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*Research Article*

# The Sub-Riemannian Limit of Curvatures for Curves and Surfaces and a Gauss-Bonnet Theorem in the Group of Rigid Motions of Minkowski Plane with General Left-Invariant Metric

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The group of rigid motions of the Minkowski plane with a general left-invariant metric is denoted by $(E(1,1), g(\lambda_1, \lambda_2))$, where $\lambda_1 \geq \lambda_2 > 0$. It provides a natural 2-parametric deformation family of the Riemannian homogeneous manifold $\mathrm{Sol}_3 = (E(1,1), g(1,1))$ which is the model space to solve geometry in the eight model geometries of Thurston. In this paper, we compute the sub-Riemannian limits of the Gaussian curvature for a Euclidean $C^2$-smooth surface in $(E(1,1), g_L(\lambda_1, \lambda_2))$ away from characteristic points and signed geodesic curvature for the Euclidean $C^2$-smooth curves on surfaces. Based on these results, we get a Gauss-Bonnet theorem in the group of rigid motions of the Minkowski plane with a general left-invariant metric.

## 1. Introduction

In [1], Proposition 2.6 stated that any left-invariant metric on the group of rigid motions of the Minkowski plane $E(1,1)$ is isometric to one of the metric $g(\lambda_1, \lambda_2, \lambda_3)$ with $\lambda_1 \geq \lambda_2 > 0$ and $\lambda_3 = 1/\lambda_1\lambda_2$. In [2], the metric $g(\lambda_1, \lambda_2, \lambda_3)$ was denoted by $g(\lambda_1, \lambda_2) = g(\lambda_1, \lambda_2, 1/\lambda_1\lambda_2)$ as we take in this paper, and the authors classified parallel surfaces in the groups of rigid motions of the Euclidean plane and the Minkowski plane. In [3], they completed the classification of parallel and totally geodesic surfaces in all three-dimensional homogeneous spaces by solving the problem in three-dimensional Lie groups with a left-invariant metric yielding a three-dimensional isometry group. In this paper, we consider $(E(1,1), g(\lambda_1, \lambda_2))$ which is the group of rigid motions of the Minkowski plane with the general left-invariant metric $g(\lambda_1, \lambda_2)$. This group is very interesting and important for the reason that it provides a natural 2-parametric deformation family of one of the Riemannian homogeneous manifold $\mathrm{Sol}_3 = (E(1,1), g(1,1))$ which is the model space to solve-geometry in the eight model geometries of Thurston.

In [4, 5], Balogh et al. used a Riemannian approximation scheme to define a notion of intrinsic Gaussian curvature for a Euclidean $C^2$-smooth surface in the Heisenberg group $H^1$ away from characteristic points, and a notion of intrinsic signed geodesic curvature for the Euclidean $C^2$-smooth curves on surfaces. These results were then used to prove a Heisenberg version of the Gauss-Bonnet theorem. They proposed an interesting question to understand to what extent similar phenomena hold in other sub-Riemannian geometric structures. In [6, 7], Wang and Wei solved this problem for the affine group, the group of rigid motions of the Minkowski plane $(E(1,1), g(1,1))$, the BCV spaces, and the twisted Heisenberg group. Recently, we got the Gauss-Bonnet theorems in the rototranslation group and the Lorentzian Sasakian space forms [8, 9]. In this paper, we try to solve this problem for the group of rigid motions of the Minkowski plane with the general left-invariant metric $g(\lambda_1, \lambda_2)$. We compute the sub-Riemannian limits of the Gaussian curvature for a Euclidean $C^2$-smooth surface in $(E(1,1), g_L(\lambda_1, \lambda_2))$ away from characteristic points and signed geodesic curvature for the Euclidean $C^2$-smooth curves on surfaces. We get a generalized Gauss-Bonnet theorem in $(E(1,1), g_L(\lambda_1, \lambda_2))$.

In Section 2, we provide a short introduction to $(E(1,1), g_L(\lambda_1,\lambda_2))$ and the notions which we will use throughout the paper, such as the Levi-Civita connection in the Riemannian approximants of $(E(1,1), g(\lambda_1,\lambda_2))$. Furthermore, we compute the sub-Riemannian limit of the curvature of curves in $(E(1,1), g_L(\lambda_1,\lambda_2))$. In Sections 3 and 4, we compute the sub-Riemannian limits of the geodesic curvature of curves on surfaces and the Riemannian Gaussian curvature of surfaces in $(E(1,1), g_L(\lambda_1,\lambda_2))$. In Section 5, we get the Gauss-Bonnet theorem in the group of rigid motions of the Minkowski plane with the general left-invariant metric.

