Hindawi
Journal of Function Spaces
Volume 2021, Article ID 1142942, 5 pages
https://doi.org/10.1155/2021/1142942

Research Article

Boundedness of Fractional Integral Operators on Hardy-Amalgam Spaces

Ka Luen Cheung, Kwok-Pun Ho, and Tat-Leung Yee

Department of Mathematics and Information Technology, The Education University of Hong Kong, 10 Lo Ping Road, Tai Po, Hong Kong

Correspondence should be addressed to Kwok-Pun Ho; vkpho@eduhk.hk

Received 24 September 2021; Accepted 15 October 2021; Published 10 November 2021

Academic Editor: Douadi Drihem

We establish the boundedness of the fractional integral operators on the Hardy-amalgam spaces.

1. Introduction

In this paper, we establish the boundedness of the fractional integral operators on Hardy-amalgam spaces.

The amalgam spaces were introduced by Wiener [1]. The amalgam spaces are important function spaces for the Fourier analysis and the mapping properties of operators [2–8]. The amalgam spaces also provide the foundation for the time-frequency analysis [9], especially, the introduction of the Wiener amalgam space and the modulation spaces.

The study of the amalgam spaces has been further extended to the Hardy-amalgam space [10, 11] and the slice spaces [12–14]. The reader is referred to [6, 12] for the boundedness of the Riesz transform, the Calderón-Zygmund operators and the intrinsic square function on the Hardy-amalgam spaces.

The above results on Hardy-amalgam spaces motivate us to investigate the mapping properties of the fractional integral operators on the Hardy-amalgam spaces.

The studies of the mapping properties of the fractional integral operators on Hardy type spaces begun from the classical Hardy spaces introduced by Stein and Weiss [15]. The mapping properties of the fractional type integrals on Hardy spaces and the weighted norm inequalities of the fractional integral operators on Hardy spaces were established in [16, 17], respectively.

In this paper, we use the extrapolation theory for amalgam spaces developed in [6] to obtain our main result. The extrapolation theory was introduced by Rubio de Francia in [18–20]. By using the extrapolation theory, we do not need to develop the atomic decomposition of the Hardy-amalgam spaces. In addition, we use the idea from [21] to refine the extrapolation theory so that we do not need to use the density argument in [6].

This paper is organized as follows. Section 2 presents the definitions of the amalgam spaces and the Hardy-amalgam spaces. It also includes the mapping properties of the fractional integral operators on weighted Hardy spaces which is an essential component for applying the extrapolation theory. The mapping properties of the fractional integral operators on the Hardy-amalgam spaces are established in Section 3.

2. Preliminaries and Definitions

In this section, we present the definitions and the preliminary results used to obtain the main result. In particular, this section contains the duality of the amalgam spaces, the boundedness of the Hardy-Littlewood maximal operator on the amalgam spaces, and the mapping properties of the fractional integral operators on the weighted Hardy spaces.

Let $\mathbb{I}$ denote the class of open connected intervals in $\mathbb{R}$. Let $|I|$ be the Lebesgue measure of $I \in \mathbb{I}$.

Let $0 < \alpha < 1$. The fractional integral operator on $\mathbb{R}$ is defined by

$$I_\alpha f(x) = \int_{\mathbb{R}} \frac{f(y)}{|x-y|^{1-\alpha}} dy. \tag{1}$$

If $0 < \alpha < 1$, $1 < p < 1/\alpha$, and $1/p = 1/q + \alpha$, then $I_\alpha : L_p \longrightarrow L_q$ is bounded.

