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Research Article

Applications of Fixed Point Theory to Investigate a System of Fractional Order Differential Equations

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We investigate a nonlinear system of pantograph-type fractional differential equations (FDEs) via Caputo-Hadamard derivative (CHD). We establish the conditions for existence theory and Ulam-Hyers-type stability for the underlying boundary value system (BVS) of FDE. We use Krasnoselskii's and Banach's fixed point theorems to obtain the desired results for the existence of solution. Stability is an important aspect from a numerical point of view we investigate here. To justify the main work, relevant examples are provided.

1. Introduction

The generalized form of ordinary calculus is called fractional calculus. This newly developed branch of mathematics has numerous applications in many scientific fields including the study of nonlinear oscillations of earthquakes, nanotechnology, and other engineering disciplines. Also, fractional derivatives and integrals have the ability to explore the dynamics of many real-world problems more comprehensively and extensively. To these characteristics of the said area, researchers in the past several decades have taken great interest to investigate FDEs for a different kind of analysis. For applications and usefulness, see [1–5]. The concerned study includes optimization, stability and numerical results, and theoretical analysis. In this regard, existence theory for different kinds of problems of FDEs has been investigated and plenty of research work has been done (see [6–8]).

One of the new emerging classes of FDEs is known as pantograph differential equations (PDEs). The work related to this new research field has been published in large numbers. Initially, pantograph differential equations (PDEs) were studied with delay terms [9, 10], material modeling [11], and modeling lasers, especially quantum dot lasers [12]. Basically, PDEs give change in terms of a dependent variable at a previous time [13]. Some beneficial research has been performed in this area [14–16]. Further, these types of FDEs occur in traffic models, control systems, population dynamics, and many natural phenomena.

In the last few decades, the stability analysis for FDEs has been established very well. Therefore. different kinds of stability notions have been constructed in literature including exponential, Mittag-Leffler, and Lyapunov. The mentioned stability concepts have been very well investigated for FDEs. Among these, UH stability analysis is an important tool that has gained the attention of researchers [17, 18]. The aforesaid UH stability has extended to other forms in large many articles [19, 20]. The UH stability analysis method has been developed for ordinary and FDEs over the last twenty years [21–23].

It is remarkable that great interest has been observed to derive various kinds of results including qualitative and numerical for higher-order problems under BCs [24–26]. Since fractional derivative has various definitions, each and every definition has its own uncharacteristic features. One of the well-known definitions is called the Caputo-Hadamard derivative. The said area has been initiated in the last few years (for detail, see [27–29]). After that, the said definition has been used in large numbers of articles. Motivated from aforesaid work, the qualitative study of a coupled system of FDEs under BCs with fractional CHD has not been investigated properly involving proportional delay term. Therefore, using the results from fixed point theory, we studied the qualitative aspects of the system of FDEs under BCs with CHD given as

$$\begin{cases} {}^{C}D^{\eth}{}_{1+}\nu(t) + f(t, \nu(\lambda t), \mathscr{Y}(t)) = 0, \\ {}^{C}D^{\eth}1_{+}\mathscr{Y}(t) + g(t, \nu(t), \mathscr{Y}(\lambda t)) = 0, \\ \nu(1) = \nu'(1) = 0 = \nu'(e), \nu(e) = \varphi(\nu), \\ \mathscr{Y}(1) = \mathscr{Y}'(1) = 0 = \mathscr{Y}'(e), \mathscr{Y}(e) = \Psi(\mathscr{Y}), \end{cases} \tag{1}$$

with $t \in [1, e] = \mathscr{H}$, $\eth \in (3, 4]$, $\lambda \in (0, 1)$ also the functions $f, g : \mathscr{H} \times \mathbf{R} \times \mathbf{R} \longrightarrow \mathbf{R}$ and $\Phi, \Psi : \mathbb{Y} \longrightarrow \mathbf{R}$ are continuous functions. The complete norm space is defined by $\mathbb{Y}, \|,\|$ under the norm $\|y\| = \max_{t\in\mathscr{H}} |y|$.

Consequently, P is a Banach space such that $\mathbf{P} = \mathbb{Y} \times \mathbb{Y}$ with norms $\|(\nu, \mathscr{Y})\| = \|\nu\| + \|\mathscr{Y}\|$ or $\|(\nu, \mathscr{Y})\| = \max\{\|\nu\|, \|\mathscr{Y}\|\}$. We established sufficient conditions under which the problem under our investigation has at least one solution. Further, some adequate results are studied to check the stability of the UH type for the corresponding solution. These results are derived by using fixed point theory and nonlinear analysis. The analysis is justified by pertinent examples.

