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Research Article

Best Proximity Point Theorems for Single and Multivalued Mappings in Fuzzy Multiplicative Metric Space

Misbah Farheen[1], **Tayyab Kamran**[1], and **Azhar Hussain**[2]

[1]*Department of Mathematics, Quaid-i-Azam University, Islamabad, Pakistan*
[2]*Department of Mathematics, University of Sargodha, Sargodha 40100, Pakistan*

Correspondence should be addressed to Misbah Farheen; fmisbah90@yahoo.com

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In this paper, we introduce fuzzy multiplicative metric space and prove some best proximity point theorems for single-valued and multivalued proximal contractions on the newly introduced space. As corollaries of our results, we prove some fixed-point theorems. Also, we present best proximity point theorems for Feng-Liu-type multivalued proximal contraction in fuzzy metric space. Moreover, we illustrate our results with some interesting examples.

1. Introduction and Preliminaries

Best proximity point is the generalization of fixed point and is useful when contraction map is not a self-map that is $T : A \longrightarrow B$ where $A \cap B = \phi$. A point $\mu \in A$ is known as best proximity point if $d(\mu, T\mu) = d(A, B)$. Fan [1] presented best approximation theorem which is stated as follows: "If K is a nonempty compact convex subset of a Hausdorff locally convex topological vector space E and $T : K \longrightarrow E$ is a continuous non-self-mapping, then there exists an element μ in K such a way that $d(\mu, T\mu) = d(T\mu, K)$." A best proximity point theorem is more applicable than best approximation theorem, as it provides optimal approximate solution. Therefore, best proximity point theory seeks attention of authors such as [2–7]. Many research works done on multivalued non-self-maps use Nadler's approach [8]. Nadler's theorem is stated as follows: "Let (M, d) be a complete metric space and T be a mapping from M into $CB(M)$, where $CB(M)$ is the collection of all closed and bounded subsets of M, such that for all $\mu, \nu \in M$, $H(T\mu, T\nu) \leq \lambda d(\mu, \nu)$ where $0 < \lambda < 1$. Then, T has a fixed point." Another way of defining multivalued contraction is approached by Feng and Liu [9]. They proved a fixed-point theorem for newly defined multivalued contraction which is stated as follows: "Let (M, d) be a complete metric space, $T : M \longrightarrow C(M)$, where $C(M)$ is the collection of all closed subsets of M, be a multivalued mapping. If there exists a constant $c \in (0, 1)$ such that for any $\mu \in M$, there is $\nu \in I_b^\mu$ (where $I_b^\mu = \{\nu \in T\mu \mid bd(\mu, \nu) \leq d(\mu, T\mu)\} \subset M$ for some $b \in (0, 1)$) satisfying $d(\nu, T\nu) \leq c\, d(\mu, \nu)$. Then, T has a fixed point in M provided that $c < b$ and $f(\mu) = d(\mu, T\mu)$ is lower semicontinuous". With the help of example, in the same article, they also have shown that Feng-Liu-type multivalued contraction is more general than Nadler's multivalued contraction. Recently, Sahin et al. [10] proved best proximity point theorem for Feng-Liu-type multivalued map.

On the other hand, fuzzy metric space was firstly defined by Kramosil and Michalek [11] and then modified by George and Veeramani [12]. The modified definition is given as follows.

Definition 1 (see [12]). A 3-tuple $(M, F_M, \star)$ is called fuzzy metric space if M is an arbitrary set, $\star$ is continuous t-norm, and F_M is a fuzzy set on $M \times M \times (0,\infty)$ satisfying the following conditions for all $\mu, \nu, \rho \in M$ and $t, s > 0$:

FM1: $F_M(\mu, \nu, t) > 0$
FM2: $F_M(\mu, \nu, t) = 1$ if and only if $\mu = \nu$
FM3: $F_M(\mu, \nu, t) = F_M(\nu, \mu, t)$
FM4: $F_M(\mu, \rho, t + s) \geq F_M(\mu, \nu, t) \star F_M(\nu, \rho, s)$
FM5: $F_M(\mu, \nu, .): (0,\infty) \longrightarrow [0, 1]$ is continuous

The $t-$norm is defined as follows.

Definition 2 (see [12]). A continuous t-norm is a binary operation $\star : [0,1]^2 \longrightarrow [0,1]$ if the pair $([0,1],\star)$ is a topological monoid, that is,

(1) $\star$ satisfies associative and commutative laws

(2) $\star$ is continuous

(3) $a\star 1 = a$, $\forall a \in [0,1]$

(4) for every $a, b, c, d \in [0,1]$, $a\star b \le c\star d$ whenever $a \le c$ and $b \le d$

Some known examples of a continuous t-norm are $a\star_1 b = \min\{a,b\}$, $a\star_2 b = (ab/\max\{a,b,\lambda\})$ for $0<\lambda<1$, $a\star_3 b = ab$, $a\star_4 b = \max\{a+b-1, 0\}$.

Many researches have been produced on fixed-point theory in fuzzy metric spaces [4, 13–19]. Vetro and Salimi [20] proved best proximity point theorem in fuzzy metric spaces. Due to the development of new calculus by Grossman and Katz [21], known as multiplicative calculus, a metric was introduced by Bashirov et al. [22] called multiplicative metric defined as follows.

Definition 3 (see [22]). Assume a nonempty set M. Regard multiplicative metric as a mapping $d : M \times M \longrightarrow \mathbb{R}$ obeying the following assertions:

M1: $d(\mu,\nu) > 1$ for all $\mu,\nu \in M$ and $d(\mu,\nu)=1$ if and only if $\mu=\nu$

M2: $d(\mu,\nu) = d(\nu,\mu)$

M3: $d(\mu,\rho) \le d(\mu,\nu)\cdot d(\nu,\mu)$ for all $\mu,\nu,\rho \in M$

Getting inspiration from all the work mentioned above, we firstly introduce fuzzy multiplicative metric space and prove some of its topological properties. Moreover, we obtain some best proximity point theorems for Feng-Liu-type multivalued non-self-maps on fuzzy multiplicative metric space.

2. Fuzzy Multiplicative Metric Spaces

This section introduces a new type of metric space which is fuzzy analogy of multiplicative metric space. We give an example to show the existence of such space.

Definition 4. A triplet $(M, F_{MM},\star)$ is termed as fuzzy multiplicative metric space if $\star$ is continuous t -norm, M is arbitrary set, and F_{MM} is fuzzy set on $M \times M \times (1,\infty)$ fulfilling the accompanying conditions for all $\mu,\nu,\rho \in M$ and $t,s>1$.

FMM1: $F_{MM}(\mu,\nu,t) > 0$

FMM2: $F_{MM}(\mu,\nu,t) = 1$ if and only if $\mu=\nu$

FMM3: $F_{MM}(\mu,\nu,t) = F_{MM}(\nu,\mu,t)$

FMM4: $F_{MM}(\mu,\rho,t.s) \ge F_{MM}(\mu,\nu,t)\star F_{MM}(\nu,\rho,s)$

FMM5: $F_{MM}(\mu,\nu,.)$: $(1,\infty) \longrightarrow [0,1]$ is continuous

Here, we have an example of fuzzy multiplicative metric which cannot be fuzzy metric.

Example 5. Let $M=\mathbb{R}^+$ and $F_{MM}(\mu,\nu,t) = ((t+1)/(t+|\mu/\nu|^*))$, consider a continuous t-norm $\star : [0,1]\times[0,1] \longrightarrow [0,1]$ as $\mu\star\nu = \mu\nu$. Then, M is fuzzy multiplicative metric space.