## 2. The Sub-Riemannian Limit of Curvature of Curves in $(E(1,1), g_L(\lambda_1,\lambda_2))$

In this section, some basic notions in the motion group of the Minkowski plane will be introduced. Let $E(1,1)$ be the motion group of the Minkowski plane:

$$E(1,1) = \left\{ \begin{pmatrix} e^z & 0 & x \\ 0 & e^{-z} & y \\ 0 & 0 & 1 \end{pmatrix} \middle| x, y, z \in \mathbb{R} \right\}. \tag{1}$$

The Lie algebra $\ell(1,1)$ is given explicitly by

$$\ell(1,1) = \left\{ \begin{pmatrix} \omega & 0 & u \\ 0 & -\omega & v \\ 0 & 0 & 0 \end{pmatrix} \middle| u, v, \omega \in \mathbb{R} \right\}. \tag{2}$$

We consider the group of rigid motions of the Minkowski plane with a general left-invariant metric, $(E(1,1), g(\lambda_1,\lambda_2))$. As a model of $(E(1,1), g(\lambda_1,\lambda_2))$, we choose the underlying manifold $\mathbb{R}^3$. On $\mathbb{R}^3$, we let

$$\begin{aligned} X_1 &= \lambda_1\lambda_2 \frac{\partial}{\partial z}, \\ X_2 &= \frac{1}{\lambda_1\sqrt{2}}\left(-e^z\frac{\partial}{\partial x} + e^{-z}\frac{\partial}{\partial y}\right), \\ X_3 &= -\frac{1}{\lambda_2\sqrt{2}}\left(e^z\frac{\partial}{\partial x} + e^{-z}\frac{\partial}{\partial y}\right). \end{aligned} \tag{3}$$

Then, we have

$$\begin{aligned} \frac{\partial}{\partial x} &= -\frac{\sqrt{2}}{2}e^{-z}(\lambda_1 X_2 + \lambda_2 X_3), \\ \frac{\partial}{\partial y} &= \frac{\sqrt{2}}{2}e^{z}(\lambda_1 X_2 - \lambda_2 X_3), \\ \frac{\partial}{\partial z} &= \frac{1}{\lambda_1\lambda_2}X_1, \end{aligned} \tag{4}$$

$$\operatorname{span}\{X_1, X_2, X_3\} = T((E(1,1), g(\lambda_1,\lambda_2))). \tag{5}$$

Let $H = \operatorname{span}\{X_1, X_2\}$ be the horizontal distribution on $(E(1,1), g(\lambda_1,\lambda_2))$. Let

$$\begin{aligned} \omega_1 &= \frac{1}{\lambda_1\lambda_2}dz, \\ \omega_2 &= \frac{\lambda_1}{\sqrt{2}}(-e^{-z}dx + e^z dy), \\ \omega &= -\frac{\lambda_2}{\sqrt{2}}(e^{-z}dx + e^z dy), \end{aligned} \tag{6}$$

be the dual coframe field. Then, $H = \ker \omega$. The Riemannian approximation scheme used in [4] can in general depend on the choice of the complement to the horizontal distribution. In the context of $(E(1,1), g(\lambda_1,\lambda_2))$, the choice is similar to $(E(1,1), g(1,1))$ in [6]. Let $L > 0$ and define a metric $g_L(\lambda_1,\lambda_2) = \omega_1\otimes\omega_1 + \omega_2\otimes\omega_2 + L\omega\otimes\omega$, so that $X_1, X_2, \widetilde{X_3} := L^{-1/2}X_3$ are orthonormal basis on $T(E(1,1), g_L(\lambda_1,\lambda_2))$ with respect to $g_L(\lambda_1,\lambda_2)$. Notice that

$$g_L(\lambda_1,\lambda_2) = \frac{1}{\lambda_1^2\lambda_2^2}dz^2 + \frac{\lambda_1^2}{2}(-e^{-z}dx + e^z dy)^2 + L\frac{\lambda_2^2}{2}(e^{-z}dx + e^z dy)^2, \tag{7}$$

and $g(\lambda_1,\lambda_2) = g_1(\lambda_1,\lambda_2)$ be the Riemannian metric on $(E(1,1), g(\lambda_1,\lambda_2))$. The approach in this paper is to define sub-Riemannian objects as limits of horizontal objects in $(E(1,1), g_L(\lambda_1,\lambda_2))$, where a family of metrics $g_L(\lambda_1,\lambda_2)$ is essentially obtained as an anisotropic blow-up of the Riemannian metric $g_1(\lambda_1,\lambda_2)$. At the heart of this approach is the fact that the intrinsic horizontal geometry does not change with $L$. In general, the metric $g_L(\lambda_1,\lambda_2)$ does not fit in the family $g(\lambda_1,\lambda_2)$ for the case of $L \neq 1$. We have

$$\begin{aligned} [X_1, X_2] &= {\lambda_2}^2 X_3, \\ [X_2, X_3] &= 0, \\ [X_1, X_3] &= {\lambda_1}^2 X_2. \end{aligned} \tag{8}$$

To compute the curvatures of curves and surfaces in the motion group of the Minkowski plane with respect to $g_L(\lambda_1,\lambda_2)$, we use the Levi-Civita connection $\nabla^L$ on $(E(1,1), g_L(\lambda_1,\lambda_2))$. A straightforward calculation shows the following proposition.