2. Preliminaries

Here, we recall some needful preliminary results.

Definition 1. For a function $\nu : (\mathscr{J}) = (1, e) \longrightarrow \mathbf{R}$, the fractional Hadamard integral is expressed as [30]:

$$I^{\eth}_{1+}\nu(t) = \frac{1}{\Gamma(\eth)} \int_1^t \left(\ln \frac{t}{\theta}\right)^{\eth-1} \nu(\theta) \frac{d\theta}{\theta}, \tag{2}$$

if the above integral exists.

Definition 2. For a function $\nu : (\mathscr{J}) \longrightarrow \boldsymbol{R}$, the fractional Hadamard derivative is denoted as [30]:

$${}^{C}D^{\eth}_{1+}\nu(t) = \sigma^k I^{k-\eth}_{1+} = \frac{1}{\Gamma(k-\eth)} \left(t\frac{d}{dt}\right)^k \int_1^t \left(\ln \frac{t}{\theta}\right)^{k-\eth-1} \nu(\theta) \frac{d\theta}{\theta}, \tag{3}$$

where $k = [\eth] + 1$ and $\sigma = t(d/dt)$.

Lemma 3 (see [30]). *Let $\nu(t) \in AC^k_\sigma[1, e]$, then for fractional differential equation (FDE)*

$${}^{C}D^{\eth}_{1+}\nu(t) = 0, \eth \in (k-1, k], \tag{4}$$

the solution is given as follows:

$$\nu(t) = \sum_{j=0}^{k-1} a_j (\ln t)^j, \quad j = 1, 2, 3, \cdots, k-1, where\, a_j \in \mathbf{R}. \tag{5}$$

Lemma 4 (see [30]). *The FDE holds the result in the following:*

$$I^{\eth}_{a+}\left[{}^{C}D_{a}{}^{\eth}{}_{+}\nu(t)\right] = \nu(t) + \sum_{j=0}^{k-1} a_j (\ln t)^j, \quad j = 1, 2, 3, \cdots, k-1, \tag{6}$$

where $k = [\eth] + 1$.

Definition 5 (see [31]). Let for operators $V_1, V_2 \ni V_1, V_2 : \mathbf{P} \longrightarrow \mathbb{Y}$, denoted by

$$\begin{cases} \nu(t) = V_1(\nu, Y)(t), \\ Y(t) = V_2(\nu, Y)(t) \end{cases} \tag{7}$$

is called UH stable if for real positive constants $ai(i = 1, 2, 3, 4)$, $\delta i(i = 1, 2)$ and for each solution $(\nu\wedge, Y\wedge) \in \mathbf{P}$, we have

$$\begin{cases} \|\nu\wedge - V_1(\nu\wedge, Y\wedge)\| \le \delta_{1,} \\ \|Y\wedge - V_2(\nu\wedge, Y\wedge)\| \le \delta_2, \end{cases} \tag{8}$$

there exist a solution $(\nu, Y) \in \mathbf{P}$ of (7),$\ni$

$$\begin{cases} \|V_1(\nu, Y) - V_1(\nu\wedge, Y\wedge)\| \le b_1\|\nu - \nu\wedge\| + b_2\|Y - Y\wedge\|, \\ \|V_2(\nu, Y) - V_2(\nu\wedge, Y\wedge)\| \le b_3\|\nu - \nu\wedge\| + b_4\|Y - Y\wedge\|. \end{cases} \tag{9}$$

Furthermore, if the matrix

$$M = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} \tag{10}$$

converges to zero, then the solution of (7) is UH stable.