Remark 6.

(1) Let $(M, F_{MM},\star)$ be a fuzzy multiplicative metric space. Whenever $F_{MM}(\mu,\nu,t) > 1-\varepsilon$ for $\mu,\nu\in M$ and $t>1$, $0<\varepsilon<1$, we can find t_0, $1<t_0<t$ such that $F_{MM}(\mu,\nu,t_0) > 1-\varepsilon$

(2) Let $\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4,\varepsilon_5 \in (0,1)$. For any $\varepsilon_1 > \varepsilon_2$, we can find ε_3 such that $\varepsilon_1\star\varepsilon_3 \ge \varepsilon_2$, and for any ε_4, we can find ε_5 such that $\varepsilon_5\star\varepsilon_5 \ge \varepsilon_4$

We now discuss some topological properties of fuzzy multiplicative metric space.

Definition 7. Let $(M, F_{MM},\star)$ be a fuzzy multiplicative metric space and $0<\varepsilon<1$, $t>1$; then, an open ball having center μ and radius ε is defined as

$$B(\mu,\varepsilon,t) = \{\nu \in M : F_{MM}(\mu,\nu,t) > 1-\varepsilon\}. \tag{1}$$

Proposition 8. *Every open ball is an open set in fuzzy multiplicative metric space.*

Proof. Consider an open ball $B(\mu,\varepsilon,t)$ and let $\nu \in B(\mu,\varepsilon,t)$. This implies that $F_{MM}(\mu,\nu,t) > 1-\varepsilon$. Since $F_{MM}(\mu,\nu,t) > 1-\varepsilon$, using Remark 6, we can find t_0, $1<t_0<t$, such that $F_{MM}(\mu,\nu,t_0) > 1-\varepsilon$. Let $\varepsilon_0 = F_{MM}(\mu,\nu,t_0) > 1-\varepsilon$. Since $\varepsilon_0 > 1-\varepsilon$, therefore by using Remark 6, we can find ε_1, $0<\varepsilon_1<1$, such that $\varepsilon_0 > 1-\varepsilon_1 > 1-\varepsilon$. Now, for a given ε_0 and ε_1 such that $\varepsilon_0 > 1-\varepsilon_1$, we can find ε_2, $0<\varepsilon_2<1$ such that $\varepsilon_0\star\varepsilon_2 \ge 1-\varepsilon_1$. Now, consider the ball $B(\nu, 1-\varepsilon_2, t/t_0)$. We claim that $B(\nu,1-\varepsilon_2,t/t_0) \subset B(\mu,\varepsilon,t)$.

Now, $\rho \in B(\nu,1-\varepsilon_2,t/t_0)$ implies that $F_{MM}(\nu,\rho,t/t_0) > \varepsilon_2$. Therefore,

$$F_{MM}(\mu,\rho,t) \ge F_{MM}(\mu,\nu,t_0)\star F_{MM}\left(\nu,\rho,\frac{t}{t_0}\right) \ge \varepsilon_0\star\varepsilon_2 \ge 1-\varepsilon_1 > 1-\varepsilon. \tag{2}$$

Therefore, $\rho \in B(\mu,\varepsilon,t)$, and hence,

$$B\left(\nu, 1-\varepsilon_2, \frac{t}{t_0}\right) \subset B(\mu,\varepsilon,t). \tag{3}$$

□ □

Proposition 9. *Let $(M, F_{MM},\star)$ be a fuzzy multiplicative metric space. Define $\tau = \{A \subset M : \mu \in A$ if and only if there exist $t>1$ and ε, $0<\varepsilon<1$ such that $B(\mu,\varepsilon,t)\subset A\}$.*

Then, τ is a topology on M.

Theorem 10. *Every fuzzy multiplicative metric space is Hausdorff.*

Proof. Assume that $(M, F_{MM}, \star)$ is a given fuzzy multiplicative metric space. Let μ, ν be two distinct points of M, and then, $0 < F_{MM}(\mu, \nu, t) < 1$. Let $F_{MM}(\mu, \nu, t) = \varepsilon$, $0 < \varepsilon < 1$. For each ε_0, $\varepsilon < \varepsilon_0 < 1$, using Remark 6, we can find ε_1 such that $\varepsilon_1 \star \varepsilon_1 \geq \varepsilon_0$. Now, consider the open balls $B(\mu, 1-\varepsilon_1, t^{1/2})$ and $B(\nu, 1-\varepsilon_1, t^{1/2})$. Clearly,

$$B\left(\mu, 1-\varepsilon_1, t^{1/2}\right) \cap B\left(\nu, 1-\varepsilon_1, t^{1/2}\right) = \phi. \tag{4}$$

For if there exists

$$\rho \in B\left(\mu, 1-\varepsilon_1, t^{1/2}\right) \cap B\left(\nu, 1-\varepsilon_1, t^{1/2}\right). \tag{5}$$

Then,

$$\varepsilon = F_{MM}(\mu, \nu, t) \geq F_{MM}\left(\mu, \rho, t^{1/2}\right) \star F_{MM}\left(\rho, \nu, t^{1/2}\right) \geq \varepsilon_1 \star \varepsilon_1 \geq \varepsilon_0 > \varepsilon, \tag{6}$$

which is a contradiction. Therefore, $(M, F_{MM}, \star)$ is Hausdorff.□ □

Definition 11. In a fuzzy multiplicative metric space $(M, F_{MM}, \star)$, a sequence $\{\mu_a\}$ is a convergent sequence which converges to μ if and only if there exist $a_1 \in \mathbb{N}$ with $F_{MM}(\mu_a, \mu, t) > 1 - \varepsilon$, for all $a \geq a_1$ and for each $\varepsilon > 0, t > 1$.

Theorem 12. *Let $(M, F_{MM}, \star)$ be a fuzzy multiplicative metric space, $\mu \in M$ and $\{\mu_a\}$ be a sequence in M. Then, $\{\mu_a\}$ converges to μ if and only if $F_{MM}(\mu_a, \mu, t) \longrightarrow 1$ as $a \longrightarrow \infty$ for each $t > 1$.*

Proof. Suppose that $\mu_a \longrightarrow \mu$. Then, for each $t > 1$ and $\varepsilon \in (0, 1)$, there exists a natural number a_1 such that $F_{MM}(\mu_a, \mu, t) > 1 - \varepsilon$ for all $a \geq a_1$. We have $1 - F_{MM}(\mu_a, \mu, t) < \varepsilon$. Hence, $F_{MM}(\mu_a, \mu, t) \longrightarrow 1$ as $a \longrightarrow \infty$.

Conversely, suppose that $F_{MM}(\mu_a, \mu, t) \longrightarrow 1$ as $a \longrightarrow \infty$. Then, for each $t > 1$ and $\varepsilon \in (0, 1)$, there exist a natural number a_1 such that $1 - F_{MM}(\mu_a, \mu, t) < \varepsilon$ for all $a \geq a_1$. In that case, $F_{MM}(\mu_a, \mu, t) > 1 - \varepsilon$. Hence, $\mu_a \longrightarrow \mu$ as $a \longrightarrow \infty$.□ □

Definition 13. Consider a sequence $\{\mu_a\}$ in a fuzzy multiplicative metric space $(M, F_{MM}, \star)$. If for each $\varepsilon > 0, t > 1$, there exist $a_1 \in \mathbb{N}$ such that $F_{MM}(\mu_a, \mu_b, t) > 1 - \varepsilon$ for all $a, b \geq a_1$, and then, $\{\mu_a\}$ is termed as Cauchy sequence in M.