**Proposition 1.** *Let $(E(1,1), g_L(\lambda_1,\lambda_2))$ be the motion group of the Minkowski plane with a general left-invariant metric, relative to the coordinate frame $X_1$, $X_2$, $\widetilde{X_3}$; then, the Levi-Civita connection on $(E(1,1), g_L(\lambda_1,\lambda_2))$ is given by*

$$\begin{aligned}
\nabla^L_{X_j} X_j &= 0, \quad 1 \le j \le 3,\\
\nabla^L_{X_1} X_2 &= \frac{\lambda_2^2 L - \lambda_1^2}{2L} X_3,\\
\nabla^L_{X_2} X_1 &= \frac{-\lambda_2^2 L - \lambda_1^2}{2L} X_3,\\
\nabla^L_{X_1} X_3 &= \frac{\lambda_1^2 - \lambda_2^2 L}{2} X_2,\\
\nabla^L_{X_3} X_1 &= \frac{-\lambda_1^2 - \lambda_2^2 L}{2} X_2,\\
\nabla^L_{X_2} X_3 = \nabla^L_{X_3} X_2 &= \frac{\lambda_1^2 + \lambda_2^2 L}{2} X_1.
\end{aligned} \tag{9}$$

*Proof.* It follows from a direct application of the Koszul identity, which here simplifies

$$2\left\langle \nabla^L_{X_i} X_j, X_k \right\rangle_L = \left\langle [X_i, X_j], X_k \right\rangle_L - \left\langle [X_j, X_k], X_i \right\rangle_L + \left\langle [X_k X_i], X_j \right\rangle_L, \tag{10}$$

where $i, j, k = 1, 2, 3$. By (8) and (9), we have

$$\begin{aligned}
2\left\langle \nabla^L_{X_j} X_j, X_k \right\rangle_L &= \left\langle [X_j, X_j], X_k \right\rangle_L - \left\langle [X_j, X_k], X_j \right\rangle_L + \left\langle [X_k, X_j], X_j \right\rangle_L\\
&= -\left\langle [X_j, X_k], X_j \right\rangle_L + \left\langle [X_k, X_j], X_j \right\rangle_L = 2\left\langle [X_k, X_j], X_j \right\rangle_L.
\end{aligned} \tag{11}$$

When $j = 1$, we compute $\langle \nabla^L_{X_1} X_1, X_k \rangle_L = \langle [X_k, X_1], X_1 \rangle_L$. It follows that $\langle \nabla^L_{X_1} X_1, X_1 \rangle_L = 0$, $\langle \nabla^L_{X_1} X_1, X_2 \rangle_L = \langle [X_2, X_1], X_1 \rangle_L = \langle -\lambda_2{}^2 X_3, X_1 \rangle_L = 0$, and $\langle \nabla^L_{X_1} X_1, X_3 \rangle_L = \langle [X_3, X_1], X_1 \rangle_L = 0$. Hence, $\nabla^L_{X_1} X_1 = 0$. Similarly, $\nabla^L_{X_2} X_2 = 0$ and $\nabla^L_{X_3} X_3 = 0$. By the following equation,

$$\begin{aligned}
2\left\langle \nabla^L_{X_1} X_2, X_k \right\rangle_L &= \langle [X_1, X_2], X_k \rangle_L - \langle [X_2, X_k], X_1 \rangle_L + \langle [X_k, X_1], X_2 \rangle_L\\
&= \left\langle \lambda_2{}^2 X_3, X_k \right\rangle_L - \left\langle [X_2, X_k], X_1 \right\rangle_L + \langle [X_k, X_1], X_2 \rangle_L,
\end{aligned} \tag{12}$$

we get $\nabla^L_{X_1} X_2 = (\lambda_2^2 L - \lambda_1^2/2L) X_3$. Other cases follow by similar computations. □ □