Theorem 6 (see [32–34]). *Let $E \ne \emptyset$ be closed convex subset of the Banach space P, and there exist two operators $F, \aleph$ such that (a) $Fx + \aleph y \in E$ whenever $x, y \in E$, (b) F is continuous and compact, and (c) $\aleph$ is contraction. So one has $z = (\nu, Y) \in E$ such that $Fz + \aleph z = z$.*

(M_1) *For all $\nu, Y \in C(H, \boldsymbol{R})$,$\exists \mathfrak{R}^*_\Phi, \mathfrak{R}^*_\psi > 0 \ni$*

$$|\Phi(\nu) - \Phi(\nu)| \le \mathfrak{R}^*_\Phi |\nu - \nu|, |\psi(Y) - \psi(Y)| \le \mathfrak{R}^*_\psi |Y - Y|. \tag{11}$$

(M_2) *For all* $v, v, Y, Y \in C(H, \mathbf{R}) \forall t \in H \exists L_f^* > 0 \ni$

$$|f(t, v(\imath t), Y(t)) - f(t, v(\imath t), Y(t))| \le L_f^*[|v - v| + |Y - Y|]. \tag{12}$$

(M_3) *For all* $v, v, Y, Y \in C(H, \mathbf{R}) \forall t \in H \exists L^*{}_g > 0 \ni$

$$|\aleph(t, v(t), Y(\imath t)) - \aleph(t, v(t), Y(\imath t))| \le L_g^*[|v - v| + |Y - Y|]. \tag{13}$$

(M_4) *There exist positive real numbers* C_f^*, D_f^*, *and* M_f^*, $M_f^* \ni$

$$\begin{aligned} |f(t, v(\imath t), Y(t))| &\le C_f^*|v| + D_f^*|Y| + M_f^*, \\ |\aleph(t, v(t), Y(\imath t))| &\le C_g^*|v| + D_g^*|Y| + M_g^*. \end{aligned} \tag{14}$$

(M_5) *There exist positive real numbers* $\kappa_i (i = 1, 2), \beta_\Phi$, $\beta_\Psi \ni$

$$|\Phi(v)| \le \kappa_1 |v| + \beta_\Phi, |\Psi(Y)| \le \kappa_2 |Y| + \beta_\Psi. \tag{15}$$

(M_6) *For simplicity, we introduce the notation as follows:*

$$\hbar(t) = 3(\ln t)^2 - 2(\ln t)^3. \tag{16}$$

3. Main Results

Theorem 7. *Let* $v \in C[1, e]$ *and* $x \in AC_\sigma^k[1, e]$, *the solution for linear problem*

$$^C D_{1+}^{\eth} v(t) = x(t), t \in H, \eth \in (3, 4], \tag{17}$$

$$v(1) = v'(1) = v'(e) = 0, v(e) = \Phi(v) \tag{18}$$

converts to the following form:

$$v(t) = \Phi(v)\hbar(t) + \frac{l}{\Gamma(\eth)} \int_1^t \left(\ln \frac{t}{\theta}\right)^{\eth - 1} x(\theta) \frac{d\theta}{\theta}. \tag{19}$$

Proof. Thanks to Lemma (4), Equation (18) obtained the form

$$v(t) = I_{1+}^{\eth} x(t) + a_0 + a_1(\ln t) + a_2(\ln t)^2 + a_3(\ln t)^3, \tag{20}$$

by making use of the considered boundary conditions $v(1) = v'(1) = 0$, we get $a_0 = a_1 = 0$ also by

$$\begin{aligned} v(e) &= \Phi(v), v'(e) = 0 \Rightarrow, \\ \Phi(v) &= a_2 + a_3, 0 = 2a_2 + 3a_3, \end{aligned} \tag{21}$$

from this, we can say that $a_2 = 3\Phi(v)$ and $a_3 = -2\Phi(v)$. By making use of a_0, a_1, a_2, and a_3 in (20), we obtain the solution as follows:

$$v(t) = \Phi(v)\hbar(t) + \frac{l}{\Gamma(\eth)} \int_1^t \left(\ln \frac{t}{\theta}\right)^{\eth - 1} x(\theta) \frac{d\theta}{\theta}. \tag{22}$$

Also, for $Y \in C[1, e]$, and $z \in AC_\sigma^k[1, e]$, the solution of

$$\begin{aligned} &^C D_{1+}^{\eth} Y(t) = z(t), t \in \mathrm{H}, \eth \in (3, 4], \\ &Y(1) = Y'(1) = Y'(e) = 0, \mathrm{Y}(e) = \Psi(Y) \end{aligned} \tag{23}$$

may be expressed as

$$y(t) = \Psi(y)\hbar(t) + \frac{l}{\Gamma(\eth)} \int_1^t \left(\ln \frac{t}{\theta}\right)^{\eth - 1} z(\theta) \frac{d\theta}{\theta}. \tag{24}$$