Theorem 14. *Let $(M, F_{MM}, \star)$ be a fuzzy multiplicative metric space, $\mu \in M$ and $\{\mu_a\}$ be a sequence in M. Then, $\{\mu_a\}$ is Cauchy if and only if $F_{MM}(\mu_a, \mu_b, t) \longrightarrow 1$ as $a, b \longrightarrow \infty$ for each $t > 1$.*

Proof. Suppose that μ_a is a Cauchy sequence in M. Then, for each $t > 1$ and $\varepsilon \in (0, 1)$, there exists a natural number a_1 such that $F_{MM}(\mu_a, \mu_b, t) > 1 - \varepsilon$ for all $a, b \geq a_1$. We have $1 - F_{MM}(\mu_a, \mu_b, t) < \varepsilon$. Hence, $F_{MM}(\mu_a, \mu_b, t) \longrightarrow 1$ as $a, b \longrightarrow \infty$.

Conversely, suppose that $F_{MM}(\mu_a, \mu_b, t) \longrightarrow 1$ as $a, b \longrightarrow \infty$. Then, for each $t > 1$ and $\varepsilon \in (0, 1)$, there exists a natural number a_1 such that $1 - F_{MM}(\mu_a, \mu_b, t) < \varepsilon$ for all $a, b \geq a_1$. In that case, $F_{MM}(\mu_a, \mu_b, t) > 1 - \varepsilon$. Hence, μ_a is a Cauchy sequence.□ □

Proposition 15. *In a fuzzy multiplicative metric space $(M, F_{MM}, \star)$, if a sequence $\{\mu_a\}$ converges in M, then $\{\mu_a\}$ is Cauchy.*

Proof. Let ε and t be real numbers with $\varepsilon \in (0, 1)$, $t > 1$. Since $\varepsilon \in (0, 1)$, there is some $\varepsilon_0 \in (0, 1)$ such that $(1 - \varepsilon_0) \star (1 - \varepsilon_0) > 1 - \varepsilon$. Also, suppose that $\{\mu_a\}$ converges in M, say it converges to $\mu \in M$. Then, there exists $a_0 \in \mathbb{N}$ such that for each $a \geq a_0$,

$$F_{MM}\left(\mu_a, \mu, t^{1/2}\right) > 1 - \varepsilon_0. \tag{7}$$

Thus, for $a > b \geq a_0$, we have

$$F_{MM}(\mu_a, \mu_b, t) \geq F_{MM}\left(\mu_a, \mu, t^{1/2}\right) \star F_{MM}\left(\mu_b, \mu, t^{1/2}\right) > (1 - \varepsilon_0) \star (1 - \varepsilon_0) > 1 - \varepsilon. \tag{8}$$

□ □

That is $\{\mu_a\}$ is a Cauchy sequence.

Definition 16. A fuzzy multiplicative metric space $(M, F_{MM}, \star)$ is termed as complete if and only if every sequence in M which is Cauchy must converge in M.

Definition 17. Let $(M, F_{MM}, \star)$ be a fuzzy multiplicative metric space. A subset A of M is closed if for each sequence $\{\mu_a\}$ in A which is convergent with $\mu_a \longrightarrow \mu$, we have $\mu \in A$.

Remark 18. Let $(M, F_{MM}, \star)$ be a complete fuzzy multiplicative metric space. A subset A of M is closed if and only if $(A, F_{MM}, \star)$ is complete.

The following lemma is the analogue of Kiany's lemma [16] in the setting of newly defined space.

Lemma 19. *Let $(M, F_{MM}, \star)$ be a fuzzy multiplicative metric space such that for $\mu, \nu \in M$, $t > 1$ and $h > 1$*

$$\lim_{a \longrightarrow \infty} \star_{i=a}^{\infty} F_{MM}\left(\mu, \nu, t^{h^i}\right) = 1. \tag{9}$$

Suppose $\{\mu_a\}$ is a sequence in M such that for all $a \in \mathbb{N}$

$$F_{MM}(\mu_a, \mu_{a+1}, t^{\alpha}) \geq F_{MM}(\mu_{a-1}, \mu_a, t), \tag{10}$$

where $0 < \alpha < 1$. Then, $\{\mu_a\}$ is a Cauchy sequence.

Proof. For each $a \in \mathbb{N}$ and $t > 1$, we have

$$F_{MM}(\mu_a, \mu_{a+1}, t) \geq F_{MM}(\mu_{a-1}, \mu_a, t^{1/\alpha}) \geq F_{MM}\left(\mu_{a-2}, \mu_{a-1}, t^{1/\alpha^2}\right) \geq \cdots \geq F_{MM}\left(\mu_0, \mu_1, t^{1/\alpha^{a-1}}\right). \tag{11}$$

Thus, for each $a \in \mathbb{N}$, we get

$$F_{MM}(\mu_a, \mu_{a+1}, t) \geq F_{MM}\left(\mu_0, \mu_1, t^{1/\alpha^{a-1}}\right). \tag{12}$$

□ □

Choosing constants $h > 1$ and $l \in \mathbb{N}$ such that $h\alpha < 1$ and $\sum_{i=l}^{\infty} 1/h^i = (1/h^l)/(1-(1/h)) < 1$. Therefore, for $b \geq a$,

$$F_{MM}(\mu_a, \mu_b, t) \geq F_{MM}\left(\mu_a, \mu_b, t^{\left(\left(1/h^l\right)+\left(1/h^{l+1}\right)+\cdots+\left(1/h^{l+b}\right)\right)}\right) \geq F_{MM}\left(\mu_a, \mu_{a+1}, t^{1/h^l}\right) \star F_{MM}\left(\mu_{a+1}, \mu_{a+2}, t^{1/h^{l+1}}\right) \text{no number},$$

$$\star \cdots \star F_{MM}\left(\mu_{b-1}, \mu_b, t^{1/h^{l+b}}\right). \tag{13}$$

Using (12) in above inequality, we have

$$F_{MM}(\mu_a, \mu_b, t) \geq F_{MM}\left(\mu_0, \mu_1, t^{1/\alpha^{a-1}h^l}\right) \star F_{MM} \cdot \left(\mu_0, \mu_1, t^{1/\alpha^a h^{l+1}}\right) \star \cdots \star F_{MM}\left(\mu_0, \mu_1, t^{1/\left(\alpha^{b-2}h^{l+b-a-2}\right)}\right). \tag{14}$$

That is

$$F_{MM}(\mu_a, \mu_b, t) \geq F_{MM}\left(\mu_0, \mu_1, t^{1/(\alpha h)^{a-1}}\right) \star F_{MM} \cdot \left(\mu_0, \mu_1, t^{1/(\alpha h)^a}\right) \star \cdots \star F_{MM}\left(\mu_0, \mu_1, t^{1/(\alpha h)^{b-2}}\right). \tag{15}$$

The above expression can be simplified as

$$F_{MM}(\mu_a, \mu_b, t) \geq \star_{i=a}^{\infty} F_{MM}\left(\mu_0, \mu_1, t^{1/(\alpha h)^{i-1}}\right). \tag{16}$$

Then, from the above, we have

$$\lim_{a,b \longrightarrow \infty} F_{MM}(\mu_a, \mu_b, t) \geq \lim_{a \longrightarrow \infty} \star_{i=a}^{\infty} F_{MM}\left(\mu_0, \mu_1, t^{1/(\alpha h)^{i-1}}\right) = 1, \tag{17}$$

for each $t > 1$. Hence, for each $t > 1$,

$$\lim_{a,b \longrightarrow \infty} F_{MM}(\mu_a, \mu_b, t) = 1, \tag{18}$$

which shows that $\{\mu_a\}$ is a Cauchy sequence.

