-1, 1]$ (see [1–8] and [9]), but they often fail in the larger domain, also if the exact solution of the problem was in a rational form. For this reason, the rational Chebyshev (RC) functions are applied in the large domain $[0, l]$ where $l \longrightarrow \infty$, which provide a major success in dealing with differential equations (DEs) defined in the open domain $[0, l]$. Many researchers studied RC functions for treating man different problems of differential, integrodifferential equations (IDEs), partial, and some other physical-engineering problems as in [10, 11] and [12]. Abbasbandy et al. [13] applied the RC collocation method to get numerical solution of the magnetohydrodynamic flow (MHF) of an incompressible viscous fluid (VF) over a stretching sheet problem. Ramadan et al. in [14, 15] and [16], Yalçınbas et al. in [17], and Parand and Razzaghi in [18] are scrupulous in the use of RC functions to express the approximate solution of high-order ordinary differential equations (ODEs) by different spectral approaches. In [19], Parand and Razzaghi introduced RC functions for solving a population growth of a species within a closed system, named as a Volterra model, where the authors converted the Volterra population model first to an equivalent nonlinear ODE; the solution is approximated by the RC functions with the unknown coefficients. The authors of [20, 21] introduced the RC function approximation with the collocation technique for solving the natural convection of the Darcian fluid (DF) about a vertical full cone embedded in porous media (PM) with a prescribed wall temperature. Ramadan et al. [22, 23] obtained an approximate solution of the applied collocation method based on RC functions to treat high-order linear IDEs with variable coefficients. In [24], the authors applied the RC collocation approach for approximating nonlinear biomathematical problems, namely, the systematic logistic growth, the Lotka-Volterra system (prey-predator model), the simple two-species Lotka-Volterra model, and the prey-predator model with limit cycle (periodic behavior).

All the aforementioned work either relied on the RC functions as a basis defined of an open interval or used it to treat a specific application or used truncated matrices. The truncation in matrices was handled for the first time by us in [15], and it was also an application on open period equations. In this work, a comparison study for solving linear ODEs defined on semi-infinite domains using a spectral collocation method is presented. The equation under investigation is a high-order nonhomogeneous linear variable coefficient ordinary differential equation defined on semi-infinite domain $[0, \infty)$. Two algorithms are considered by the collocation method with the RC functions as basis functions. The matrices of derivative for RC functions introduced with regular definition with truncation and improved one, which lead us to two different schemes. The collocation points that are used here are suggested by us, and they are defined on the interval $[0, \infty)$, which deal with the boundary value problems defend on $[0, \infty)$ or if one of the mixed conditions tends to infinity. So, we suggested new collocation points that are valid for dealing with a problem of this type.

2. Preliminaries

In this section, the definition and some properties of the RC functions are listed; also, the convergence for RC functions will be discussed.

The Chebyshev polynomials $T_n(x)$ are an orthonormal system in the closed interval [-1, 1], where the weight function for $T_n(x)$ is $w_T(x) = 1/\sqrt{1-x^2}$, and they may be generated using the recurrence formulae:

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x), \quad n \geq 1, \tag{1}$$

with the initials

$$T_0(x) = 1, T_1(x) = x. \tag{2}$$

For more details about $T_n(x)$, see Ref. [8].

2.1. The RC Functions. The RC functions are orthonormal on the open interval $[0, \infty)$, defined as

$$R_n(\chi) = T_n\left(\frac{x-1}{x+1}\right), \quad x = \cos\varphi, \varphi \in [0, \pi], \tag{3}$$

and they form an orthonormal set of functions with respect to the weight function $w_R(x) = x^{-1/2}(x+1)^{-1}$, and they can be generated with the aid of the following recurrence formulae:

$$R_{n+1}(x) = 2\left(\frac{x-1}{x+1}\right)R_n(x) - R_{n-1}(x), \quad n \geq 1, \tag{4}$$

with the initials

$$R_0(x) = 1, R_1(x) = \frac{x-1}{x+1}, \tag{5}$$

and the property of the orthogonality is

$$\int_0^\infty R_r(x)R_s(x)w_R(x)dx = \frac{c_s\pi}{2}\delta_{rs}, \tag{6}$$

where $c_0 = 2$, $c_s = 1$ for all $s \geq 1$ and δ_{rs} is the Kronecker delta function.

Let $\Omega = \{x : 0 \leq x < \infty\}$, and we note that $R_n(x)$ is the eigen function of the singular Sturm-Liouville problem of the following form:

$$w_R^{-1}(x)\frac{d}{dx}\left[w_R^{-1}(x)\frac{d}{dx}R_n(x)\right] + n^2R_n(x) = 0, \quad x \in \Omega. \tag{7}$$

2.2. Function Spaces. In this subsection, the order of convergence for RC functions will be discussed; let us begin with assuming that

$$L_w^2(\Omega) = \left\{\xi : \|\xi\|_w = \left(\int_0^\infty |\xi(x)|^2 w_R(x)dx\right)^{1/2} < \infty\right\}, \tag{8}$$

represent the space functions, and the inner product is denoted here as

$$\langle \alpha, \beta \rangle_{w_R}, \tag{9}$$

such that

$$\langle \varphi, \varphi \rangle_w = \left(\|\varphi\|_{w_R}\right)^2. \tag{10}$$

Subsequently, from the property of orthogonality (relation (6)), we get the fact that RC functions form a set of orthonormal basis of $L_w^2(\Omega)$. Also, let us define the normed spaces $H_w^r(\Omega)$ and $H_{w,A}^r(\Omega)$ as

$$H_{w_R}^r(\Omega) = \left\{\xi : \|\xi\|_{r,w_R} = \left(\sum_{k=0}^{r}\left\|\frac{d^k}{dx^k}\xi\right\|_{w_R}\right)^{1/2} < \infty\right\},$$
$$H_{w_R,\theta}^r(\Omega) = \left\{\xi : \|\xi\|_{r,w_R,\theta} = \left(\sum_{k=0}^{r}\left\|(x+1)^{r/2+k}\frac{d^k}{dx^k}\xi\right\|_{w_R}\right)^{1/2} < \infty\right\}, \tag{11}$$

where $r \geq 0$ and k is a positive integer constant, and we let θ be the Sturm-Liouville operator in (7), namely, it may be written as

$$\theta\xi = -w_R^{-1}\left(w_R^{-1}\xi'\right)'. \tag{12}$$

Let N be a positive integer such that $N < \infty$, and $\mathfrak{R}_N = \text{span}\{R_0, R_1, \cdots, R_N\}$.