□

Corollary 8. *The solution of the concerned problem (1) is expressed as follows:*

$$\begin{cases} v(t) = \Phi(v)\hbar(t) + \dfrac{l}{\Gamma(\eth)} \displaystyle\int_1^t \left(\ln \frac{t}{\theta}\right)^{\eth - 1} f(\theta, v(\imath\theta), Y(\theta)), \dfrac{d\theta}{\theta}, \\ y(t) = \Psi(y)\hbar(t) + \dfrac{l}{\Gamma(\eth)} \displaystyle\int_1^t \left(\ln \frac{t}{\theta}\right)^{\eth - 1} g(\theta, v(\theta), Y(\imath\theta)), \dfrac{d\theta}{\theta}. \end{cases} \tag{25}$$

Theorem 9. *Consider two functions* f, g *possesses continuity and then the solution of (25) is* $(v, Y) \in \mathbf{P}$, *if* $f(v, Y)$ *is the solution of (1).*

Proof. If (v, Y) is the solution of (25), then by the differentiation of both sides of (25), we have (1). However, if (v, Y) is a solution of (1), then (v, Y) is the solution of (25). □

Let $\mathbf{F}_1, \mathbf{F}_2 : \mathbf{P} \longrightarrow \mathbf{P} \ni$

$$\begin{aligned} \mathbf{F}_1(v, Y) &= \Phi(v)\hbar(t) + \frac{l}{\Gamma(\eth)} \int_1^t \left(\ln \frac{t}{\theta}\right)^{\eth - 1} f(\theta, v(\imath\theta), Y(\theta)) \frac{d\theta}{\theta}, \\ \mathbf{F}_2(v, Y) &= \Psi(y)\hbar(t) + \frac{l}{\Gamma(\eth)} \int_1^t \left(\ln \frac{t}{\theta}\right)^{\eth - 1} g(\theta, v(\theta), Y(\imath\theta)) \frac{d\theta}{\theta}, \end{aligned} \tag{26}$$

and $\mathbf{F}(v, Y) = \begin{pmatrix} \mathbf{F}_1(v, Y) \\ \mathbf{F}_2(v, Y) \end{pmatrix}$. Hence, solution of (25) is a fixed point of $\mathbf{F}$.

Theorem 10. *If* $\Delta < 1$, *with the help of assumptions* $(M_1) - (M_3)$, *system (1) has at most one solution, where*

$$\Delta = \max \left\{ \mathfrak{R}_\Phi^* + \frac{L_f^*}{\Gamma(\eth + 1)}, \mathfrak{R}_\Psi^* + \frac{L_g^*}{\Gamma(\eth + 1)} \right\}. \tag{27}$$

Proof. Let $(\nu, Y), (\nu, Y) \in \mathbf{P}$ and for all $t \in H$, we have

$$\begin{aligned}
&\|\mathbf{F}_1(\nu, Y) - \mathbf{F}_1(\nu, Y)\| \\
&\quad \le \max_{t\in H}|\Phi(\nu) - \Phi(\nu)|\hbar(t) + \max_{t\in H}\frac{l}{\Gamma(\eth)} \\
&\qquad \cdot \int_1^t \left(\ln \frac{t}{\theta}\right)^{\eth-1} |(f(\theta, \nu(\jmath\theta), Y(t) - f(\theta, \nu(\jmath\theta), Y(t)))| \frac{d\theta}{\theta} \\
&\quad \le \max_{t\in H}\mathfrak{R}_\Phi^*|\nu - \nu|\hbar(t) + \max_{t\in H}(\ln t)^{\eth}\frac{L_f^*}{\Gamma(\eth+1)}(|\nu - \nu| + |Y - Y|) \\
&\quad \le \left(\mathfrak{R}_\Phi^* + \frac{L_f^*}{\Gamma(\eth+1)}\right)\|\nu - \nu\| + \frac{L_f^*}{\Gamma(\eth+1)}\|Y - Y\| \\
&\quad \le \Delta_1(\|Y - Y\| + \|Y - Y\|),
\end{aligned} \tag{28}$$

where

$$\Delta_1 = \mathfrak{R}_\Phi^* + \frac{L_f^*}{\Gamma(\eth+1)}. \tag{29}$$

In a similar way, we obtain

$$\|\mathbf{F}_2(\nu, Y) - \mathbf{F}_2(\nu, Y)\| \le \Delta_2(\|\nu - \nu\| + \|Y - Y\|), \tag{30}$$