Definition 20. Consider a fuzzy multiplicative metric space $(M, F_{MM}, \star)$ and $A, B \subset M$; then, for all $t > 1$,

$$A_0 = \{\mu \in A : F_{MM}(\mu, \nu, t) = F_{MM}(A, B, t) \text{ for some } \nu \in B\},$$
$$B_0 = \{\nu \in B : F_{MM}(\mu, \nu, t) = F_{MM}(A, B, t) \text{ for some } \mu \in A\}, \tag{19}$$

where

$$F_{MM}(A, B, t) = \text{Sup}\{F_{MM}(\mu, \nu, t), \mu \in A, \nu \in B\}, \tag{20}$$

for all $t > 1$.

Definition 21. Let $(M, F_{MM}, \star)$ be a fuzzy multiplicative metric space and $A, B \subset M$. If every sequence $\{\mu_a\}$ of A, fulfilling the condition that $F_{MM}(\nu, \mu_a, t) \longrightarrow F_{MM}(\nu, A, t)$ for some ν in B and for all $t > 1$, has a convergent subsequence, then A is termed as approximatively compact with respect to B.

3. Best Proximity Point Theorems in Fuzzy Multiplicative Metric Spaces

In the present section, we prove some best proximity point theorems for single-valued and multivalued proximal contractions. First, we define the analogous of proximal contractions in the setting of fuzzy multiplicative metric space and then proceed to the main results.

Definition 22. Let $(M, F_{MM}, \star)$ be a fuzzy multiplicative metric space and $A, B \subset M$. A mapping $T : A \longrightarrow B$ is named as multiplicative contraction of first kind if there exists $\alpha \in [0, 1)$, such that for all $u, \nu, \mu, \nu \in A$

$$F_{MM}(u, T\mu, t) = F_{MM}(A, B, t), \tag{21}$$

$$F_{MM}(\nu, T\nu, t) = F_{MM}(A, B, t) \Rightarrow F_{MM}(u, \nu, t^\alpha) \geq F_{MM}(\mu, \nu, t). \tag{22}$$

Theorem 23. *Let $(M, F_{MM}, \star)$ be a complete fuzzy multiplicative metric space and $A, B \subset M$ such that B is approximatively compact with respect to A. Assume that $\lim_{t \longrightarrow \infty} F_{MM}(\mu, \nu, t) = 1$, $T : A \longrightarrow B$ be multiplicative contraction of first kind and $T(A_0) \subset B_0$. Then, T possesses best proximity point.*

Proof. Let $\mu_0 \in A_0$ then for $T\mu_0 \in TA_0 \subset B_0$, there exist $\mu_1 \in A_0$ such that

$$F_{MM}(\mu_1, T\mu_0, t) = F_{MM}(A, B, t). \tag{23}$$

Further, since $T\mu_1 \in TA_0 \subset B_0$, there exist $\mu_2 \in A_0$ such that

$$F_{MM}(\mu_2, T\mu_1, t) = F_{MM}(A, B, t). \tag{24}$$

Similarly, for $T\mu_2 \in TA_0 \subset B_0$, there exist $\mu_3 \in A_0$ such that

$$F_{MM}(\mu_3, T\mu_2, t) = F_{MM}(A, B, t). \quad (25)$$

By continuing the similar steps, we get

$$F_{MM}(\mu_{a+1}, T\mu_a, t) = F_{MM}(A, B, t) \text{ for all } a \in \mathbb{N}. \quad (26)$$

By successive application of fuzzy multiplicative contraction, we have for all $a \in \mathbb{N} \cup \{0\}$

$$F_{MM}(\mu_a, \mu_{a+1}, t^\alpha) \geq F_{MM}(\mu_{a-1}, \mu_a, t) \geq F_{MM}\left(\mu_{a-2}, \mu_{a-1}, t^{1/\alpha}\right) \geq F_{MM}\left(\mu_{a-3}, \mu_{a-2}, t^{1/\alpha^2}\right) \geq \cdots \geq F_{MM}\left(\mu_0, \mu_1, t^{1/\alpha^{a-1}}\right). \quad (27)$$

For any $q \in \mathbb{N}$,

$$F_{MM}\left(\mu_a, \mu_{a+q}, t\right) \geq F_{MM}\left(\mu_a, \mu_{a+1}, t^{1/q}\right) \star F_{MM} \cdot \left(\mu_{a+1}, \mu_{a+2}, t^{1/q}\right) \star \cdots \star F_{MM}\left(\mu_{a+q-1}, \mu_{a+q}, t^{1/q}\right). \quad (28)$$

Using (27) in above inequality, we obtain

$$F_{MM}\left(\mu_a, \mu_{a+q}, t\right) \geq F_{MM}\left(\mu_0, \mu_1, t^{1/q\alpha^a}\right) \star F_{MM} \cdot \left(\mu_0, \mu_1, t^{1/q\alpha^{a+1}}\right) \star \cdots \star F_{MM}\left(\mu_0, \mu_1, t^{1/q\alpha^{a+q-1}}\right). \quad (29)$$

By assumption, $\lim_{t \longrightarrow \infty} F_{MM}(\mu, \nu, t) = 1$, we get that

$$\lim_{a \longrightarrow \infty} F_{MM}\left(\mu_a, \mu_{a+q}, t\right) = 1 \star 1 \star \cdots \star 1 = 1. \quad (30)$$

Hence, $\{\mu_a\}$ is a Cauchy sequence. The completeness of fuzzy multiplicative metric space $(M, F_{MM}, \star)$ implies that $\{\mu_a\}$ converges to $\mu^* \in A$, that is,

$$\lim_{a \longrightarrow \infty} F_{MM}(\mu_a, \mu^*, t) = 1, \quad (31)$$

for all $t > 1$. Notice that

$$F_{MM}(\mu, B, t) \geq F_{MM}(\mu, T\mu_a, t) \geq F_{MM}\left(\mu, \mu_{a+1}, t^{1/2}\right) \star F_{MM}\left(\mu_{a+1}, T\mu_a, t^{1/2}\right) = F_{MM}\left(\mu, \mu_{a+1}, t^{1/2}\right) \star F_{MM}(A, B, t) \geq F_{MM}\left(\mu, \mu_{a+1}, t^{1/2}\right) \star F_{MM}(\mu, B, t). \quad (32)$$

□ □

Therefore, $F_{MM}(\mu, T\mu_a, t) \longrightarrow F_{MM}(\mu, B, t)$ as $a \longrightarrow \infty$. Since B is approximatively compact with respect to A, so $\{T\mu_a\}$ has a convergent subsequence $\{T\mu_{a_k}\}$ converging to some $\rho \in B$. Further, for each $k \in \mathbb{N}$, we have

$$F_{MM}(A, B, t) \geq F_{MM}(\mu, \rho, t) \geq F_{MM}\left(\mu, \mu_{a_{k+1}}, t^{1/3}\right) \star F_{MM} \cdot \left(\mu_{a_{k+1}}, T\mu_{a_k}, t^{1/3}\right) \star F_{MM}\left(T\mu_{a_k}, \rho, t^{1/3}\right) = F_{MM}\left(\mu, \mu_{a_{k+1}}, t^{1/3}\right) \star F_{MM}\left(A, B, t^{1/3}\right) \star F_{MM}\left(T\mu_{a_k}, \rho, t^{1/3}\right). \quad (33)$$