Theorem 1. *For any $r \geq 0$, and c is a generic positive constant independent of any variable, and $\phi \in \mathfrak{R}_N$, then*

$$\|\xi\|_{r,w_R} \leq cN^{2r}\|\xi\|_{w_R}. \tag{13}$$

The proof of Theorem 1 is given in [25].

Since the set of RC functions is orthonormal and a complete set, we assume that $f(x)$ is defined over the space Ω; then, it be may expanded in terms of RC functions as

$$f(x) = \sum_{i=0}^{\infty} a_i R_i(x), \tag{14}$$

where

$$a_i = \frac{\langle f, R_i\rangle_{w_R}}{\left(\|R_i\|_{w_R}\right)^2} = \frac{2}{c_s\pi}\int_0^\infty f(x)R_i(x)w_R(x)dx. \tag{15}$$

Infinite series expression (14) represents as a spectral truncated approximation as follows

$$f_N(x) = \sum_{i=0}^{N} a_i R_i(x). \tag{16}$$

The order of convergence for the RC function approximation will be obtained using several orthonormal projections. From (16), it is clear that f_N is the orthogonal projection of f onto $\mathfrak{R}_N$ with respect to the inner product (10). For all of the above, especially Theorem 1, the following theorem is presented and contains the order of convergence of RC functions.

Theorem 2. *For any function f such that $f \in H^r_{w_R,\theta}(\Omega)$, where $r \geq 0$, there exists a positive constant c independent of N such that*

$$\|f_N - f\|_{r,w_R} \leq cN^{-r}\|f\|_{r,w_R,\theta}. \tag{17}$$

The complete proof of Theorem 2 is found in [13] (or see Ref. [25] for more details); this theorem shows that the RC approximation has exponential convergence.

2.3. Operational Matrix. This subsection introduces the form of operational matrix for the RC functions; the derivative of the vector $\mathbf{R}(x) = [R_0(x)R_1(x)R_2(x)\cdots R_N(x)]$ can be expressed by

$$\mathbf{R}'(x) = \frac{d\mathbf{R}(x)}{dx} \approx \mathbf{R}(x)\mathbf{D}^T, \tag{18}$$

where $\mathbf{D}$ is $(N+1)\times(N+1)$ operational differentiation matrix. The elements of $\mathbf{D}$ are found by differentiating (4) and using $R_1(x) = (x-1)/(x+1)$, then

$$R'_{n+1}(x) = 2(R_1(x)\cdot R_n(x))' - R'_{n-1}(x), \tag{19}$$

also using the multiplication relation:

$$R_m(x)\cdot R_n(x) = \frac{1}{2}\left[R_{m+n} + R_{|m-n|}\right]. \tag{20}$$

The approximation sign in (18) made by a truncation to the last column of $\mathbf{D}$ (by consideration that $R'_i(x) = 0$, for $i > N$) to get an invertible square operational matrix $\mathbf{D}$ (see Ref. [14]). The structure of the matrix $\mathbf{D}$ is obtained as a lower-Heisenberg matrix. The matrix $\mathbf{D}$ can be expressed as $\mathbf{D} = \mathbf{D}_1 + \mathbf{D}_2$, where $\mathbf{D}_1$ is a tridiagonal matrix which is obtained from

$$\mathbf{D}_1 = \text{diag.}\left(\frac{7}{4}(i-1), -(i-1), \frac{1}{4}(i-1)\right), \quad i = 1, 2, \cdots, N+1, \tag{21}$$

and the entire elements of matrix $\mathbf{D}_2$ are d_{ij}, obtained from

$$d_{21} = -1, d_{ij} = \begin{cases} 0, & j \geq i-1, \\ k(i-1)c_j, & j < i-1. \end{cases} \tag{22}$$

In addition to $k = (-1)^{i+j+1}$, $c_1 = 1$ and $c_j = 2$ for $j \geq 2$.

Consequently, the k^{th}-order derivative of the row matrix $\mathbf{R}(x)$, which is given in (18), is obtained as

$$\frac{d^k\mathbf{R}(x)}{dx^k} = [\mathbf{R}(x)]^{(k)} \approx \mathbf{R}(x)\left(\mathbf{D}^T\right)^k, \tag{23}$$

And we note here that $\mathbf{R}^{(0)}(x) = \mathbf{R}(x)$, $R'_i(x) = 0$, for $i > N$. Definitions (18) and (23) were introduced for the first time in [18, 19], and many works have used them; see, for example, Refs. [20–22], [23, 24], and [17].

3. The Improved Differentiation of the RC Functions

In the present section, an improved definition of differentiation for the RC functions is introduced. There was a need to find an improvement to definition (23), because in the higher derivatives, when using this definition, a week approximation is obtained. The truncated definition (18) and the k^{th}-order derivative (23) give us a regular truncated differentiation of the RC functions (RRC). Generally, the derivative of the rational or fractional functions increases the order of the denominator (in contrast to polynomials that reduce the order at differentiation), so the truncation increases as the order of the derivative increases (more than one column in Equation (18)). For example, the fourth-order derivative of the vector $\mathbf{R}(x)$ at $N = 3$, the truncated terms in the row vector $\mathbf{R}(x)$ are (the last four terms) R_4, R_5, R_6, R_7. This will lead to unsatisfied approximating in high-order DEs using the presented RRC definition (18). Therefore, an improved definition of the derivative of RC functions will be proposed next.

First, a vector will be inserted into (18) to treat the truncated terms, which will improve the regular definition. This manner will be called an improved derivative of the RC functions and will be indicated by IRC.