where

$$\Delta_2 = \mathfrak{R}_\Psi^* + \frac{L_g^*}{\Gamma(\eth+1)}. \tag{31}$$

Hence, from (28) and (30), one has

$$\begin{aligned}
\|F(\nu, Y) - F(\nu, Y)| &\le \max\,(\Delta_1, \Delta_2)(\|\nu - \nu\| + \|Y - Y\|) \\
&= \Delta(\|\nu - \nu\| + \|Y - Y\|),
\end{aligned} \tag{32}$$

where $\Delta = \max_{t\in H}\{\Delta_1, \Delta_2\}$. Hence, it is obvious that $\mathbf{F}$ is contraction; therefore, (1) has a unique result. □

Theorem 11. *In the light of hypotheses* (M_1), (M_4), *and* (M_5) *together with condition* $\max\{Y_1, Y_2\} < 1$, *system (1) has a minimum of one solution.*

Proof. Let

$$\begin{aligned}
J_1 &= \kappa_1 + \kappa_2 + \frac{\left(C_f^* + D_f^* + C_g^* + D_g^*\right)}{\Gamma(\eth+1)}, \\
J_2 &= \beta_\Phi + \beta_\Psi + \frac{\left(M_f^* + M_g^*\right)}{\Gamma(\eth+1)}.
\end{aligned} \tag{33}$$

We define a subset $\mathbf{B}$ of $\mathbf{P}$ which is closed. That is,

$$\mathbf{B} = \{(\nu, Y) \in \mathbf{P} : \|(\nu, Y)\| \le \rho\}, \text{for} \rho \ge \max\left\{\frac{J_2}{1 - J_1}\right\}. \tag{34}$$

Let us define the following operators as

$$\begin{aligned}
\aleph_1(\nu, Y) &= \frac{1}{\Gamma(\eth)}\int_1^t \left(\ln \frac{t}{\theta}\right)^{\eth-1} f(\theta, \nu(\jmath\theta), Y(\theta))\frac{d\theta}{\theta}, \\
\aleph_2(\nu, Y) &= \frac{1}{\Gamma(\eth)}\int_1^t \left(\ln \frac{t}{\theta}\right)^{\eth-1} g(\theta, \nu(\theta), Y(\jmath\theta))\frac{d\theta}{\theta}, \\
\mathbf{S}_1\nu(t) &= \Phi(\nu)\hbar(t), \\
\mathbf{S}_2 Y(t) &= \Psi(Y)\hbar(t).
\end{aligned} \tag{35}$$

It is obvious that $\tilde{T}_1 = \aleph_1 + \mathbf{S}_1$, $\tilde{T}_2 = \aleph_2 + \mathbf{S}_2$. Further, we prove that

$$\tilde{T}(\nu, Y) = \aleph(\nu, Y) + \mathbf{S}(\nu, Y) \in \mathbf{B}, \text{for all } (\nu, Y) \in \mathbf{B}. \tag{36}$$

For any $(\nu, Y) \in \mathbf{B}$, we have

$$\begin{aligned}
|T_1(\nu, Y)| &= \left|\Phi(\nu)\hbar(t) + \frac{l}{\Gamma(\eth)}\int_1^t \left(\ln \frac{t}{\theta}\right)^{\eth-1} f(\theta, \nu(\jmath\theta), Y(\theta))\frac{d\theta}{\theta}\right| \\
&\le \max_{t\in H} + |\Phi(\nu)\hbar(t)| + \max_{t\in H}\frac{1}{\Gamma(\eth)} \\
&\quad \cdot \int_1^t \left(\ln \frac{t}{\theta}\right)^{\eth-1} |f(\theta, \nu(\jmath\theta), Y(\theta))|\frac{d\theta}{\theta} \\
&\le \max_{t\in H}\hbar(t)(\kappa_1|\nu| + \beta_\Phi) \\
&\quad + \max_{t\in H}\left(C_f^*|\nu| + D_f^*|Y| + M_f^*\right)\frac{1}{\Gamma(\eth+1)}\left(\ln \frac{t}{a}\right)^{\eth} \\
&\le \kappa_1\rho + \beta_\Phi + \frac{\left(C_f^*\rho + D_f^*\rho + M_f^*\right)}{\Gamma(\eth+1)} \le \frac{\rho}{2}.
\end{aligned} \tag{37}$$