*Definition 2.* Let $\gamma : [a, b] \longrightarrow (E(1,1), g_L(\lambda_1, \lambda_2))$ be a Euclidean $C^1$-smooth curve. We say that $\gamma$ is regular, if $\dot\gamma \neq 0$ for every $t \in [a, b]$. Moreover, we say that $\gamma(t)$ is a horizontal point of $\gamma$ if

$$\begin{aligned}
\omega(\dot\gamma(t)) &= -\frac{\lambda_2}{\sqrt{2}} (e^{-\gamma_3} dx + e^{\gamma_3} dy) \left( \dot\gamma_1(t) \frac{\partial}{\partial x} + \dot\gamma_2(t) \frac{\partial}{\partial y} + \dot\gamma_3(t) \frac{\partial}{\partial z} \right)\\
&= -\frac{\lambda_2 \sqrt{2}}{2} (e^{-\gamma_3} \dot\gamma_1(t) + e^{\gamma_3} \dot\gamma_2(t)) = 0,
\end{aligned} \tag{13}$$

where $\gamma(t) = (\gamma_1(t), \gamma_2(t), \gamma_3(t))$.

*Definition 3.* Let $\gamma : [a, b] \longrightarrow (E(1,1), g_L(\lambda_1, \lambda_2))$ be a Euclidean $C^2$-smooth regular curve in the Riemannian manifold $(E(1,1), g_L(\lambda_1, \lambda_2))$. The curvature $\kappa^L_\gamma$ of $\gamma$ at $\gamma(t)$ is defined as follows:

$$\kappa^L_\gamma := \sqrt{\frac{\left\| \nabla^L_{\dot\gamma} \dot\gamma \right\|_L^2}{\|\dot\gamma\|_L^4} - \frac{\left\langle \nabla^L_{\dot\gamma} \dot\gamma, \dot\gamma \right\rangle_L^2}{\|\dot\gamma\|_L{}^6}}. \tag{14}$$

**Proposition 4.** *Let $\gamma : [a, b] \longrightarrow (E(1,1), g_L(\lambda_1, \lambda_2))$ be a Euclidean $C^2$-smooth regular curve in the Riemannian manifold $(E(1,1), g_L(\lambda_1, \lambda_2))$. Then, we have*

$$\begin{aligned}
\kappa^L_\gamma = \Bigg\{ \Bigg\{ & \left[ \frac{1}{\lambda_1 \lambda_2} \ddot\gamma_3(t) + \frac{\lambda_1{}^3 \sqrt{2} + \lambda_1 \lambda_2{}^2 \sqrt{2} L}{2} (-e^{-\gamma_3} \dot\gamma_1(t) + e^{\gamma_3} \dot\gamma_2(t)) \omega(\dot\gamma(t)) \right]^2\\
&+ \left[ \frac{\lambda_1 \sqrt{2}}{2} (\ddot\gamma_2 e^{\gamma_3} + \dot\gamma_2 \dot\gamma_3 e^{\gamma_3} - \ddot\gamma_1 e^{-\gamma_3} + \dot\gamma_1 \dot\gamma_3 e^{-\gamma_3}) - L\omega(\dot\gamma(t)) \dot\gamma_3(t) \right]^2\\
&+ L \left[ \frac{d}{dt} (\omega(\dot\gamma(t))) - \frac{\lambda_1{}^2 \sqrt{2}}{2\lambda_2 L} (-e^{-\gamma_3} \dot\gamma_1(t) + e^{\gamma_3} \dot\gamma_2(t)) \dot\gamma_3(t) \right]^2 \Bigg\}\\
&\cdot \left\{ \left[ \frac{1}{\lambda_1 \lambda_2} \dot\gamma_3(t) \right]^2 + \left[ \frac{\lambda_1 \sqrt{2}}{2} (-e^{-\gamma_3} \dot\gamma_1(t) + e^{\gamma_3} \dot\gamma_2(t)) \right]^2 + L(\omega(\dot\gamma(t)))^2 \right\}^{-2}\\
&- \Bigg\{ \frac{1}{\lambda_1 \lambda_2} \dot\gamma_3(t) \left[ \frac{1}{\lambda_1 \lambda_2} \ddot\gamma_3(t) + \frac{\lambda_1{}^3 \sqrt{2} + \lambda_1 \lambda_2{}^2 \sqrt{2} L}{2} (-e^{-\gamma_3} \dot\gamma_1(t) + e^{\gamma_3} \dot\gamma_2(t)) \omega(\dot\gamma(t)) \right]\\
&+ \frac{\lambda_1 \sqrt{2}}{2} (-e^{-\gamma_3} \dot\gamma_1(t) + e^{\gamma_3} \dot\gamma_2(t))\\
&\cdot \left[ \frac{\lambda_1 \sqrt{2}}{2} (\ddot\gamma_2 e^{\gamma_3} + \dot\gamma_2 \dot\gamma_3 e^{\gamma_3} - \ddot\gamma_1 e^{-\gamma_3} + \dot\gamma_1 \dot\gamma_3 e^{-\gamma_3}) - L\omega(\dot\gamma(t)) \dot\gamma_3(t) \right]\\
&+ L\omega(\dot\gamma(t)) \left[ \frac{d}{dt} (\omega(\dot\gamma(t))) - \frac{\lambda_1{}^2 \sqrt{2}}{2\lambda_2 L} (-e^{-\gamma_3} \dot\gamma_1(t) + e^{\gamma_3} \dot\gamma_2(t) \dot\gamma_3(t)) \right] \Bigg\}^2\\
&\cdot \left\{ \left[ \frac{1}{\lambda_1 \lambda_2} \dot\gamma_3(t) \right]^2 + \left[ \frac{\lambda_1 \sqrt{2}}{2} (-e^{-\gamma_3} \dot\gamma_1(t) + e^{\gamma_3} \dot\gamma_2(t)) \right]^2 + L(\omega(\dot\gamma(t)))^2 \right\}^{-3} \Bigg\}^{1/2}.
\end{aligned} \tag{15}$$