Additionally, the first-order derivative of the row matrix $R(x)$ is

$$\mathbf{R}'(x) = \frac{d\mathbf{R}(x)}{dx} = \mathbf{R}(x)\mathbf{D}^T + \mathbf{Z}(x), \tag{24}$$

where

$$\mathbf{Z}(x) = [0 \quad 0 \quad 0 \quad \cdots \quad 0 \quad d_{N+1,N+1}R_{N+1}(x)]_{1\times(N+1)}. \tag{25}$$

Theorem 3. *The k^{th} derivative of the matrix vector $\mathbf{R}(x)$, in terms of itself, is defined as*

$$[\mathbf{R}(x)]^{(k)} = \mathbf{R}(x)\left(\mathbf{D}^T\right)^k + \sum_{i=0}^{k-1} \mathbf{Z}^{(i)}(x)\left(\mathbf{D}^T\right)^{k-i-1}, \quad k \geq 1, \tag{26}$$

where

$$\mathbf{Z}^{(k)}(x) = \left[0 \quad 0 \quad 0 \quad \cdots \quad 0 \quad d_{N+1,N+2}\mathbf{R}^{(k)}_{N+1}(x)\right]. \tag{27}$$

Proof. Using the assumption (24) as the first derivative, and differentiating (24), then we get

$$\mathbf{R}''(x) = \mathbf{R}'(x)\mathbf{D}^T + \mathbf{Z}'(x), \tag{28}$$

or

$$\begin{aligned}\mathbf{R}''(x) &= \left\{\mathbf{R}(x)\mathbf{D}^T + \mathbf{Z}(x)\right\}\mathbf{D}^T + \mathbf{Z}'(x)\\ &= \mathbf{R}(x)\left(\mathbf{D}^T\right)^2 + \mathbf{Z}(x)\mathbf{D}^T + \mathbf{Z}'(x),\end{aligned} \tag{29}$$

and by induction, we get the k^{th} derivative as relation (26).□

□

As a special case, if $\mathbf{Z}(x) = 0$, it leads us to the regular RC definition in relations (18) and (23). The introduced definition (24) and the k^{th}-order derivative (26) give us an improved differentiation of the RC functions.

4. Problem Statement

In this study, the form of high-order ODEs which represents a linear nonhomogeneous with variable coefficients defined on a semi-infinite domain is

$$\sum_{k=0}^{m} Q_k(x)f^{(k)}(x) = g(x), \quad 0 \leq x < \infty, \tag{30}$$

which forms m^{th}-order ODEs; the previous forms of DEs are subjected to the following conditions:

$$f^{(k)}(\gamma_i) = \lambda_i, \quad 0 \leq \gamma_i < \infty, i = 0, 1, \cdots, m-1, \tag{31}$$

where the $Q_k(x)$ and $g(x)$ are well-defined functions on Ω, and γ_j and λ_i are constants (initial value problem), where γ_j may tend to ∞ (boundary value problems).

Now, we consider that the approximate solution $f_N(x)$ according to (16) for the exact solution $f(x)$ of equation (30) in the vector form as

$$f_N(x) = \sum_{n=0}^{N} a_n R_n(x) = \mathbf{R}(x)\mathbf{A}, \tag{32}$$

$$f_N^{(k)}(x) = \sum_{n=0}^{N} a_n (R_n(x))^{(k)} = \mathbf{R}^{(k)}(x)\mathbf{A}, \tag{33}$$

where

$$\mathbf{A} = [a_0 \ a_1 \ \cdots \ a_N]. \tag{34}$$

5. Fundamental Relation-Based Matrix Forms

In the beginning, we provide the fundamental matrix relation of the solution of (30) by two schemes using the RC collocation approach.

Assuming that the solution$f(x)$of (30) can be expressed as relation (32), which is a truncated RC series, then$f(x)$ and itsk^{th}derivative$f^{(k)}(x)$are written in the matrix forms (32) and (33) such that$k = 0, 1, \cdots, m$,and$m \leq N$, where$f^{(0)}(x) \equiv f(x)$,$a_0, a_1, \cdots, a_N$are the RC coefficients to be determined later.

Now, let the collocation points x_s as

$$x_s = \frac{1 + \cos(s\pi/N)}{1 - \cos(s\pi/N)}, \quad s = 0, 1, \cdots, N, \tag{35}$$

and at the end points $(s = 0, s = N)x_0 \longrightarrow \infty, x_N = 0$, namely,

$$\begin{aligned}R_n(x) &= 1 \text{ when } x \longrightarrow \infty, \text{for all } n,\\ R_n(x) &= \cos(n\pi) = (-1)^n \text{ when } x \longrightarrow 0, \text{ for all } n.\end{aligned} \tag{36}$$

Permanently, the RC functions are convergent at both end points 0 and ∞; in addition, the presence of infinity in the collocation points $(x_0 \longrightarrow \infty)$ does not cause a failure in the substitution.

Hence, upon substituting these points (35) into (30), one obtains

$$\sum_{k=0}^{m} Q_k(x_s)f^{(k)}(x_s) = g(x_s), \quad s = 0, 1, 2, \cdots, N. \tag{37}$$

The matrix form of the obtained system (37) is written farther as

$$\sum_{k=0}^{m} \mathbf{Q}_k \mathbf{F}^{(k)} = \mathbf{G}, \tag{38}$$

where

$$\mathbf{Q}_k = \begin{bmatrix} Q_k(x_0) & 0 & \cdots & 0 \\ 0 & Q_k(x_1) & \cdots & 0 \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & \cdots & Q_k(x_N) \end{bmatrix},$$

$$\mathbf{F}^{(k)} = \begin{bmatrix} f^{(k)}(x_0) \\ f^{(k)}(x_1) \\ \vdots \\ f^{(k)}(x_N) \end{bmatrix}, \quad \mathbf{G} = \begin{bmatrix} g(x_0) \\ g(x_1) \\ \vdots \\ g(x_N) \end{bmatrix}. \tag{39}$$

5.1. The RRC Scheme. From (23), we know that the first scheme RRC gives us a derivative of RC functions from the k^{th} order; thus, equation (38) takes the form

$$\mathbf{F}^{(k)} = \mathbf{R}\left(\mathbf{D}^T\right)^k \mathbf{A}, \tag{40}$$

where

$$\mathbf{R} = \begin{bmatrix} \mathbf{R}(x_0) \\ \mathbf{R}(x_1) \\ \mathbf{R}(x_2) \\ \vdots \\ \mathbf{R}(x_N) \end{bmatrix} = \begin{bmatrix} R_0(x_0) & R_1(x_0) & \cdots & R_N(x_0) \\ R_0(x_1) & R_1(x_1) & \cdots & R_N(x_1) \\ R_0(x_2) & R_0(x_2) & \cdots & R_0(x_2) \\ \vdots & \vdots & \ddots & \vdots \\ R_0(x_N) & R_1(x_N) & \cdots & R_N(x_N) \end{bmatrix}. \tag{41}$$