Definition 1. Let $0 < p, q < \infty$. The amalgam space (L^p, l^q) consists of all Lebesgue measurable functions f satisfying

$$\|f\|_{(L^p,l^q)} = \left(\sum_{n\in\mathbb{Z}} \left(\int_n^{n+1} |f(x)|^p dx \right)^{q/p} \right)^{1/q} < \infty. \tag{2}$$

Obviously, when $1 \le p = q < \infty$, then $(L^p, l^q) = L^p$. Moreover, when $1 \le p, q < \infty$, (L^p, l^q) is a Banach space ([7], Theorem 1). It is easy to see that if $p \in (0,\infty)$ and $0 < q_1 \le q_2 < \infty$, we have

$$(L^p, l^{q_1}) \subset (L^p, l^{q_2}). \tag{3}$$

In addition, when $0 < p_1 \le p_2 < \infty$ and $q \in (0,\infty)$, we have

$$(L^{p_2}, l^q) \subset (L^{p_1}, l^q). \tag{4}$$

The amalgam spaces had been extended to the slice spaces introduced and studied in [13, 22] and Orlicz slice spaces in [14]. For the mapping properties of operators on slice spaces and Orlicz-slice spaces, the reader is referred to [14, 23].

The following result present the dual space of (L^p, l^q).

Proposition 2. *Let $1 < p, q < \infty$. The dual space of (L^p, l^q) is $(L^{p'}, l^{q'})$.*

The reader is referred to [7] for the proof of the above result. Particularly, we have the Hölder inequality for the amalgam space

$$\int_{\mathbb{R}} |f(x)g(x)| dx \le \|f\|_{(L^p,l^q)} \|g\|_{(L^{p'},l^{q'})}. \tag{5}$$

We now study the boundedness of the Hardy-Littlewood maximal operator on (L^p, l^q). For any locally integrable function f, the Hardy-Littlewood maximal operator is defined as

$$M(f)(x) = \sup_{I \ni x} \frac{1}{|I|} \int_I |f(y)| dy, \tag{6}$$

where the supremum is taken over all interval $I \in \mathbb{I}$ containing x.

We now recall the boundedness of the Hardy-Littlewood maximal operator on the amalgam spaces from [2].

Theorem 3. *Let $1 < p, q < \infty$. The Hardy-Littlewood maximal operator M is bounded on (L^p, l^q).*

The reader is referred to [2], Theorems 4.2 and 4.5 for the proof of the above result when $p \ne q$. When $p = q$, it follows from the well-known result that M is bounded on the Lebesgue space L^p.

We now recall the definition of the Muckenhoupt weight functions A_p.

Definition 4. For $1 < p < \infty$, a locally integrable function $\omega : \mathbb{R} \longrightarrow [0,\infty)$ is said to be an A_p weight if

$$[\omega]_{A_p} = \sup_{I\in\mathbb{I}} \left(\frac{1}{|I|} \int_I \omega(x) dx \right) \left(\frac{1}{|I|} \int_I \omega(x)^{-p'/p} dx \right)^{p/p'} < \infty, \tag{7}$$

where $p' = p/(p-1)$. A locally integrable function $\omega : \mathbb{R} \longrightarrow [0,\infty)$ is said to be an A_1 weight if

$$\frac{1}{|I|} \int_I \omega(y) dy \le C\omega(x), a.e. x \in I, \tag{8}$$

for some constants $C > 0$. The infimum of all such C is denoted by $[\omega]_{A_1}$. We define $A_\infty = \cup_{p\ge1} A_p$.

For any $p \in (0,\infty)$ and $u : \mathbb{R} \longrightarrow [0,\infty)$, the weighted Lebesgue space L^p_u consists of all Lebesgue measurable functions f satisfying

$$\|f\|_{L^p_u} = \left(\int_{\mathbb{R}} |f(x)|^p u(x) dx \right)^{1/p} < \infty. \tag{9}$$

Let $p \in (1,\infty)$. It is well known that M is bounded on $L^p(u)$ if and only if $u \in A_p$.

Let $F = \{\|\cdot\|_{\alpha_i,\beta_i}\}$ be any finite collection of semi-norms on $\mathcal{S}$ and

$$\mathcal{S}_F = \left\{ \psi \in \mathcal{S} : \|\psi\|_{\alpha_i,\beta_i} \le 1, \text{for all } \|\cdot\|_{\alpha_i,\beta_i} \in \mathcal{F} \right\}. \tag{10}$$

For any $f \in \mathcal{S}'$, write

$$M_F f(x) = \sup_{\psi\in s_F} \sup_{t>0} |(f * \psi_t)(x)|, \tag{11}$$

where for any $t > 0$, write $\psi_t(x) = t^{-1}\psi(x/t)$.