In a similar way, we obtain

$$|\tilde{T}_2(\nu, \mathscr{Y})| \le \kappa_2\rho + \beta_\Psi + \frac{\left(C_g^*\rho + D_g^*\rho + M_g^*\right)}{\Gamma(\eth+1)} \le \frac{\rho}{2}. \tag{38}$$

The preceding calculations imply that $\|\tilde{T}(\nu, \mathscr{Y})\| \le \rho$, which clarify that $\tilde{T}(\mathbf{B}) \subseteq \mathbf{B}$. For $(\nu, Y), (\nu, Y) \in \mathbf{B}$, we can write it as

$$\begin{aligned}
\|\mathbf{S}_1(\nu) - \mathbf{S}_1(\nu)\| &\le \max_{t\in H}[|\Phi(\nu) - \Phi(\nu)|] \\
&\le \max_{t\in H}\mathfrak{R}_\Phi^*\hbar(t)\|\nu - \nu\| \\
&\le Y_1\|\nu - \nu\|.
\end{aligned} \tag{39}$$

We can also prove that

$$\|\mathbf{S}_2(\mathscr{Y}) - \mathbf{S}_2(\mathscr{Y})\| \le \mathscr{Y}_2\|\mathscr{Y} - \mathscr{Y}\|, \tag{40}$$

where

$$Y_1 = \mathfrak{R}^*_{\Phi},$$
$$Y_2 = \mathfrak{R}^*_{\Psi}. \quad (41)$$

Clearly, (39) and (40) assure the contraction of **S**. Now, we need to show the relative compactness of $\aleph$. Now, as f and g are continuous, hence $\aleph$ is continuous too. For $(\nu, Y) \in \mathbf{B}$, we have

$$\begin{aligned} |\aleph_1(\nu, \mathcal{Y})| &\le \max_{t\in H} \frac{l}{\Gamma(\eth)} \int_1^t \left(\ln \frac{t}{\theta}\right)^{\eth-1} |f(\theta, \nu(\imath\theta), \mathcal{Y}(\theta))| \frac{d\theta}{\theta} \\ &\le \frac{\left(C^*_f \rho + D^*_f \rho + M^*_f\right)}{\Gamma(\eth+1)}. \end{aligned} \quad (42)$$

In the same way, one can get

$$|\aleph_2(\nu, \mathcal{Y})| \le \frac{\left(C^*_g \rho + D^*_g \rho + M^*_g\right)}{\Gamma(\eth+1)}. \quad (43)$$

Therefore, from (42) and (43), it implies

$$\rho \ge \|\aleph(\nu, \mathcal{Y})\|. \quad (44)$$

Hence, from (44), the boundedness of $\aleph$ can also be deduced on **B**. Take any $(\nu, \mathcal{Y}) \in \mathbf{B}$. Subsequently, for t_1, $t_2 \in \mathcal{H}$ with $t_1 \le t_2 \in [1, e]$, one has

$$\begin{aligned} &|\aleph_1(\nu(t_1), \mathcal{Y}(t_1)) - \aleph_1(\nu(t_2), \mathcal{Y}(t_2)| \\ &\quad\le \frac{1}{\Gamma(\eth)} \int_1^{t_1} \left(\left(\ln \frac{t_1}{\theta}\right)^{\eth-1} - \left(\ln \frac{t_2}{\theta}\right)^{\eth-1}\right) \\ &\qquad \cdot |f(\theta, \nu(\imath\theta), \mathcal{Y}(\theta))| \frac{d\theta}{\theta} + \frac{1}{\Gamma(\eth)} \int_{t_1}^{t_2} \left(\ln \frac{t_2}{\theta}\right)^{\eth-1} \\ &\qquad \cdot |f(\theta, \nu(\imath\theta), \mathcal{Y}(\theta))| \frac{d\theta}{\theta} \\ &\quad\le \frac{1}{\Gamma(\eth+1)} \left(C^*_f |\nu| + D^*_f |\mathcal{Y}| + M^*_f\right) \\ &\qquad \times \left((\ln t_2)^{\eth} + 2\left(\ln \frac{t_2}{t_1}\right)^{\eth} - (\ln t_1)^{\eth}\right). \end{aligned} \quad (45)$$