Hence, from (38) and (40), one obtains the fundamental matrix equation for (30) as

$$\sum_{k=0}^{m} \mathbf{Q}_k \mathbf{R}\left(\mathbf{D}^T\right)^k \mathbf{A} = \mathbf{G}. \tag{42}$$

Also, we obtain the matrix forms corresponding to condition (31) as follows: setting $x = \gamma_j$ in (33), we get the fundamental matrix form corresponding to the condition (31):

$$\mathbf{R}(\gamma_i)\left(\mathbf{D}^T\right)^k \mathbf{A} = \lambda_i, \quad 0 \le \gamma_i < \infty, i = 0, 1, \cdots, m-1. \tag{43}$$

5.2. The IRC Scheme. We studied the improved and regular differentiating RC functions in the preceding section. Now, we deduce the fundamental matrix relation by the IRC scheme.

Substituting relation (26) into (40), we get

$$\mathbf{F}^{(k)} = \left\{ \mathbf{R}\left(\mathbf{D}^T\right)^k + \sum_{i=0}^{k-1} \mathbf{Z}^{(i)} \left(\mathbf{D}^T\right)^{k-i-1} \right\} \mathbf{A}, \tag{44}$$

where

$$\mathbf{Z} = \begin{bmatrix} \mathbf{Z}(x_0) \\ \mathbf{Z}(x_1) \\ \vdots \\ \mathbf{Z}(x_N) \end{bmatrix} = \begin{bmatrix} Z_0(x_0) & Z_1(x_0) & \cdots & Z_N(x_0) \\ Z_0(x_1) & Z_1(x_1) & \cdots & Z_N(x_1) \\ \vdots & \vdots & \ddots & \vdots \\ Z_0(x_N) & Z_1(x_N) & \cdots & Z_N(x_N) \end{bmatrix}. \tag{45}$$

Hence, from (38) and (44), the fundamental matrix equation for (30) is obtained as

$$\sum_{k=0}^{m} \mathbf{Q}_k \left\{ \mathbf{R}\left(\mathbf{D}^T\right)^k + \sum_{i=0}^{k-1} \mathbf{Z}^{(i)} \left(\mathbf{D}^T\right)^{k-i-1} \right\} \mathbf{A} = \mathbf{G}. \tag{46}$$

Similarly, the matrix form corresponding to the condition (31) using (26) is obtained as

$$\left\{ \mathbf{R}(\gamma_i)\left(\mathbf{D}^T\right)^k + \sum_{i=0}^{k-1} \mathbf{Z}^{(i)}(\gamma_i) \left(\mathbf{D}^T\right)^{k-i-1} \right\} \mathbf{A} = \lambda_i, \tag{47}$$

for $i = 0, 1, \cdots, m-1$, so that $0 \le \gamma_j < \infty$.

6. Method of Solution

Due to the collocation method, the regular (42) and the improved (46) fundamental matrix equations for the proposed problem (30) correspond to a system of algebraic equations with $(N+1)$ equations for the $(N+1)$ unknown RC coefficients $a_0, a_1, \cdots, a_N$..

One writes matrix equations (42) and (46) compactly as

$$\mathbf{S}\mathbf{A} = \mathbf{G}, \tag{48}$$

or in the augmented form as

$$[\mathbf{S}\,;\mathbf{G}]. \tag{49}$$

Equations (43) and (47) obtain the matrix form for the condition (31); also, they are written compactly as

$$\mathbf{H}_i \mathbf{A} = [\lambda_i], \tag{50}$$

so that $\mathbf{S}$ and $\mathbf{H}_i$ for RRC are defined by

$$\begin{aligned} \mathbf{S} = \left[s_{pq}\right] &= \sum_{k=0}^{m} \mathbf{Q}_k \mathbf{R}\left(\mathbf{D}^T\right)^k, \quad p, q = 0, 1, \cdots, N, \\ \mathbf{H}_i = [\, h_{i0} \quad h_{i1} \quad \cdots \quad h_{\mathrm{iN}} \,] &= \mathbf{R}\left(\gamma_j\right)\left(\mathbf{D}^T\right)^k, \end{aligned} \tag{51}$$

while $\mathbf{S}$ and $\mathbf{H}_i$ for IRC are defined by

$$\mathbf{S} = [s_{pq}] = \sum_{k=0}^{m} \mathbf{Q}_k \left\{ \mathbf{R}(\mathbf{D}^T)^k + \sum_{i=0}^{k-1} \mathbf{Z}^{(i)} (\mathbf{D}^T)^{k-i-1} \right\},$$
$$\mathbf{H}_i = [h_{i0} \quad h_{i1} \quad \cdots \quad h_{iN}] = \left\{ \mathbf{R}(\gamma_j)(\mathbf{D}^T)^k + \sum_{i=0}^{k-1} \mathbf{Z}^{(i)}(\gamma_j)(\mathbf{D}^T)^{k-i-1} \right\}. \quad (52)$$

Hence, the approximate solution of (30) under the condition (31) may be obtained by exchanging the rows of matrices (50) by the last (or first) m rows of the matrix (49), then getting the required augmented matrix as

$$[\tilde{\mathrm{S}};\tilde{\mathbf{G}}] = \begin{bmatrix} s_{00} & s_{01} & \cdots & s_{0N} & ; & g(x_0) \\ s_{10} & s_{11} & \cdots & s_{1N} & ; & g(x_1) \\ \cdots & \cdots & \cdots & \cdots & ; & \cdots \\ s_{N-m,0} & s_{N-m,1} & \cdots & s_{N-m,N} & ; & g(x_{N-m}) \\ h_{00} & h_{01} & \cdots & h_{0N} & ; & \lambda_0 \\ h_{10} & h_{11} & \cdots & h_{1N} & ; & \lambda_1 \\ \cdots & \cdots & \cdots & \cdots & ; & \cdots \\ h_{m-1,0} & h_{m-1,1} & \cdots & h_{m-1,N} & ; & \lambda_{m-1} \end{bmatrix}. \quad (53)$$

If rank $\tilde{\mathbf{S}}$ is equal to rank $[\tilde{\mathbf{S}};\tilde{\mathbf{G}}]$, then the algebraic system has a solution, and if the two ranks are equal to $N+1$, then the solution is unique, the inverse matrix method is used here to solve the system, and one may write the matrix equation (49) as

$$\mathbf{A} = (\tilde{\mathbf{S}})^{-1}\tilde{\mathbf{G}}. \quad (54)$$

Therefore, the RC coefficients a_n, $n = 0, 1, \cdots, N$ are uniquely determined.