Definition 5. Let $p, q \in (0,\infty)$. The Hardy-amalgam space (H^p, l^q) consists of all $f \in \mathcal{S}'$ satisfying

$$\|f\|_{(H^p,l^q)} = \|M_F f\|_{(L^p,l^q)} < \infty. \tag{12}$$

We use the grand maximal function to define the Hardy-

amalgam spaces while in [6], we use the Littlewood-Paley function. In view of the definition of (L^p, l^q) and [6], Proposition 2, (L^p, l^q) is a ball quasi-Banach function space. Thus, [24], Section 5 shows that they are equivalent definition for the Hardy-amalgam spaces. For simplicity, we refer the reader to [25, 26] for the definitions of ball quasi-Banach function spaces and the Hardy spaces built on the ball quasi-Banach function spaces. In particular, the Orlicz-slice Hardy spaces were introduced in [14]. The intrinsic square function characterization of the Orlicz slice Hardy spaces was established in [27]. The mapping properties of the maximal Bochner-Riesz means, the parametric Marcinkiewicz integrals, and the multiplier operators on the Orlicz-slice Hardy spaces were given in [23].

For the atomic decomposition and the dual space of the Hardy-amalgam spaces, see [10, 11].

We need to use the weighted Hardy spaces to establish our main result. Thus, we also recall the definition of the weighted Hardy spaces.

Let $p \in (0,\infty)$. For any weighted function $\omega : \mathbb{R} \longrightarrow (0, \infty)$, the weighted Hardy space $H^p(\omega)$ consists of all $f \in \mathcal{S}'$ satisfying

$$\|f\|_{H^p(\omega)} = \|M_F f\|_{L^p(\omega)} < \infty. \tag{13}$$

For the studies of the weighted Hardy spaces, the reader is referred to [28, 29].

We now present the boundedness of the fractional integral operators on weighted Hardy spaces.

Theorem 6. *Let $0 < p < 1/\alpha$ and $1/q = (1/p) - \alpha$. Then, $\omega \in A_\infty$ if and only if*

$$\|I_\alpha f\|_{H^q(\omega)} \le C\|f\|_{H^p(\omega^{p/q})}, \tag{14}$$

for some $C > 0$.

For the proof of the above theorem, see [17], Corollary 6.2 and Theorem 8.1.

3. Main Result

The main result of this paper, the boundedness of the fractional integral operators on the Hardy-amalgam spaces is established in this section.

Theorem 7. *Let $0 < \alpha < 1$ and $0 < p, q < 1/\alpha$. Suppose that*

$$\frac{1}{p} - \frac{1}{r} = \frac{1}{q} - \frac{1}{s} = \alpha. \tag{15}$$

We have a constant $C > 0$ such that for any $f \in (H^p, l^q)$,

$$\|I_\alpha f\|_{(H^r, l^s)} \le C\|f\|_{(H^p, l^q)}. \tag{16}$$

Proof. Let $\beta \in (0, \min(p, q))$. Define θ by

$$\frac{1}{\beta} - \frac{1}{\theta} = \alpha. \tag{17}$$

As $0 < \beta < \min(p, q) < 1/\alpha$, θ is well defined.

In view of (15), we have $(1/\beta) - (1/\theta) = \alpha = (1/p) - (1/r)$. Consequently,

$$\frac{1}{r} - \frac{1}{\theta} = \frac{1}{p} - \frac{1}{\beta} < 0. \tag{18}$$

Since $\beta < p$, we have $r > \theta$. Similarly, we have $s > \theta$. Thus, $(r/\theta)'$ and $(s/\theta)'$ are well defined.