Letting $k \longrightarrow \infty$, we get $F_{MM}(\mu, \rho, t) = F_{MM}(A, B, t)$, which implies that $\mu \in A_0$ and $T(A_0) \subseteq B_0$ implies that $T\mu \in B_0$, there exist $\mu^* \in A$, such that $F_{MM}(\mu^*, T\mu, t) = F_{MM}(A, B, t)$. From this and equation (26) implies that

$$F_{MM}(\mu_{a+1}, \mu^*, t) \geq F_{MM}\left(\mu_a, \mu, t^{1/\alpha}\right). \quad (34)$$

Applying limit $a \longrightarrow \infty$ to above inequality gives $F_{MM}(\mu, \mu^*, t) = 1$ which implies that $\mu = \mu^*$. Hence, $F_{MM}(\mu, T\mu, t) = F_{MM}(A, B, t)$, which shows that T possesses best proximity point μ.

Example 24. Let $M = \mathbb{R}^+ \times \mathbb{R}^+$ and $F_{MM}(\mu, \nu, t) = (t+1)/(t + d(\mu, \nu))$ where $d(\mu, \nu) = |\mu_1/\nu_1|^* . |\mu_2/\nu_2|^*$ for $\mu = (\mu_1, \mu_2)$ and $\nu = (\nu_1, \nu_2)$. Then, $(M, F_{MM}, \star)$ is complete fuzzy multiplicative metric space with $\star : [0, 1]^2 \longrightarrow [0, 1]$ defined as $a \star b = ab$. Let $A = \{(1, \mu) : \mu \in \mathbb{R}^+\}$ and $B = \{(2, \nu) : \nu \in \mathbb{R}^+\}$ then A and B are closed subsets of M and $F_{MM}(A, B, t) = (t+1)/(t+2)$, $A_0 = A$, $B_0 = B$. Define $T : A \longrightarrow B$ as

$$T(1, \mu) = \left(2, \frac{\mu^2}{2}\right). \quad (35)$$

Let $\mu = (1, \mu), \nu = (1, \nu) \in A$ and then $u = (1, \mu^2/2)$ and $v = (1, \nu^2/2) \in A$ such that $F_{MM}(u, T\mu, t) = F_{MM}(A, B, t) = F_{MM}(v, T\nu, t)$. It can be easily checked that T is proximal contraction in fuzzy multiplicative metric space M with $\alpha = 2/3$. Also, the condition $\lim_{t \longrightarrow \infty} F_{MM}(\mu, \nu, t) = 1$ holds.

Since all statements of Theorem 23 hold, therefore T possesses best proximity point. We can see that $(1, 2)$ is best proximity point of T.

If $A = B = M$ in Theorem 23, then we obtain the following corollary which is the fixed-point theorem for fuzzy multiplicative contraction in fuzzy multiplicative metric space.

Corollary 25. *Let $(M, F_{MM}, \star)$ be a complete fuzzy multiplicative metric space. A mapping $T : M \longrightarrow M$ satisfying $F_{MM}(\mu, \nu, t^\alpha) \geq F_{MM}(\mu, \nu, t)$ has fixed point provided that $\lim_{t \longrightarrow \infty} F_{MM}(\mu, \nu, t) = 1$.*

Now, we prove a best proximity theorem for Feng-Liu-type multivalued contraction in fuzzy multiplicative metric space.

Theorem 26. *Let $(M, F_{MM}, \star)$ be complete fuzzy multiplicative metric space. $A, B \subseteq M$ be two nonempty closed subsets of M having P-property and $A_0 \neq \phi$. Let $T : A \longrightarrow C(B)$ be a mapping such that $T(A_0) \subseteq B_0$ and for all $\mu \in A_0$ and $\nu \in$*

$T\mu$, there exist $\rho \in A_0$ satisfying

$$F_{MM}(\nu, \rho, t) = F_{MM}(A, B, t) \text{ and } F_{MM}(\nu, T\rho, t^c) \geq F_{MM}(\mu, \rho, t), \quad (36)$$

for some $c \in (0, 1)$ and $t > 1$. Assume that $(M, F_{MM}, \star)$ satisfy

$$\lim_{a \longrightarrow \infty} \star_{i=a}^{\infty} F_{MM}\left(\mu, \nu, t^{h^i}\right) = 1, \quad (37)$$

for every $\mu, \nu \in M$, $t > 1$ and $h > 1$. Then, T has best proximity point in A provided that $f(\mu, \nu) = F_{MM}(\nu, T\mu, t)$ is upper semicontinuous.

Proof. Let $\mu_0 \in A_0$ be arbitrary point. Choose $\nu_0 \in T\mu_0$. Then, by assumption, there exist $\mu_1 \in A_0$ such that

$$\begin{aligned} F_{MM}(\nu_0, \mu_1, t) &= F_{MM}(A, B, t), \\ F_{MM}(\nu_0, T\mu_1, t^c) &\geq F_{MM}(\mu_0, \mu_1, t). \end{aligned} \quad (38)$$

Presently, let $b \in (c, 1)$, and then, we can discover $\nu_1 \in T\mu_1$ such that

$$F_{MM}(\nu_0, \nu_1, t) \geq F_{MM}\left(\nu_0, T\mu_1, t^b\right). \quad (39)$$

Again by assumption, there exist $\mu_2 \in A_0$ such that

$$\begin{aligned} F_{MM}(\nu_1, \mu_2, t) &= F_{MM}(A, B, t), \\ F_{MM}(\nu_1, T\mu_2, t^c) &\geq F_{MM}(\mu_1, \mu_2, t). \end{aligned} \quad (40)$$

Also, we can find $\nu_2 \in T\mu_2$ such that

$$F_{MM}(\nu_1, \nu_2, t) \geq F_{MM}\left(\nu_1, T\mu_2, t^b\right). \quad (41)$$

□ □

Proceeding in similar manner, we develop two sequences $\{\mu_a\}$ and $\{\nu_a\}$ in A and B, respectively, with $\mu_a \in A_0$, $\nu_a \in T\mu_a$ and

$$F_{MM}(\nu_a, \mu_{a+1}, t) = F_{MM}(A, B, t), \quad (42)$$

$$F_{MM}(\nu_a, T\mu_{a+1}, t^c) \geq F_{MM}(\mu_a, \mu_{a+1}, t), \quad (43)$$

$$F_{MM}(\nu_a, \nu_{a+1}, t) \geq F_{MM}(\nu_a, T\mu_{a+1}, t), \quad (44)$$

for all $a \in \mathbb{N}$ and $t > 1$. Then again, since A and B have P -property, so from inequality (43), we get

$$F_{MM}(\mu_a, \mu_{a+1}, t) = F_{MM}(\nu_{a-1}, \nu_a, t). \quad (45)$$

Therefore, from inequality (44), we have

$$F_{MM}(\mu_a, \mu_{a+1}, t) = F_{MM}(\nu_{a-1}, \nu_a, t) \geq F_{MM}\left(\nu_{a-1}, T\mu_a, t^b\right). \quad (46)$$