7. Test Examples

In the present section, three numerical test examples are given to explain the applicability of the proposed two techniques. Using own codes written in MATHEMATICA 10.0. package, the numerical results and figures are presented, as shown in the illustrative comparison tables.

The absolute error is given to compare the efficiency of the proposed schemes, given by $e_N = |f^i_{\text{Exact}} - f^i_{\text{Approximate}}|$, and evaluated at selected points for some N. The error norms L_2 and L_∞ calculated in an interval $x \in [0, b]$ are given by

$$L_b^2 = \sqrt{h \sum_i^l \left(f^i_{\text{Exact}} - f^i_{\text{Approximate}}\right)^2},$$

$$L_b^\infty = \text{Max}\left|f^i_{\text{Exact}} - f^i_{\text{Approximate}}\right|, \quad (55)$$

for the h step size along the interval $x \in [0, b]$. All numeric calculations are carried out on a regular machine Intel(R) Core(TM) i7 CPU, 3.2 GHz.

Example 1. Let us assume the following fourth-order boundary value problem

$$f^{(4)}(x) - \frac{1}{4}(1+x)^{-2}f''(x) + \frac{1}{2}(1+x)^{-4}f(x) = \frac{x^2 - 238x + 713}{(x+1)^6},$$
$$x \in [0,\infty), \quad (56)$$

with $f(0) = 0, f'(1) = 1/2, f''(1) = 1/2, f(x) \longrightarrow 2$ when $x \longrightarrow \infty$.

We have

$$m = 4, Q_0(x) = \frac{1}{2(1+x)^4}, Q_1(x) = 0, Q_2(x) = -\frac{1}{4(1+x)^2},$$
$$Q_3(x) = 0, Q_4(x) = 1, g(x) = \frac{x^2 - 238x + 713}{(x+1)^6}. \quad (57)$$

Thus, for $N = 5$, the numeric collocation points according to (35) are

$$x_1 = \frac{1 + 1/4(1+\sqrt{5})}{1 - 1/4(1+\sqrt{5})},$$
$$x_2 = \frac{1 - 1/4(1-\sqrt{5})}{1 + 1/4(1-\sqrt{5})},$$
$$x_3 = \frac{1 + 1/4(1-\sqrt{5})}{1 - 1/4(1-\sqrt{5})},$$
$$x_4 = \frac{1 - 1/4(1+\sqrt{5})}{1 + 1/4(1+\sqrt{5})},$$
$$x_0 \longrightarrow \infty, x_5 = 0. \quad (58)$$

The fundamental matrix equation of problem using RRC is

$$\left\{\mathbf{Q}_0\mathbf{R} + \mathbf{Q}_1\mathbf{R}\mathbf{D}^T + \mathbf{Q}_2\mathbf{R}(\mathbf{D}^T)^2 + \mathbf{Q}_3\mathbf{R}(\mathbf{D}^T)^3 + \mathbf{Q}_4\mathbf{R}(\mathbf{D}^T)^4\right\}\mathbf{A} = \mathbf{G}, \quad (59)$$

while the fundamental matrix equation of problem using IRC is

$$\begin{aligned}\Big\{&\mathbf{Q}_0\mathbf{R}+\mathbf{Q}_1\left(\mathbf{R}\mathbf{D}^T+\mathbf{Z}\right)+\mathbf{Q}_2\left(\mathbf{R}\left(\mathbf{D}^T\right)^2+\mathbf{Z}\mathbf{D}^T\right)\\&+\mathbf{Z}'+\mathbf{Q}_3\left(\mathbf{R}\left(\mathbf{D}^T\right)^3+\mathbf{Z}\left(\mathbf{D}^T\right)^2+\mathbf{Z}'\mathbf{D}^T+\mathbf{Z}''\right)\\&+\mathbf{Q}_4\left(\mathbf{R}\left(\mathbf{D}^T\right)^4+\mathbf{Z}\left(\mathbf{D}^T\right)^3+\mathbf{Z}'\left(\mathbf{D}^T\right)^2+\mathbf{Z}''\mathbf{D}^T+\mathbf{Z}'''\right)\Big\}\mathbf{A}=\mathbf{G},\end{aligned}\tag{60}$$

where $\mathbf{Q}_0, \mathbf{Q}_2, \mathbf{Q}_4, \mathbf{R}, \mathbf{D}^T, \mathbf{Z}, \mathbf{Z}', \mathbf{Z}''$, and $\mathbf{Z}'''$ are matrices with a given size of 6×6, for this example at $N=5$,

$$\mathbf{Q}_0=\begin{bmatrix}0&0&0&0&0&0\\0&0.0000415747&0&0&0&0\\0&0&0.00712393&0&0&0\\0&0&0&0.0917553&0&0\\0&0&0&0&0.334673&0\\0&0&0&0&0&0.5\end{bmatrix},$$

$$\mathbf{Q}_4=\begin{bmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\end{bmatrix},$$

$$\mathbf{Q}_2=\begin{bmatrix}0&0&0&0&0&0\\0&-0.00227966&0&0&0&0\\0&0&-0.0298411&0&0&0\\0&0&0&-0.107095&0&0\\0&0&0&0&-0.204534&0\\0&0&0&0&0&-0.25\end{bmatrix},$$

$$\mathbf{D}^T=\begin{bmatrix}0&3/4&-2&3&-4&5\\0&-1&7/2&-6&8&10\\0&1/4&-2&21/4&-8&10\\0&0&1/2&-3&7&-10\\0&0&0&3/4&-4&35/4\\0&0&0&0&1&-5\end{bmatrix},$$