*In particular, if $\gamma(t)$ is a horizontal point of $\gamma$,*

$$\begin{aligned}
\kappa^L_\gamma = \Bigg\{ \Bigg\{ & \left[ \frac{1}{\lambda_1 \lambda_2} \ddot\gamma_3(t) \right]^2 + \left[ \frac{\lambda_1 \sqrt{2}}{2} (\ddot\gamma_2 e^{\gamma_3} + \dot\gamma_2 \dot\gamma_3 e^{\gamma_3} - \ddot\gamma_1 e^{-\gamma_3} + \dot\gamma_1 \dot\gamma_3 e^{-\gamma_3}) \right]^2\\
&+ L \left[ \frac{d}{dt} (\omega(\dot\gamma(t))) - \frac{\lambda_1{}^2 \sqrt{2}}{2\lambda_2 L} (-e^{-\gamma_3} \dot\gamma_1(t) + e^{\gamma_3} \dot\gamma_2(t)) \dot\gamma_3(t) \right]^2 \Bigg\}\\
&\cdot \left\{ \left[ \frac{1}{\lambda_1 \lambda_2} \dot\gamma_3(t) \right]^2 + \left[ \frac{\lambda_1 \sqrt{2}}{2} (-e^{-\gamma_3} \dot\gamma_1(t) + e^{\gamma_3} \dot\gamma_2(t)) \right]^2 \right\}^{-2}\\
&- \left[ \frac{1}{\lambda_1 \lambda_2} \dot\gamma_3(t) \left( \frac{1}{\lambda_1 \lambda_2} \ddot\gamma_3(t) \right) + \frac{\lambda_1{}^2}{2} (-e^{-\gamma_3} \dot\gamma_1(t) + e^{\gamma_3} \dot\gamma_2(t)) \right.\\
&\cdot (\ddot\gamma_2 e^{\gamma_3} + \dot\gamma_2 \dot\gamma_3 e^{\gamma_3} - \ddot\gamma_1 e^{-\gamma_3} + \dot\gamma_1 \dot\gamma_3 e^{-\gamma_3}) \Big]^2 \cdot \Bigg\{ \left[ \frac{1}{\lambda_1 \lambda_2} \dot\gamma_3(t) \right]^2\\
&+ \left[ \frac{\lambda_1 \sqrt{2}}{2} (-e^{-\gamma_3} \dot\gamma_1(t) + e^{\gamma_3} \dot\gamma_2(t)) \right]^2 \Bigg\}^{-3} \Bigg\}^{1/2}.
\end{aligned} \tag{16}$$

*Proof.* By (4), we have

$$\begin{aligned}\dot{\gamma}(t) &= \dot{\gamma}_1(t)\frac{\partial}{\partial x} + \dot{\gamma}_2(t)\frac{\partial}{\partial y} + \dot{\gamma}_3(t)\frac{\partial}{\partial z}\\ &= -\frac{\sqrt{2}}{2}e^{-\gamma_3}(\lambda_1 X_2 + \lambda_2 X_3)\dot{\gamma}_1(t) + \frac{\sqrt{2}}{2}e^{\gamma_3}(\lambda_1 X_2 - \lambda_2 X_3)\dot{\gamma}_2(t)\\ &\quad + \frac{1}{\lambda_1\lambda_2}X_1\dot{\gamma}_3(t) = \frac{1}{\lambda_1\lambda_2}\dot{\gamma}_3(t)X_1\\ &\quad + \left(-\frac{\sqrt{2}}{2}e^{-\gamma_3}\lambda_1\dot{\gamma}_1(t) + \frac{\sqrt{2}}{2}e^{\gamma_3}\lambda_1\dot{\gamma}_2(t)\right)X_2\\ &\quad + \left(-\frac{\sqrt{2}}{2}e^{-\gamma_3}\lambda_2\dot{\gamma}_1(t) + \frac{\sqrt{2}}{2}e^{\gamma_3}\lambda_2\dot{\gamma}_2(t)\right)X_3\\ &= \frac{1}{\lambda_1\lambda_2}\dot{\gamma}_3(t)X_1 + \frac{\lambda_1\sqrt{2}}{2}(-e^{-\gamma_3}\dot{\gamma}_1(t) + e^{\gamma_3}\dot{\gamma}_2(t))X_2 + \omega(\dot{\gamma}(t))X_3.\end{aligned} \tag{17}$$