From inequality (44), we have

$$F_{MM}(\nu_{a-1}, T\mu_a, t) \geq F_{MM}\left(\mu_{a-1}, \mu_a, t^{1/c}\right). \quad (47)$$

Combining inequalities (46) and (47), we get

$$F_{MM}(\mu_a, \mu_{a+1}, t) \geq F_{MM}\left(\mu_{a-1}, \mu_a, t^{b/c}\right), \quad (48)$$

for all $a \geq 1$ and $t > 1$.
Let $k = c/b$ and then $0 < k < 1$. The inequality (48) gives

$$F_{MM}\left(\mu_a, \mu_{a+1}, t^k\right) \geq F_{MM}(\mu_{a-1}, \mu_a, t), \quad (49)$$

for $0 < k < 1$ and $t > 1$. By our assumption (37) and Lemma 19, $\{\mu_a\}$ is Cauchy sequence.
Now, from inequalities (44) and (46), we have

$$\begin{aligned} &F_{MM}(\nu_a, T\mu_{a+1}, t^c) \geq F_{MM}(\mu_a, \mu_{a+1}, t) \geq F_{MM}\left(\nu_{a-1}, T\mu_a, t^b\right) \\ &\quad \Rightarrow F_{MM}(\nu_a, T\mu_{a+1}, t) \geq F_{MM}\left(\nu_{a-1}, T\mu_a, t^{b/c}\right). \end{aligned} \quad (50)$$

Also, from inequalities (44) and (56), we have

$$\begin{aligned} &F_{MM}\left(\nu_a, \nu_{a+1}, t^{1/b}\right) \geq F_{MM}(\nu_a, T\mu_{a+1}, t) \geq F_{MM}\left(\nu_{a-1}, T\mu_a, t^{b/c}\right) \\ &\quad \Rightarrow F_{MM}(\nu_a, \nu_{a+1}, t^c) \geq F_{MM}(\nu_{a-1}, T\mu_a, t), \end{aligned} \quad (51)$$

for $0 < c < 1$ and $t > 1$. Hence, $\{\nu_a\}$ is Cauchy sequence.
As subsets A and B are closed in M, therefore $\{\mu_a\}$, $\{\nu_a\}$ converges to points of A and B, respectively. Thus, there exist $\mu^* \in A$ and $\nu^* \in B$ such that $\mu_a \longrightarrow \mu^*$ and $\nu_a \longrightarrow \nu^*$ as $a \longrightarrow \infty$.
Letting $a \longrightarrow \infty$ in inequality (43), we have

$$F_{MM}(\mu^*, \nu^*, t) = F_{MM}(A, B, t), \quad (52)$$

for $t > 1$. The inequality (56) shows that the sequence $f(\mu_a, \nu_a) = F_{MM}(\nu_a, T\mu_a, t)$ is increasing and it converges to 1. Since $f(\mu, \nu)$ is upper semicontinuous, so

$$1 = \limsup_{a \longrightarrow \infty} f(\mu_a, \nu_a) \leq f(\mu^*, \nu^*) \leq 1 \quad (53)$$

implies to the fact that $f(\mu^*, \nu^*) = 1$, that is, $F_{MM}(\nu^*, T\mu^*, t) = 1$, and hence, $\nu^* \in T\mu^*$. Therefore,

(FM1)

$$\begin{aligned} F_{MM}(A, B, t) &\geq F_{MM}(\mu^*, T\mu^*, t) \geq F_{MM}(\mu^*, \nu^*, t) \\ &= F_{MM}(A, B, t), \end{aligned} \quad (54)$$

that is, $F_{MM}(\mu^*, T\mu^*, t) = F_{MM}(A, B, t)$. This shows that T possesses best proximity point μ^*.

If $A = B = M$ in Theorem 26, then we obtain the following corollary which is the fixed-point theorem for Feng-Liu-type contraction in fuzzy multiplicative metric space.

Corollary 27. *Let $(M, F_{MM}, \star)$ be complete fuzzy multiplicative metric space. Let $T : M \longrightarrow C(M)$ be a mapping, for all $\mu \in M$ and $\nu \in I_b^\mu$ (where $I_b^\mu = \{\nu \in T\mu \mid F_{MM}(\mu, \nu, t) \geq F_{MM}(\mu, T\mu, t^b)\} \subset M$ for some $b \in (0, 1)$) satisfying*

$$F_{MM}(\nu, T\nu, t^c) \geq F_{MM}(\mu, \nu, t), \tag{55}$$

for some $c \in (0, 1)$ and $t > 1$. Then, T possesses fixed point provided that $c < b$ and $f(\mu) = F_{MM}(\mu, T\mu, t)$ is upper semicontinuous.

4. Best Proximity Point Theorems of Feng-Liu-Type Mappings in Fuzzy Metric Space

Getting motivation from the work of Sahin et al. [10], we prove the following result.

Theorem 28. *Let $(M, F_M, \star)$ be complete fuzzy metric space. $A, B \subseteq M$ be closed and nonempty having P-property and $A_0 \neq \phi$. Let $T : A \longrightarrow C(B)$ be a mapping such that $T(A_0) \subseteq B_0$ and for all $\mu \in A_0$ and $\nu \in T\mu$ there exist $\rho \in A_0$ satisfying*

$$\begin{aligned} &F_M(\nu, \rho, t) = F_M(A, B, t),\\ &F_M(\nu, T\rho, ct) \geq F_M(\mu, \rho, t), \end{aligned} \tag{56}$$

for some $c \in (0, 1)$ and $t > 0$. Assume that $(M, F_M, \star)$ satisfy

$$\lim_{a \longrightarrow \infty} *_{i=a}^{\infty} F_M\left(\mu, \nu, th^i\right) = 1, \tag{57}$$

for every $t > 0, h > 1$ and $\mu, \nu \in M$. Then, T possesses best proximity point in A provided that $f(\mu, \nu) = F_M(\nu, T\mu, t)$ is upper semicontinuous.

Proof. Let $\mu_0 \in A_0$ be arbitrary point. Choose $\nu_0 \in T\mu_0$. Then, by assumption, there exist $\mu_1 \in A_0$ such that $F_M(\nu_0, \mu_1, t) = F_M(A, B, t)$ and $F_M(\nu_0, T\mu_1, ct) \geq F_M(\mu_0, \mu_1, t)$.

Presently, let $b \in (c, 1)$, then we can discover $\nu_1 \in T\mu_1$ such that

$$F_M(\nu_0, \nu_1, t) \geq F_M(\nu_0, T\mu_1, bt). \tag{58}$$

Again by assumption, there exist $\mu_2 \in A_0$ such that $F_M(\nu_1, \mu_2, t) = F_M(A, B, t)$ and $F_M(\nu_1, T\mu_2, ct) \geq F_M(\mu_1, \mu_2, t)$.