$$\mathbf{R}=\begin{bmatrix}1&1&1&1&1&1\\1&\frac{1}{4}\left(1+\sqrt{5}\right)&\frac{1}{4}\left(-1+\sqrt{5}\right)&\frac{1}{4}\left(1-\sqrt{5}\right)&\frac{1}{4}\left(-1-\sqrt{5}\right)&-1\\1&\frac{1}{4}\left(-1+\sqrt{5}\right)&\frac{1}{4}\left(-1-\sqrt{5}\right)&\frac{1}{4}\left(-1-\sqrt{5}\right)&\frac{1}{4}\left(-1+\sqrt{5}\right)&1\\1&\frac{1}{4}\left(1-\sqrt{5}\right)&\frac{1}{4}\left(-1-\sqrt{5}\right)&\frac{1}{4}\left(1+\sqrt{5}\right)&\frac{1}{4}\left(-1+\sqrt{5}\right)&-1\\1&\frac{1}{4}\left(-1-\sqrt{5}\right)&\frac{1}{4}\left(-1+\sqrt{5}\right)&\frac{1}{4}\left(-1+\sqrt{5}\right)&\frac{1}{4}\left(-1-\sqrt{5}\right)&1\\1&-1&1&-1&1&-1\end{bmatrix},$$

Table 1: Comparing the CPU time used by seconds for RRC and IRC schemes.

N	CPU time used by RRC scheme	CPU time used by IRC scheme
4	0.094	0.155
5	0.11	0.19
6	0.156	0.241

$$\mathbf{Z}=\begin{bmatrix}0&0&0&0&0&\frac{5}{4}\\0&0&0&0&0&\frac{5}{16}\left(-1-\sqrt{5}\right)\\0&0&0&0&0&\frac{5}{16}\left(-1+\sqrt{5}\right)\\0&0&0&0&0&\frac{5}{16}\left(-1+\sqrt{5}\right)\\0&0&0&0&0&\frac{5}{16}\left(-1-\sqrt{5}\right)\\0&0&0&0&0&\frac{5}{4}\end{bmatrix},$$

$$\mathbf{Z}'=\begin{bmatrix}0&0&0&0&0&0&\\0&0&0&0&0&\frac{15}{32}\left(-7+3\sqrt{5}\right)&\\0&0&0&0&0&\frac{-75}{32}\left(-3+\sqrt{5}\right)&0\\0&0&0&0&0&\frac{-15}{32}\left(7+3\sqrt{5}\right)&\\0&0&0&0&0&\frac{75}{32}\left(3+\sqrt{5}\right)&\\0&0&0&0&0&-90&\end{bmatrix},$$

$$\mathbf{Z}''=\begin{bmatrix}0&0&0&0&0&0&\\0&0&0&0&0&\frac{15}{64}\left(-2+\sqrt{5}\right)&\\0&0&0&0&0&\frac{75}{64}\left(90-41\sqrt{5}\right)&0\\0&0&0&0&0&\frac{15}{64}\left(-2-\sqrt{5}\right)&\\0&0&0&0&0&\frac{75}{64}\left(90+41\sqrt{5}\right)&\\0&0&0&0&0&2280&\end{bmatrix},$$

$$\mathbf{Z}'''=\begin{bmatrix}0&0&0&0&0&0\\0&0&0&0&0&\frac{15}{128}\left(901-403\sqrt{5}\right)\\0&0&0&0&0&\frac{375}{128}\left(-1237-553\sqrt{5}\right)\\0&0&0&0&0&\frac{15}{128}\left(901+403\sqrt{5}\right)\\0&0&0&0&0&\frac{375}{128}\left(-1237-553\sqrt{5}\right)\\0&0&0&0&0&-40020\end{bmatrix}.\tag{61}$$

Table 2: Comparison between absolute errors with $\lambda = 1$ given by two proposed schemes.

x	RRC scheme e_{20}	RRC scheme e_{30}	IRC scheme e_{20}	IRC scheme e_{30}
0	1	1	1	1
0.5	2.23497×10^{-7}	1.59521×10^{-9}	4.61847×10^{-8}	3.66704×10^{-9}
1	4.90784×10^{-7}	5.74586×10^{-10}	5.93028×10^{-7}	6.13636×10^{-9}
1.5	4.68739×10^{-9}	2.52918×10^{-9}	2.2526×10^{-8}	1.4928×10^{-9}
2	1.78495×10^{-7}	1.33078×10^{-8}	1.8386×10^{-6}	9.67304×10^{-9}
2.5	1.43122×10^{-6}	1.99709×10^{-8}	1.11684×10^{-7}	9.44468×10^{-9}
3	1.35397×10^{-6}	7.57517×10^{-9}	2.11749×10^{-6}	2.75718×10^{-9}
3.5	2.16835×10^{-6}	3.82668×10^{-8}	1.29148×10^{-6}	1.00546×10^{-8}
5	3.534×10^{-6}	2.72901×10^{-8}	1.27908×10^{-6}	2.26605×10^{-8}
7.5	5.73755×10^{-6}	1.88112×10^{-7}	2.23981×10^{-7}	6.70631×10^{-8}
10	7.31028×10^{-6}	2.76595×10^{-7}	4.59453×10^{-6}	3.03553×10^{-9}

Table 3: Comparison between absolute errors with $\lambda = 2$ given by two proposed schemes.

x	RRC scheme e_{20}	RRC scheme e_{30}	IRC scheme e_{20}	IRC scheme e_{30}
0	1	1	1	1
0.5	3.31737×10^{-6}	3.91279×10^{-8}	4.85215×10^{-6}	2.86433×10^{-7}
1	1.21648×10^{-5}	2.30618×10^{-7}	6.13597×10^{-6}	9.11931×10^{-8}
1.5	1.68999×10^{-5}	3.20837×10^{-7}	2.99644×10^{-5}	6.97906×10^{-8}
2	2.57922×10^{-5}	3.48555×10^{-7}	2.71531×10^{-5}	8.9908×10^{-7}
2.5	2.07512×10^{-5}	4.35107×10^{-7}	8.88987×10^{-6}	1.01733×10^{-7}
3	3.12895×10^{-5}	6.64248×10^{-7}	4.44485×10^{-5}	3.8419×10^{-7}
3.5	1.99079×10^{-5}	2.69503×10^{-7}	1.5211×10^{-5}	1.08309×10^{-6}
5	1.14161×10^{-6}	1.06864×10^{-6}	2.31729×10^{-5}	6.46631×10^{-7}
7.5	5.4514×10^{-5}	1.43524×10^{-6}	2.57514×10^{-5}	1.97968×10^{-7}
10	2.13571×10^{-4}	2.89479×10^{-6}	4.48328×10^{-5}	2.05962×10^{-6}