By Proposition 1 and (17), we have

$$\begin{aligned}\nabla^L_{\dot{\gamma}}X_1 &= \frac{1}{\lambda_1\lambda_2}\dot{\gamma}_3(t)\nabla^L_{X_1}X_1 + \frac{\lambda_1\sqrt{2}}{2}(-e^{-\gamma_3}\dot{\gamma}_1(t) + e^{\gamma_3}\dot{\gamma}_2(t))\nabla^L_{X_2}X_1\\ &\quad + \omega(\dot{\gamma}(t))\nabla^L_{X_3}X_1\\ &= \frac{-\lambda_1^2 - \lambda_2^2 L}{2}\omega(\dot{\gamma}(t))X_2 + \frac{-\lambda_1\lambda_2^2\sqrt{2}L - \lambda_1^3\sqrt{2}}{4L}\\ &\quad \cdot(-e^{-\gamma_3}\dot{\gamma}_1(t) + e^{\gamma_3}\dot{\gamma}_2(t))X_3,\end{aligned} \tag{18}$$

$$\begin{aligned}\nabla^L_{\dot{\gamma}}X_2 &= \frac{1}{\lambda_1\lambda_2}\dot{\gamma}_3(t)\nabla^L_{X_1}X_2 + \frac{\lambda_1\sqrt{2}}{2}(-e^{-\gamma_3}\dot{\gamma}_1(t) + e^{\gamma_3}\dot{\gamma}_2(t))\nabla^L_{X_2}X_2 + \omega(\dot{\gamma}(t))\nabla^L_{X_3}X_2\\ &= \frac{\lambda_1^2 + \lambda_2^2 L}{2}\omega(\dot{\gamma}(t))X_1 + \frac{\lambda_2^2 L - \lambda_1^2}{2\lambda_1\lambda_2 L}\dot{\gamma}_3(t)X_3,\end{aligned} \tag{19}$$

$$\begin{aligned}\nabla^L_{\dot{\gamma}}X_3 &= \frac{1}{\lambda_1\lambda_2}\dot{\gamma}_3(t)\nabla^L_{X_1}X_3 + \frac{\lambda_1\sqrt{2}}{2}(-e^{-\gamma_3}\dot{\gamma}_1(t) + e^{\gamma_3}\dot{\gamma}_2(t))\nabla^L_{X_2}X_3 + \omega(\dot{\gamma}(t))\nabla^L_{X_3}X_3\\ &= \frac{\lambda_1^3\sqrt{2} + \lambda_1\lambda_2^2\sqrt{2}L}{4}(-e^{-\gamma_3}\dot{\gamma}_1(t) + e^{\gamma_3}\dot{\gamma}_2(t))X_1 + \frac{\lambda_1^2 - \lambda_2^2 L}{2\lambda_1\lambda_2}\dot{\gamma}_3(t)X_2.\end{aligned} \tag{20}$$