From the previous inequality, we can claim that (45) approaches to zero on $t_1 \longrightarrow t_2$. As $\aleph_1$ possesses the properties of continuity and boundedness, it clearly means that $\aleph_1$ possesses uniform boundedness. Therefore, $\|\aleph_1(\nu(t_2), \mathcal{Y}(t_2)) - \aleph_1(\nu(t_1), \mathcal{Y}(t_1))\| \longrightarrow 0$ as t_1 tends to t_2. Similarly, $\|\aleph_2(\nu(t_2), \mathcal{Y}(t_2)) - \aleph_2(\nu(t_1), \mathcal{Y}(t_1))\| \longrightarrow 0$ as t_1 tends to t_2. Hence, all the assumptions of at least one solution for system (1) are achieved. □

4. Stability Results

Theorem 12. *Under the hypothesis $(M_1) - (M_3)$ together with condition $\Delta < 1$, the considered system has UH stable solution.*

Proof. Let for arbitrary solutions $(\nu, \mathcal{Y}), (\nu, \mathcal{Y}) \in \mathbf{P}$, and for all $t \in \mathcal{H}$, we have

$$\begin{aligned} &\|\mathbf{F}_1(\nu, \mathcal{Y}) - \mathbf{F}_1(\nu, \mathcal{Y})\| \\ &\quad\le \max_{t\in H} |\Phi(\nu) - \Phi(\nu)| \hbar(t) + \max_{t\in H} \frac{1}{\Gamma(\eth)} \int_1^t \left(\ln \frac{t}{\theta}\right)^{\eth-1} \\ &\qquad \cdot |(f(\theta, \nu(\imath\theta), \mathcal{Y}(t) - f(\theta, \nu(\imath\theta), \mathcal{Y}(t)))| \frac{d\theta}{\theta} \\ &\quad\le \max_{t\in H} \hbar(t) \mathfrak{R}^*_{\Phi} |\nu - \nu| + \max_{t\in H} (\ln t)^{\eth} \frac{L^*_f}{\Gamma(\eth+1)} (|\nu - \nu| + |\mathcal{Y} - \mathcal{Y}|) \\ &\quad\le \left(\mathfrak{R}^*_{\Phi} + \frac{L^*_f}{\Gamma(\eth+1)}\right) \|\nu - \nu\| + \frac{L^*_f}{\Gamma(\eth+1)} \|\mathcal{Y} - \mathcal{Y}\| \\ &\quad\le b_1 \|\mathcal{Y} - \mathcal{Y}\| + b_2 \|\mathcal{Y} - \mathcal{Y}\|, \end{aligned} \quad (46)$$

where

$$b_1 = \mathfrak{R}^*_{\Phi} + \frac{L^*_f}{\Gamma(\eth+1)},$$
$$b_2 = \frac{L^*_f}{\Gamma(\eth+1)}. \quad (47)$$

Similarly, one has

$$\|\mathbf{F}_2(\nu, \mathcal{Y}) - \mathbf{F}_2(\nu, \mathcal{Y})\| \le b_3 \|\mathcal{Y} - \mathcal{Y}\| + b_4 \|\mathcal{Y} - \mathcal{Y}\|, \quad (48)$$

where

$$b_3 = \mathfrak{R}^*_{\Psi} + \frac{L^*_g}{\Gamma(\eth+1)},$$
$$b_4 = \frac{L^*_g}{\Gamma(\eth+1)}. \quad (49)$$

So, from (46) and (48), we get

$$\begin{aligned} &\|\mathbf{F}_1(\nu, \mathcal{Y}) - \mathbf{F}_1(\nu, \mathcal{Y})\| \\ &\quad\le b_1 \|\mathcal{Y} - \mathcal{Y}\| + b_2 \|\mathcal{Y} - \mathcal{Y}\|, \|\mathbf{F}_2(\nu, \mathcal{Y}) - \mathbf{F}_2(\nu, \mathcal{Y})\| \\ &\quad\le b_3 \|\mathcal{Y} - \mathcal{Y}\| + b_4 \|\mathcal{Y} - \mathcal{Y}\|. \end{aligned} \quad (50)$$

Using (50), we have

$$\mathcal{M} = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}. \quad (51)$$

Since $\mathcal{M}$ converges to zero, hence the result of (1) is UH stable. □

5. Applications

Example 13. Taking a coupled system as.