Also, we can find $\nu_2 \in T\mu_2$ such that

$$F_M(\nu_1, \nu_2, t) \geq F_M(\nu_1, T\mu_2, bt). \tag{59}$$

□ □

Proceeding in a similar manner, we develop two sequences $\{\mu_a\}$ and $\{\nu_a\}$ in A and B, respectively, with $\mu_a \in A_0$, $\nu_a \in T\mu_a$ and

$$F_M(\nu_a, \mu_{a+1}, t) = F_M(A, B, t), \tag{60}$$

$$F_M(\nu_a, T\mu_{a+1}, ct) \geq F_M(\mu_a, \mu_{a+1}, t), \tag{61}$$

$$F_M(\nu_a, \nu_{a+1}, t) \geq F_M(\nu_a, T\mu_{a+1}, t), \tag{62}$$

for all $a \in \mathbb{N}$ and $t > 0$. Then again, since A and B have P-property, so from inequality (61), we get

$$F_M(\mu_a, \mu_{a+1}, t) = F_M(\nu_{a-1}, \nu_a, t). \tag{63}$$

Therefore, from inequality (62), we have

$$F_M(\mu_a, \mu_{a+1}, t) = F_M(\nu_{a-1}, \nu_a, t) \geq F_M(\nu_{a-1}, T\mu_a, bt). \tag{64}$$

From inequality (62), we have

$$F_M(\nu_{a-1}, T\mu_a, t) \geq F_M\left(\mu_{a-1}, \mu_a, \frac{1}{c}t\right). \tag{65}$$

Combining inequalities (64) and (65), we get

$$F_M(\mu_a, \mu_{a+1}, t) \geq F_M\left(\mu_{a-1}, \mu_a, \frac{b}{c}t\right), \tag{66}$$

for all $a \geq 1$ and $t > 0$.

Let $k = c/b$ and then $0 < k < 1$. The inequality (66) gives

$$F_M(\mu_a, \mu_{a+1}, kt) \geq F_M(\mu_{a-1}, \mu_a, t), \tag{67}$$

for $0 < k < 1$ and $t > 0$. By our assumption (57) and Lemma 19, $\{\mu_a\}$ is Cauchy sequence.

Now, from inequalities (62) and (64), we have

$$\begin{aligned} &F_M(\nu_a, T\mu_{a+1}, ct) \geq F_M(\mu_a, \mu_{a+1}, t) \geq F_M(\nu_{a-1}, T\mu_a, bt)\\ &\quad \Rightarrow F_M(\nu_a, T\mu_{a+1}, t) \geq F_M\left(\nu_{a-1}, T\mu_a, \frac{b}{c}t\right). \end{aligned} \tag{68}$$

Also, from inequalities (62) and (68), we have

$$\begin{aligned} &F_M\left(\nu_a, \nu_{a+1}, \frac{1}{b}t\right) \geq F_M(\nu_a, T\mu_{a+1}, t) \geq F_M\left(\nu_{a-1}, T\mu_a, \frac{b}{c}t\right)\\ &\quad \Rightarrow F_M(\nu_a, \nu_{a+1}, ct) \geq F_M(\nu_{a-1}, T\mu_a, t), \end{aligned} \tag{69}$$

for $0 < c < 1$ and $t > 0$. Hence, $\{\nu_a\}$ is Cauchy sequence.

As subsets A and B are closed in M, so $\{\mu_a\}, \{\nu_a\}$ converges to points of A and B, respectively. Thus, there is some $\mu^* \in A$ and $\nu^* \in B$ such that $\mu_a \longrightarrow \mu^*$ and $\nu_a \longrightarrow \nu^*$ as $a \longrightarrow \infty$.

Letting $a \longrightarrow \infty$ in equation (61), we have

$$F_M(\mu^*, \nu^*, t) = F_M(A, B, t), \tag{70}$$

for $t > 0$. The inequality (68) shows that the sequence $f(\mu_a$

$, \nu_a) = F_M(\nu_a, T\mu_a, t)$ is increasing sequence, so it converges to 1. Since $f(\mu, \nu)$ is upper semicontinuous, so

$$1 = \limsup_{a \longrightarrow \infty} f(\mu_a, \nu_a) \leq f(\mu^*, \nu^*) \leq 1 \tag{71}$$

implies to the fact that $f(\mu^*, \nu^*) = 1$, that is, $F_M(\nu^*, T\mu^*, t) = 1$, and hence, $\nu^* \in T\mu^*$.

Therefore,

$$F_M(A, B, t) \geq F_M(\mu^*, T\mu^*, t) \geq F_M(\mu^*, \nu^*, t) = F_M(A, B, t), \tag{72}$$

that is, $F_M(\mu^*, T\mu^*, t) = F_M(A, B, t)$. This shows that T possesses best proximity point μ^*.

Example 29. Let $J = \{0, 1\} \cup \{1/2^a : a \in \mathbb{N}\}$ and $M = J \times J$, $F_M(\mu, \nu, t) = t/(t + d(\mu, \nu))$ and $d(\mu, \nu) = |\mu_1 - \nu_1| + |\mu_2 - \nu_2|$ for $\mu = (\mu_1, \mu_2)$ and $\nu = (\nu_1, \nu_2) \in M$. Then, $(M, F_M, \star)$ is complete fuzzy metric space where $\star : [0, 1]^2 \longrightarrow [0, 1]$ defined by $a \star b = ab$. Let $A = \{(0, 1/2^a) \colon a \in \mathbb{N}\} \cup \{(0, 0), (0, 1)\}$ and $B = \{(1, 1/2^a) \colon a \in \mathbb{N}\} \cup \{(1, 0), (1, 1)\}$. Then, $A_0 = A$, $B_0 = B$ and $F_M(A, B, t) = t/(t + 1)$. Define $T : A \longrightarrow C(B)$ as

$$T(1, \mu) = \begin{cases} \left\{ \left(0, \dfrac{1}{2^{a+1}}\right), (0, 1) \right\} \text{ if } \mu = \dfrac{1}{2^a}, & a = 0, 1, 2, \cdots, \\ \left\{ (0, 0), \left(0, \dfrac{1}{2}\right) \right\} \text{ if } \mu = 0. \end{cases} \tag{73}$$

For all $\mu, \nu \in M, \lim_{a \longrightarrow \infty} \star_{i=a}^{\infty} F_M(\mu, \nu, th^i) = 1$ which implies that M satisfies 16. Let $\mu = (1, 1/2^a) \in A_0$ and $\nu = (0, 1/2^{a_1}) \in T\mu = (1, 1/2^a)$; then, for $\rho = (1, 1/2^{a_1}) \in A_0$, we have $F_M(\nu, \rho, t) = F_M(A, B, t)$ and $F_M(\nu, T\rho, t) = 1 \geq F_M(\mu, \nu, t)$.

Also,

$$f(\mu, \nu) = F_M(\nu, T\mu, t) = \frac{t}{t + d(\nu, T\mu)} = \begin{cases} \dfrac{t}{t + \left(1/2^{a+1}\right)} \text{ for } \mu = \left(1, \dfrac{1}{2^a}\right) \\ 1 \text{ for } \mu = (1, 0), (1, 1) \end{cases} \tag{74}$$

is continuous. Since all conditions of Theorem 28 are satisfied, so best proximity point for T exists. Furthermore, for $u = (1, 1/2^a), \nu = (1, 0) \in A_0$ $H_{F_M}(T(1, 1/2^a), T(1, 0), ct) = ct/(ct + (1/2))$ and $F_M((1, 1/2^a), (1, 0), t) = t/(t + (1/2^a))$.

Assume that for $c \in (0, 1)$, $H_{F_M}(T(1, 1/2^a), T(1, 0), ct) \geq F_M((1, 1/2^a), (1, 0), t)$. That is

$$\frac{ct}{ct + (1/2)} \geq \frac{t}{t + (1/2^a)}, \tag{75}$$

which implies that $c \geq 2^{a-1}$ for $a \in \mathbb{N}$ which is a contradiction. This shows that T does not satisfy the contraction condition of Nadler's multivalued mapping.