By (17) and (18), we have

$$\begin{aligned}\nabla^L_{\dot{\gamma}}\dot{\gamma} &= \nabla^L_{\dot{\gamma}}\left[\frac{1}{\lambda_1\lambda_2}\dot{\gamma}_3(t)X_1 + \frac{\lambda_1\sqrt{2}}{2}(-e^{-\gamma_3}\dot{\gamma}_1(t) + e^{\gamma_3}\dot{\gamma}_2(t))X_2 + \omega(\dot{\gamma}(t))X_3\right]\\ &= \dot{\gamma}\left[\frac{1}{\lambda_1\lambda_2}\dot{\gamma}_3(t)\right]X_1 + \frac{1}{\lambda_1\lambda_2}\dot{\gamma}_3(t)\nabla^L_{\dot{\gamma}}X_1 + \dot{\gamma}\left[\frac{\lambda_1\sqrt{2}}{2}(-e^{-\gamma_3}\dot{\gamma}_1(t) + e^{\gamma_3}\dot{\gamma}_2(t))\right]X_2\\ &\quad + \frac{\lambda_1\sqrt{2}}{2}(-e^{-\gamma_3}\dot{\gamma}_1(t) + e^{\gamma_3}\dot{\gamma}_2(t))\nabla^L_{\dot{\gamma}}X_2 + \dot{\gamma}[\omega(\dot{\gamma}(t))]X_3 + \omega(\dot{\gamma}(t))\nabla^L_{\dot{\gamma}}X_3\\ &= \left[\frac{1}{\lambda_1\lambda_2}\ddot{\gamma}_3(t) + \frac{\lambda_1^3\sqrt{2} + \lambda_1\lambda_2^2\sqrt{2}L}{2}(-e^{-\gamma_3}\dot{\gamma}_1(t) + e^{\gamma_3}\dot{\gamma}_2(t))\omega(\dot{\gamma}(t))\right]X_1\\ &\quad + \left[\frac{\lambda_1\sqrt{2}}{2}(\ddot{\gamma}_2 e^{\gamma_3} + \dot{\gamma}_2\dot{\gamma}_3 e^{\gamma_3} - \ddot{\gamma}_1 e^{-\gamma_3} + \dot{\gamma}_1\dot{\gamma}_3 e^{-\gamma_3}) - L\omega(\dot{\gamma}(t))\dot{\gamma}_3(t)\right]X_2\\ &\quad + \left[\frac{d}{dt}(\omega(\dot{\gamma}(t))) - \frac{\lambda_1^2\sqrt{2}}{2\lambda_2 L}(-e^{-\gamma_3}\dot{\gamma}_1(t) + e^{\gamma_3}\dot{\gamma}_2(t))\dot{\gamma}_3(t)\right]X_3.\end{aligned} \tag{21}$$

By (14), (17), and (21), we get

$$\begin{aligned}\left\|\nabla^L_{\dot{\gamma}}\dot{\gamma}\right\|^2_L &= \left[\frac{1}{\lambda_1\lambda_2}\ddot{\gamma}_3(t) + \frac{\lambda_1^3\sqrt{2} + \lambda_1\lambda_2^2\sqrt{2}L}{2}(-e^{-\gamma_3}\dot{\gamma}_1(t) + e^{\gamma_3}\dot{\gamma}_2(t))\omega(\dot{\gamma}(t))\right]^2\\ &\quad + \left[\frac{\lambda_1\sqrt{2}}{2}(\ddot{\gamma}_2 e^{\gamma_3} + \dot{\gamma}_2\dot{\gamma}_3 e^{\gamma_3} - \ddot{\gamma}_1 e^{-\gamma_3} + \dot{\gamma}_1\dot{\gamma}_3 e^{-\gamma_3}) - L\omega(\dot{\gamma}(t))\dot{\gamma}_3(t)\right]^2\\ &\quad + L\left[\frac{d}{dt}(\omega(\dot{\gamma}(t))) - \frac{\lambda_1^2\sqrt{2}}{2\lambda_2 L}(-e^{-\gamma_3}\dot{\gamma}_1(t) + e^{\gamma_3}\dot{\gamma}_2(t))\dot{\gamma}_3(t)\right]^2,\end{aligned}$$

$$\|\dot{\gamma}\|^4_L = \left\{\left[\frac{1}{\lambda_1\lambda_2}\dot{\gamma}_3(t)\right]^2 + \left[\frac{\lambda_1\sqrt{2}}{2}(-e^{-\gamma_3}\dot{\gamma}_1(t) + e^{\gamma_3}\dot{\gamma}_2(t))\right]^2 + L(\omega(\dot{\gamma}(t)))^2\right\}^2,$$

$$\begin{aligned}\left\langle\nabla^L_{\dot{\gamma}}\dot{\gamma}, \dot{\gamma}\right\rangle^2_L &= \left\{\frac{1}{\lambda_1\lambda_2}\dot{\gamma}_3(t)\left[\frac{1}{\lambda_1\lambda_2}\ddot{\gamma}_3(t) + \frac{\lambda_1^3\sqrt{2} + \lambda_1\lambda_2^2\sqrt{2}L}{2}\right.\right.\\ &\quad \left.\cdot(-e^{-\gamma_3}\dot{\gamma}_1(t) + e^{\gamma_3}\dot{\gamma}_2(t))\omega(\dot{\gamma}(t))\right]\\ &\quad + \frac{\lambda_1\sqrt{2}}{2}(-e^{-\gamma_3}\dot{\gamma}_1(t) + e^{\gamma_3}\dot{\gamma}_2(t))\\ &\quad \cdot\left[\frac{\lambda_1\sqrt{2}}{2}(\ddot{\gamma}_2 e^{\gamma_3} + \dot{\gamma}_2\dot{\gamma}_3 e^{\gamma_3} - \ddot{\gamma}_1 e^{-\gamma_3} + \dot{\gamma}_1\dot{\gamma}_3 e^{-\gamma_3})\right.\\ &\quad \left. - L\omega(\dot{\gamma}(t))\dot{\gamma}_3(t)\right] + L\omega(\dot{\gamma}(t))\\ &\quad \left.\cdot\left[\frac{d}{dt}(\omega(\dot{\gamma}(t))) - \frac{\lambda_1^2\sqrt{2}}{2\lambda_2 L}(-e^{-\gamma_3}\dot{\gamma}_1(t) + e^{\gamma_3}\dot{\gamma}_2(t))\dot{\gamma}_3(t)\right]\right\}^2,\end{aligned}$$