$$\begin{cases} {}^{C}D_{1+}^{3.8}\nu(t) + \dfrac{\sin|\nu(0.3t)| + \cos|(\mathscr{Y}t)| + e^{t} + 4}{(t^{2}+10)^{3}} = 0, & t \in \mathscr{H}, \\ {}^{C}D_{1+}^{3.8}\mathscr{Y}(t) + \dfrac{t^{3} + |\nu(t)| - |\mathscr{Y}(0.3t)|}{(e^{t}+50)} = 0, & t \in \mathscr{H}, \\ \nu(1) = \nu'(1) = 0 = \nu'(e), \nu(e) = \dfrac{\sin|\nu|}{30}, \\ \mathscr{Y}(1) = \mathscr{Y}'(1) = 0 = \mathscr{Y}'(e), \mathscr{Y}(e) = \dfrac{\cos(\mathscr{Y})}{29}. \end{cases} \tag{52}$$

From above, $\eth = 3.8$, $\jmath = 0.3$ after calculation, we have $L_f^* = 0.0001$, $L_g^* = 0.02$, $\mathfrak{R}_\Phi^* = 0.033$, $\mathfrak{R}_\Psi^* = 0.034$,

$$\begin{aligned} \Delta_1 &= 0.0331, \\ \Delta_2 &= 0.0351. \end{aligned} \tag{53}$$

It is obvious that max $\{\Delta_1, \Delta_2\} = 0.0351 < 1$. So (52) has a unique solution by Theorem 10. Moreover, from the values of b_i, $(i = 1, 2, 3, 4)$, we have

$$\mathscr{M} = \begin{bmatrix} 0.0331 & 0.0001 \\ 0.0351 & 0.0011 \end{bmatrix}, \tag{54}$$

after calculation, the eigenvalues are $\eth_1 = 0.0332$, $\eth_2 = 0.0010$. Therefore $\Lambda(\mathscr{M}) = 0.0332 < 1$. Thus, the given system is HU stable by using Theorem 12.

Example 14. Consider the following problem:

$$\begin{cases} {}^{C}D_{1+}^{3.7}\,\nu(t) + \dfrac{\arctan(t)}{10 + |\nu(0.4t)|} = 0, & t \in \mathscr{H}, \\ {}^{C}D_{1+}^{3.7}\mathscr{Y}(t) + \dfrac{\ln t}{8 + |\mathscr{Y}(0.4t)|} = 0, & t \in \mathscr{H}, \\ \nu(1) = \nu'(1) = 0 = \nu'(e), \nu(e) = \dfrac{|\nu| + t^{2}}{60}, \\ \mathscr{Y}(1) = \mathscr{Y}'(1) = 0 = \mathscr{Y}'(e), \mathscr{Y}(e) = \dfrac{\sin|\mathscr{Y}|}{25}. \end{cases} \tag{55}$$

From above, $\eth = 3.7$, $\jmath = 0.4$ after calculation, we have $L_f^* = 0.1218$, $L_g^* = 0.125$, $\mathfrak{R}_\Phi^* = 0.016$, $\mathfrak{R}_\Psi^* = 0.04$,

$$\begin{aligned} \Delta_1 &= 0.0239, \\ \Delta_2 &= 0.0481. \end{aligned} \tag{56}$$

It is obvious that max $\{\Delta_1, \Delta_2\} = 0.0481 < 1$. So (55) has a unique solution by Theorem 10. Moreover, from the values of b_i $(i = 1, 2, 3, 4)$, we have

$$\mathscr{M} = \begin{bmatrix} 0.0239 & 0.0079 \\ 0.0481 & 0.0081 \end{bmatrix}, \tag{57}$$

after calculation, the eigenvalues are $\eth_1 = 0.0351$, $\eth_2 = -0.0102$. Therefore, $\Lambda(\mathscr{M}) = 0.0351 < 1$. Thus, the given system is HU stable by using Theorem 12.

6. Conclusion

In this research work, nonlinear BVPs of FDEs containing proportional delay with CHD operator have been successfully investigated. We have utilized the techniques of fixed point theory and nonlinear analysis, to develop the existence and stability results for the proposed system. Through some examples, the main results have been justified. In the future, one can investigate the aforementioned system of FDEs for more complicated boundary conditions.

Data Availability

The data used in this research work is contained in paper.

Conflicts of Interest

There are no conflict of interest that exist.

Authors' Contributions

An equal contribution has been done by all the authors.

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