Then, we obtain the augmented matrix (49) with respect to RRC as

$$\left[\tilde{\mathbf{S}}\,;\tilde{\mathbf{G}}\right] = \begin{bmatrix} 1 & -1 & 1 & -1 & 1 & -1 & ; & 0 \\ 0 & 0.5 & 0 & -1.5 & 0 & 2.5 & ; & 0.5 \\ 1 & 0.83333 & 0.38888 & -0.18518 & -0.69753 & -0.97736 & ; & 1.22222 \\ 1 & 1 & 1 & 1 & 1 & 1 & ; & 2 \\ 0.33467 & -28.72592 & 405.80168 & -2818.03109 & 12516.65380 & -33329.69198 & ; & 376.69653 \\ 0.5 & -47.5 & 760.03125 & -6080.75 & 30513.0625 & -86383.3125 & ; & 713 \end{bmatrix}. \tag{62}$$

Then, we solve the equation (49) to find the RC coefficients in the matrix form:

$$\mathbf{A} = [\,-0.0124202 \quad 1.00234 \quad 1.01636 \quad -0.00117103 \quad -0.00393679 \quad -0.00117103\,], \tag{63}$$

Table 4: Comparing the L_{10}^2 and L_{10}^∞ on interval $x \in [0, 10]$ for $\lambda = 1$.

N	RRC scheme		IRC scheme	
	L_{10}^2	L_{10}^∞	L_{10}^2	L_{10}^∞
20	1.70683×10^{-9}	7.31028×10^{-6}	6.74223×10^{-10}	4.59453×10^{-6}
30	2.44349×10^{-12}	2.76595×10^{-7}	1.06979×10^{-15}	9.79285×10^{-8}

Table 5: Comparing the L_{10}^2 and L_{10}^∞ on interval $x \in [0, 10]$ for $\lambda = 2$.

N	RRC scheme		IRC scheme	
	L_{10}^2	L_{10}^∞	L_{10}^2	L_{10}^∞
20	1.45682×10^{-6}	2.13571×10^{-4}	6.41968×10^{-8}	5.10902×10^{-5}
30	2.67643×10^{-10}	2.89479×10^{-6}	1.35486×10^{-11}	2.05962×10^{-6}

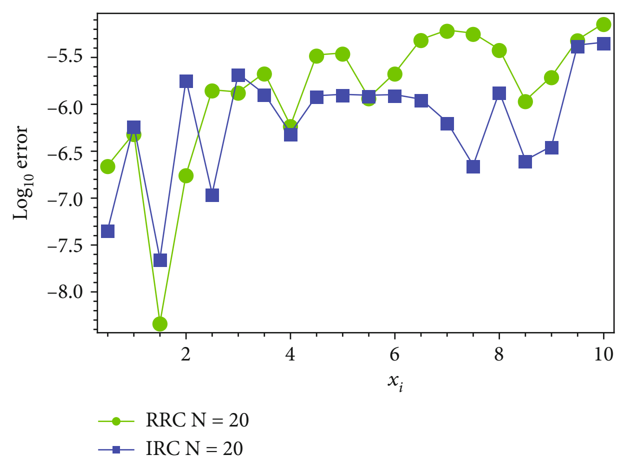

Figure 1: The absolute errors for the proposed two schemes when $N = 20$ with $\lambda = 1$.

Table 6: Comparing the CPU time used for RRC and IRC schemes.

N	RRC scheme		IRC scheme	
	$\lambda = 1$	$\lambda = 2$	$\lambda = 1$	$\lambda = 2$
20	1.173	1.172	1.938	1.985
30	3.89	3.858	4.591	4.61

while the augmented matrix (49) with respect to IRC as

$$\left[\tilde{S}\,;\tilde{\mathbf{G}}\right] = \begin{bmatrix} 1 & -1 & 1 & -1 & 1 & -1 & ; & 0 \\ 0 & 0.5 & 0 & -1.5 & 0 & 2.5 & ; & 0.5 \\ 1 & 0.83333 & 0.38888 & -0.18518 & -0.69753 & -0.97736 & ; & 1.22222 \\ 1 & 1 & 1 & 1 & 1 & 1 & ; & 2 \\ 0.33467 & -28.72592 & 405.42246 & -2799.72562 & 12543.47181 & -40972.42064 & ; & 376.69653 \\ 0.5 & -47.5 & 760.5 & -6159.5 & 33696.5 & -141135.5 & ; & 713 \end{bmatrix}. \tag{64}$$

Table 7: Comparison between absolute errors given by two proposed schemes.

X	RRC scheme		IRC scheme	
	e_{20}	e_{30}	e_{20}	e_{30}
0	1	1	1	
0.1	2.20484×10^{-8}	4.0709×10^{-9}	5.80324×10^{-9}	7.13895×10^{-10}
0.7	8.86658×10^{-9}	6.84228×10^{-8}	1.84652×10^{-8}	6.01463×10^{-8}
2.4	5.81316×10^{-7}	2.90517×10^{-7}	2.83721×10^{-6}	1.27556×10^{-7}
3.8	3.56672×10^{-6}	5.43355×10^{-7}	2.68921×10^{-6}	2.70375×10^{-8}
5	4.19425×10^{-6}	6.64317×10^{-7}	1.59222×10^{-6}	2.63469×10^{-8}
6.7	4.01577×10^{-5}	8.70798×10^{-7}	4.96175×10^{-6}	5.53538×10^{-8}
13.8	1.06869×10^{-4}	9.22562×10^{-7}	1.33586×10^{-5}	7.21039×10^{-7}
14.4	8.5269×10^{-4}	8.87894×10^{-7}	2.31729×10^{-5}	5.63274×10^{-8}
18.6	1.26741×10^{-4}	1.59984×10^{-6}	1.20205×10^{-5}	1.12601×10^{-6}
20	1.87513×10^{-4}	1.10906×10^{-6}	1.20881×10^{-4}	9.73054×10^{-7}

Table 8: Comparing the L_{10}^2 and L_{10}^∞ on interval $x \in [0, 10]$ for $\lambda = -1$.