For any nonnegative function $h \in (L^{(r/\theta)'}, l^{(s/\theta)'})$, define

$$\mathcal{R}h(x) = \sum_{k=0}^{\infty} \frac{M^k h(x)}{2^k \|M\|^k_{\left(L^{(r/\theta)'}, l^{(s/\theta)'}\right)}}. \tag{19}$$

The Rubio de Francia operator $\mathcal{R}$ possesses the following properties:

$$h(x) \le \mathcal{R}h(x), \tag{20}$$

$$\|\mathcal{R}h\|_{\left(L^{(r/\theta)'}, l^{(s/\theta)'}\right)} \le 2\|h\|_{\left(L^{(r/\theta)'}, l^{(s/\theta)'}\right)}, \tag{21}$$

$$[\mathcal{R}h]_{A_1} \le 2\|M\|_{\left(L^{(r/\theta)'}, l^{(s/\theta)'}\right)}. \tag{22}$$

The above properties are valid because $(r/\theta)', (s/\theta)' > 1$, and Theorem 3 assures that M is bounded on $(L^{(r/\theta)'}, l^{(s/\theta)'})$.

The identity in (18) shows that

$$\frac{\beta}{\theta}(p/\beta)' = \frac{\beta}{\theta}\frac{p}{p-\beta} = \frac{r}{r-\theta} = (r/\theta)'. \tag{23}$$

Similarly, the identity $(1/s) - (1/\theta) = (1/q) - (1/\beta)$ shows that

$$\frac{\beta}{\theta}(q/\beta)' = \frac{\beta}{\theta}\frac{q}{q-\beta} = \frac{s}{s-\theta} = \left(\frac{s}{\theta}\right)'. \tag{24}$$

Therefore,

$$\|g\|^{\beta/\theta}_{\left(L^{(r/\theta)'}, l^{(s/\theta)'}\right)} = \left\| |g|^{\beta/\theta} \right\|_{\left(L^{(p/\beta)'}, l^{(q/\beta)'}\right)}. \tag{25}$$

For any $f \in (H^p, l^q)$ and $h \in (L^{(r/\theta)'}, l^{(s/\theta)'})$, (5), (20), and (25) assert that

$$\begin{aligned}
&\int_{\mathbb{R}} (M_F f(x))^\beta (\mathcal{R}h(x))^{\beta/\theta} dx \\
&\le \left\| (M_F f)^\beta \right\|_{\left(L^{p/\beta}, l^{q/\beta}\right)} \left\| (\mathcal{R}h)^{\beta/\theta} \right\|_{\left(L^{(p/\beta)'}, l^{(q/\beta)'}\right)} \\
&= \|(M_F f)\|^\beta_{(L^p, l^q)} \|\mathcal{R}h\|^{\beta/\theta}_{\left(L^{(r/\theta)'}, l^{(s/\theta)'}\right)} = \|f\|^\beta_{(H^p, l^q)} \|h\|^{\beta/\theta}_{\left(L^{(r/\theta)'}, l^{(s/\theta)'}\right)}.
\end{aligned} \tag{26}$$

Consequently, for any $h \in (L^{(r/\theta)'}, l^{(s/\theta)'})$, we find that

$$(H^p, l^q) \hookrightarrow H^\beta\left((\mathscr{R}h)^{\beta/\theta}\right). \tag{27}$$

Moreover, (22) yields that $\mathscr{R}h \in A_1$. By applying Theorem 6 with $\omega = \mathscr{R}h$, (20) and (27) yield

$$\begin{aligned}\int_{\mathbb{R}} (M_F I_\alpha f(x))^\theta |h(x)| dx &\le \int_{\mathbb{R}} (M_F I_\alpha f(x))^\theta \mathscr{R}h(x) dx \\ &= \|I_\alpha f\|^\theta_{H^\theta(\mathscr{R}h)} \le C\|f\|^\theta_{H^\beta((\mathscr{R}h))^{\beta/\theta}} \\ &= C\left(\int_{\mathbb{R}} (M_F f(x))^\beta (\mathscr{R}h(x))^{\beta/\theta} dx\right)^{\theta/\beta},\end{aligned} \tag{28}$$

for some $C > 0$.