As corollary of Theorem 28, we obtain a result which was proved in [23]. We get the corollary by taking $A = B = M$.

Corollary 30. *Let $(M, F_M, \star)$ be complete fuzzy metric space. Let $T : M \longrightarrow C(M)$ be a mapping, for all $\mu \in M$ and $\nu \in I_b^\mu$ (where $I_b^\mu = \{\nu \in T\mu \mid F_M(\mu, \nu, t) \geq F_M(\mu, T\mu, bt)\} \subset M$ for some $b \in (0, 1)$) satisfying*

$$F_M(\nu, T\nu, ct) \geq F_M(\mu, \nu, t), \tag{76}$$

for some $c \in (0, 1)$ and $t > 1$. Then, T possesses fixed point provided that $c < b$ and $f(\mu) = F_M(\mu, T\mu, t)$ is upper semicontinuous.

5. Conclusion

Zadeh [24] introduced the notion of fuzzy logic to cope with the problem of uncertainty that occurs essentially while studying real-life problem. Many researchers found easiness to study the phenomenon of different fields that were too complex to be analyzed by conventional techniques. Fuzzy metric introduced by Kaleva and Seikkala [25] measures the imprecision of distance between elements. Fuzzy metric has been applied in variety of applications like color image filtering [26] and in engineering methods [15]. Multiplicative calculus has its great applications in various fields, few of which are in biomedical image analysis [27] and contour detection in images [28]. In this paper, we introduced fuzzy multiplicative metric space and proved some best proximity point and fixed-point results in this new framework. The above discussion shows the possible applications in this framework in the future.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no competing interests concerning the publication of this article.

References

[1] K. Fan, "Extensions of two fixed point theorems of F. E. Browder," *Mathematische Zeitschrift*, vol. 112, no. 3, pp. 234–240, 1969.

[2] A. Abkar and M. Gabeleh, "Best proximity points for asymptotic cyclic contraction mappings," *Nonlinear Analysis*, vol. 74, no. 18, pp. 7261–7268, 2011.

[3] M. A. Alghamdi and N. Shahzad, "Best proximity point results in geodesic metric spaces," *Fixed Point Theory and Applications*, vol. 2012, Article ID 234, 2012.

[4] S. S. Basha, "Extensions of Banach's contraction principle," *Numerical Functional Analysis and Optimization*, vol. 31, no. 5, pp. 569–576, 2010.

[5] S. S. Basha, "Best proximity point theorems," *Journal of Approximation Theory*, vol. 163, no. 11, pp. 1772–1781, 2011.

[6] A. Eldred and P. Veeramani, “Existence and convergence of best proximity points,” *Journal of Mathematical Analysis and Applications*, vol. 323, no. 2, pp. 1001–1006, 2006.

[7] M. Jleli and B. Samet, “Best proximity point for $\alpha - \psi$-proximal contraction type mappings and applications,” *Bulletin des Sciences Mathematiques*, vol. 137, no. 8, pp. 977–995, 2013.

[8] S. B. Nadler, “Multi-valued contraction mappings,” *Pacific Journal of Mathematics*, vol. 30, no. 2, pp. 475–488, 1969.

[9] Y. Feng and S. Liu, “Fixed point theorems for multi-valued contractive mappings and multi-valued Caristi type mappings,” *Journal of Mathematical Analysis and Applications*, vol. 317, no. 1, pp. 103–112, 2006.

[10] H. Sahin, M. Aslantas, and I. Altun, “Feng-Liu type approach to best proximity point results for multivalued mappings,” *Journal of Fixed Point Theory and Applications*, vol. 22, no. 11, 2020.

[11] I. Kramosil and J. Michalek, “Fuzzy metric and statistical metric spaces,” *Kybernetica*, vol. 11, pp. 326–334, 1975.

[12] A. George and P. Veeramani, “On some results in fuzzy metric spaces,” *Fuzzy Sets and Systems*, vol. 64, no. 3, pp. 395–399, 1994.

[13] M. S. Ashraf, R. Ali, and N. Hussain, “New fuzzy fixed point results in generalized fuzzy metric spaces with applications to integral equations,” *IEEE Acess*, vol. 8, pp. 91653–91660, 2020.

[14] M. Grabiec, “Fixed points in fuzzy metric spaces,” *Fuzzy Sets and Systems*, vol. 27, no. 3, pp. 385–389, 1988.

[15] V. Gregori, S. Morillas, and A. Sapena, “Examples of fuzzy metrics and applications,” *Fuzzy Sets and Systems*, vol. 170, no. 1, pp. 95–111, 2011.

[16] F. Kiany and A. Amini-Harandi, “Fixed point and end point theorems for set valued fuzzy contraction maps in fuzzy metric spaces,” *Fixed Point Theory and Applications*, vol. 2011, no. 1, 9 pages, 2011.

[17] D. Mihet, “On fuzzy contractive mappings in fuzzy metric spaces,” *Fuzzy Sets and Systems*, vol. 158, no. 8, pp. 915–921, 2007.

[18] Z. Raza, N. Saleem, and M. Abbas, “Optimal coincidence points of proximal quasi-contraction mappings in non-Archimedean fuzzy metric spaces,” *Journal of Nonlinear Sciences and Applications*, vol. 9, no. 6, pp. 3787–3801, 2016.

[19] D. Turkoglu and M. Sangurlu, “Fixed point theorems for fuzzy ψ- contractive mappings in fuzzy metric spaces,” *Journal of Intelligent Fuzzy Systems*, vol. 26, no. 1, pp. 137–142, 2014.

[20] C. Vetro and P. Salimi, “Best proximity point results in non-Archimedean fuzzy metric spaces,” *Fuzzy Information and Engineering*, vol. 5, no. 4, pp. 417–429, 2013.

[21] M. Grossman and R. Katz, *Non-Newtonian Calculus*, Lee Press Pigeon Cove, MA, 1972.

[22] A. E. Bashirov, E. M. Kurpinar, and A. Ozyapici, “Multiplicative calculus and its applications,” *Journal of Mathematical Analysis and Applications*, vol. 337, no. 1, pp. 36–48, 2008.

[23] B. Ali and M. Abbas, “Fixed point theorems for multivalued contractive mappings in fuzzy metric spaces,” *American Journal of Applied Mathematics*, vol. 3, no. 3, pp. 41–45, 2015.

[24] L. A. Zadeh, “Fuzzy sets,” *Information and Control*, vol. 8, no. 3, pp. 338–353, 1965.

[25] O. Kaleva and S. Seikkala, “On fuzzy metric spaces,” *Fuzzy Sets and Systems*, vol. 12, no. 3, pp. 215–229, 1984.

[26] S. Morillas, V. Gregori, G. Peris-Fajarnes, and A. Sapena, “New adaptive vector filter using fuzzy metrics,” *Journal of Electronic Imaging*, vol. 16, no. 3, article 033007, 2007.

[27] L. Florack and H. van Assen, “Multiplicative calculus in biomedical image analysis,” *Journal of Mathematical Imaging and Vision*, vol. 42, no. 1, pp. 64–75, 2012.

[28] M. Mora, F. Cardova-Lepe, and R. Del-Valle, “A non-Newtonian gradient for contour detection in images with multiplicative noise,” *Pattern Recognition Letters*, vol. 33, no. 10, pp. 1245–1256, 2012.