N	RRC scheme		IRC scheme	
	L_{10}^2	L_{10}^∞	L_{10}^2	L_{10}^∞
20	1.12301×10^{-6}	8.5269×10^{-4}	4.667×10^{-7}	1.20881×10^{-4}
30	3.28475×10^{-11}	8.16818×10^{-6}	8.92851×10^{-12}	2.83766×10^{-6}

Similarly, we compute the RC coefficients in the matrix form:

$$\mathbf{A} = [\,0 \quad 1 \quad 1 \quad 0 \quad 0 \quad 0\,]. \tag{65}$$

Substituting by the RC coefficients of the RRC scheme in Equation (32), we get an approximate solution of this problem, while, if we use the RC coefficients of the IRC scheme in Equation (32), we get the exact solution $2x(x-3)/(x+1)^2$. We note that the RRC scheme gives the exact solution at $N = 6$ while the IRC scheme gives the exact solution at $N = 4$, which shows that the IRC scheme obtains better accuracy than the RRC due to the truncation found in RRC (18). In addition, the CPU time used for the RRC and IRC schemes at different $N$is listed in Table 1. Table 1 shows that the time used for calculating the solution by RRC is less than the time for IRC, because of the added truncated terms in coding. Therefore, the time for getting the exact solution by IRC (better accuracy at $N = 4$) is close to that given by RRC (at $N = 6$). The following examples show the comparison of the two schemes strongly with numeric table's results.

Example 2. Consider Laguerre-eigen problem [26]

$$xf''(x) + (x+1)f'(x) + \lambda f(x) = 0, \quad x \in [0,\infty), \tag{66}$$

Table 9: Comparing the CPU time used by seconds for RRC and IRC schemes $\lambda = -1$.

N	CPU time used by RRC scheme	CPU time used by IRC scheme
20	0.844	0.971
30	2.345	2.657

has the exact eigen solutions

$$\begin{aligned} f(x) &= 1, \quad \lambda = 0, \\ f(x) &= e^{-x}L_n(x), \quad \lambda = n+1, n \geq 0, \end{aligned} \tag{67}$$

where $L_n(x)$ are the well-known Laguerre polynomials of degree n; Equation (66) is a boundary value problem, so the conditions are "natural" at both endpoints where $f(0) = 1$ and $f(x) = 1$ when $x \longrightarrow \infty$. By applying the proposed two schemes as in the previous example, we obtain the solution $f(x) = 1$, which is the exact eigen solution when $\lambda = 0$. For $N = 3$, the RRC scheme gives the exact eigen solution, while the IRC is satisfied with $N = 2$ to find this solution. In Tables 2 and 3, the resulting values for $N = 20$ and 30 using the present two schemes together with the exact values of the solution $f(x) = e^{-x}L_n(x)$, $\lambda = n+1$, $n \geq 0$, are tabulated with different values of λ. The error reduces when the series increased. The numeric values of the error norms L_2 and L_∞ in interval $x \in [0, 10]$ is given in Tables 4 and 5 with different values of λ. Additionally, the absolute errors e_N for the two schemes when $N = 20$ are plotted in Figure 1. In Table 6, the CPU time used for RRC and IRC schemes at different N shows that IRC needs time more than RRC because of the calculation of the added terms of B.

Example 3. Consider Whittaker's equation eigen problem [26] of the form

$$f''(x) + \left[\frac{-1}{4} + \frac{1}{f(x)} + \frac{\lambda}{f(x)}\right] f(x) = 0, \quad x \in [0,\infty), \quad (68)$$

where λ is the eigen value; it represents a special case of Whittaker's equation. The exact given solution of (68) is $f(x) = e^{-0.5x} L_n^1(x)$, where $\lambda = n, n \geq 0$, is an integer and $L_n^k(x)$ is the well-known associated Laguerre polynomials of k^{th} order and degree n. If $\lambda = -1$, (68) gives a special case or linear Whittaker's equation as

$$4f''(x) - f(x) = 0, \quad x \in [0,\infty). \quad (69)$$

The exact solution is $f(x) = e^{-0.5x}$ with the boundary conditions $f(0) = 1$ and $f(x) = 1$ when $x \longrightarrow \infty$.

By the same procedure, the RC collocation method using the proposed two schemes is applied to solve (69) with the subjected boundary conditions. In Table 7, the numerical result for $N = 20$ and 30 using the proposed schemes is compared with the exact values of $f(x) = e^{-0.5x}$. The computation of L_2 and L_∞ on interval $x \in [0, 10]$ is given in Table 8. In Table 9, the CPU time used for RRC and IRC schemes at different N shows that IRC needs time more than RRC.

8. Conclusion

A rational Chebyshev (RC) spectral collocation technique is considered in this paper to solve high-order ordinary differential equations (ODEs) defined on a semi-infinite domain using the proposed two schemes. Two definitions of the derivative of the RC functions are introduced, namely, the regular and improved definitions. Due to the two definitions, two schemes are presented for solving the proposed ODEs with variable coefficients in the semi-infinite interval. According to the convergence of the RC functions at the infinity, the proposed technique deals with the boundary value problem which is defined on semi-infinite domains easily. Furthermore, an intriguing advantage of this approach is the ability to find the analytical exact solutions if the equation has a solution in a rational function form. To demonstrate the applicability of the proposed approach, three illustrative examples are given. The calculated numerical values and comparisons proved that the improved scheme is better with more calculation than the regular scheme which is based on the truncation in the definition. The method may extend to the case of nonlinear DEs with variable coefficients, which the authors are investigating.

Data Availability

The authors declare that there is no data associated with this research.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

All authors carried out the proofs and conceived of the study. All authors read and approved of the final form of the manuscript.

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