Consequently, (26) gives

$$\int_{\mathbb{R}} (M_F I_\alpha f(x))^\theta |h(x)| dx \le C\|f\|^\theta_{(H^p, l^q)} \|h\|_{\left(L^{(r/\theta)'}, l^{(s/\theta)'}\right)}. \tag{29}$$

Proposition 2 guarantees that

$$\begin{aligned}&\sup\left\{\int_{\mathbb{R}} (M_F I_\alpha f(x))^\theta |h(x)| dx : \|h\|_{\left(L^{(r/\theta)'}, l^{(s/\theta)'}\right)} \le 1\right\} \\ &= \left\|(M_F I_\alpha f)^\theta\right\|_{\left(L^{r/\theta}, l^{s/\theta}\right)} = \|M_F I_\alpha f\|^\theta_{(L^r, l^s)} = \|I_\alpha f\|^\theta_{(H^r, l^s)}.\end{aligned} \tag{30}$$

Therefore, by taking the supremum over $\|h\|_{\left(L^{(r/\theta)'}, l^{(s/\theta)'}\right)} \le 1$ on both sides of (29), we have a constant $C > 0$ such that for any $f \in (H^p, l^q)$

$$\|I_\alpha f\|_{(H^r, l^s)} \le C\|f\|_{(H^p, l^q)}. \tag{31}$$

□

By using the idea from [21], we can get rid of the density argument used in [6]. Moreover, the above method have been applied in [23, 30–32] to study the mapping properties of the Calderón-Zygmund operators and some sublinear operators on the Hardy local Morrey spaces with variable exponents, the Orlicz-slice Hardy spaces, the Herz-Hardy spaces with variable exponents, and the Hardy-Morrey spaces with variable exponents, respectively.

Particularly, when $p = q$ and $r = s$, the above result becomes the well-known result of the boundedness of the fractional integral operators on Hardy spaces established in [15].

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

References

[1] N. Wiener, "On the representation of functions by trigonometrical integrals," *Mathematische Zeitschrift*, vol. 24, no. 1, pp. 575–616, 1926.

[2] C. Carton-Lebrun, H. Heinig, and H. Hofmann, "Integral operators and weighted amalgams," *Studia Mathematica*, vol. 109, no. 2, pp. 133–157, 1994.

[3] M. Cowling, S. Meda, and R. Pasquale, "Riesz potentials and amalgams," *Annales de l'institut Fourier*, vol. 49, no. 4, pp. 1345–1367, 1999.

[4] J. Fournier and J. Stewart, "Amalgams of L^p and l^q," *Bulletin of the American Mathematical Society*, vol. 13, no. 1, pp. 1–22, 1985.

[5] K.-P. Ho, "Dilation operators and Integral operators on amalgam (L_p, l_q)," *Ricerche di Matematica*, vol. 68, no. 2, pp. 661–677, 2019.

[6] K.-P. Ho, "Singular integrals and sublinear operators on amalgam spaces and Hardy-amalgam spaces," *Mathematica Scandinavica*.

[7] F. Holland, "Harmonic analysis on amalgams of L^p and l^q," *Journal of the London Mathematical Society*, vol. 10, pp. 295–305, 1975.

[8] N. Kikuchi, E. Nakai, N. Tomita, K. Yabuta, and T. Yoneda, "Calderon-Zygmund operators on amalgam spaces and in the discrete case," *Journal of Mathematical Analysis and Applications*, vol. 335, no. 1, pp. 198–212, 2007.

[9] K. Gröchenig, "Foundations of time-frequency analysis," in *Applied and Numerical Harmonic Analysis*, Springer, 2001.

[10] Z. Ablé and J. Feuto, "Atomic decomposition of Hardy-amalgam spaces," *Journal of Mathematical Analysis and Applications*, vol. 455, no. 2, pp. 1899–1936, 2017.

[11] Z. Ablé and J. Feuto, "Duals of Hardy amalgam spaces and norm inequalities," *Analysis Mathematica*, vol. 45, no. 4, pp. 647–686, 2019.

[12] A.-T. Assaubay, J. Betancor, A. Castro, and J. Farińa, "Riesz transforms, Cauchy-Riemann systems, and Hardy-amalgam spaces," *Banach Journal of Mathematical Analysis*, vol. 13, no. 3, pp. 697–725, 2019.

[13] P. Auscher and C. Prisuelos-Arribas, "Tent space boundedness via extrapolation," *Mathematische Zeitschrift*, vol. 286, no. 3-4, pp. 1575–1604, 2017.

[14] Y. Zhang, D. Yang, W. Yuan, and S. Wang, "Real-variable characterizations of Orlicz-slice Hardy spaces," *Analysis and Applications*, vol. 17, no. 4, pp. 597–664, 2019.

[15] E. Stein and G. Weiss, "On the theory of harmonic functions of several variables: I. The theory of H^p-spaces," *Acta Mathematica*, vol. 103, no. 1–2, pp. 25–62, 1960.

[16] S. Krantz, "Fractional integration on Hardy spaces," *Studia Mathematica*, vol. 73, no. 2, pp. 87–94, 1982.

[17] J. O. Strömberg and R. Wheeden, "Fractional integrals on weighted H^p and L^p spaces," *Transactions of the American Mathematical Society*, vol. 287, no. 1, pp. 293–321, 1985.

[18] R. de Francia, "Factorization and extrapolation of weights," *Bulletin of the American Mathematical Society*, vol. 7, no. 2, pp. 393–395, 1982.

[19] J. Rubio de Francia, "A new technique in the theory of A_p weights," *Topics in modern harmonic analysis*, vol. 1, pp. 571–579, 1982.

[20] J. Rubio de Francia, "Factorization theory and A_p weights," *American Journal of Mathematics*, vol. 106, pp. 533–547, 1984.

[21] K.-P. Ho, *Boundedness of operators and inequalities on Morrey-Banach spaces*Publications of the Research Institute for Mathematical Sciences.

[22] P. Auscher and M. Mourgoglou, "Representation and uniqueness for boundary value elliptic problems via first order systems," *Revista matemática iberoamericana*, vol. 35, no. 1, pp. 241–315, 2019.

[23] K.-P. Ho, "Operators on Orlicz-slice spaces and Orlicz-slice Hardy spaces," *Journal of Mathematical Analysis and Applications*, vol. 503, no. 1, article 125279, 2021.

[24] D.-C. Chang, S. Wang, D. Yang, and Y. Zhang, "Littlewood-Paley characterizations of Hardy-type spaces associated with ball quasi-Banach function spaces," http://arxiv.org/abs/1911.04953.

[25] Y. Sawano, K.-P. Ho, D. Yang, and S. Yang, "Hardy spaces for ball quasi-Banach function spaces," *Dissertationes Mathematicae*, vol. 525, pp. 1–102, 2017.

[26] Y. Sawano, "Theory of Besov spaces," in *Developments in Mathematics*, Springer, Singapore, 2018.

[27] X. Yan, D. Yang, and W. Yuan, "Intrinsic square function characterizations of Hardy spaces associated with ball quasi-Banach function spaces," *Frontiers of Mathematics in China*, vol. 15, no. 4, pp. 769–806, 2020.

[28] J. García-Cuerva, "Weighted H^p spaces," *Dissertations Math*, vol. 162, pp. 1–63, 1979.

[29] J. Stömberg and A. Torchinsky, "Weighted Hardy spaces," in *Weighted Hardy Spaces, vol. 1381 of Lecture Notes in Mathematics*, Springer, Verlag, 1989.

[30] K.-P. Ho, "Singular integral operators and sublinear operators on Hardy local Morrey spaces with variable exponents," *Bulletin des Sciences Mathématiques*, vol. 171, article 103033, 2021.

[31] K.-P. Ho, "Sublinear operators on Herz-Hardy spaces with variable exponents," *Mathematische Nachrichten*, 2021.

[32] K.-P. Ho, "Calderón-Zygmund operators, Bochner-Riesz means and parametric Marcinkiewicz integrals on Hardy-Morrey spaces with variable exponents," *Kyoto Journal of Mathematics*.