$$\|\dot{\gamma}\|^6_L = \left\{\left[\frac{1}{\lambda_1\lambda_2}\dot{\gamma}_3(t)\right]^2 + \left[\frac{\lambda_1\sqrt{2}}{2}(-e^{-\gamma_3}\dot{\gamma}_1(t) + e^{\gamma_3}\dot{\gamma}_2(t))\right]^2 + L(\omega(\dot{\gamma}(t)))^2\right\}^3. \tag{22}$$

By the definition of $\kappa^L_\gamma$, we get Proposition 4. □ □

*Definition 5.* Let $\gamma : [a, b] \longrightarrow (E(1,1), g_L(\lambda_1, \lambda_2))$ be a Euclidean $C^2$-smooth regular curve in the Riemannian manifold $(E(1,1), g_L(\lambda_1, \lambda_2))$, we define the intrinsic curvature $\kappa^\infty_\gamma$ of $\gamma$ at $\gamma(t)$ to be

$$\kappa^\infty_\gamma := \lim_{L\longrightarrow\infty}\kappa^L_\gamma, \tag{23}$$

if the limit exists.

We introduce the following notation: for continuous functions $f_1, f_2 : (0,+\infty) \longrightarrow \mathbb{R}$,

$$f_1(L) \sim f_2(L), \quad \text{as } L \longrightarrow +\infty \Leftrightarrow \lim_{L\longrightarrow\infty}\frac{f_1(L)}{f_2(L)} = 1. \tag{24}$$

**Proposition 6.** *Let $\gamma : [a, b] \longrightarrow (E(1,1), g_L(\lambda_1, \lambda_2))$ be a Euclidean $C^2$-smooth regular curve in the Riemannian manifold $(E(1,1), g_L(\lambda_1, \lambda_2))$. Then, we have*

$$\kappa^\infty_\gamma = \frac{\sqrt{(\lambda_1^2\lambda_2^4/2)(-e^{-\gamma_3}\dot{\gamma}_1(t) + e^{\gamma_3}\dot{\gamma}_2(t))^2 + (\dot{\gamma}_3(t))^2}}{|\omega(\dot{\gamma}(t))|}, \quad \text{if } \omega(\dot{\gamma}(t)) \neq 0.$$

$$\begin{aligned}\kappa_\gamma^\infty = &\left\{\left\{\left[\frac{1}{\lambda_1\lambda_2}\ddot{\gamma}_3(t)\right]^2 + \left[\frac{\lambda_1\sqrt{2}}{2}(\ddot{\gamma}_2 e^{\gamma_3} + \dot{\gamma}_2\dot{\gamma}_3 e^{\gamma_3} - \ddot{\gamma}_1 e^{-\gamma_3} + \dot{\gamma}_1\dot{\gamma}_3 e^{-\gamma_3})\right]^2\right\}\right.\\ &\cdot\left\{\left[\frac{1}{\lambda_1\lambda_2}\dot{\gamma}_3(t)\right]^2 + \left[\frac{\lambda_1\sqrt{2}}{2}(-e^{-\gamma_3}\dot{\gamma}_1(t) + e^{\gamma_3}\dot{\gamma}_2(t))\right]^2\right\}^{-2}\\ &-\left[\frac{1}{\lambda_1\lambda_2}\dot{\gamma}_3(t)\left(\frac{1}{\lambda_1\lambda_2}\ddot{\gamma}_3(t)\right) + \frac{\lambda_1^2}{2}(-e^{-\gamma_3}\dot{\gamma}_1(t) + e^{\gamma_3}\dot{\gamma}_2(t))\right.\\ &\left.\cdot(\ddot{\gamma}_2 e^{\gamma_3} + \dot{\gamma}_2\dot{\gamma}_3 e^{\gamma_3} - \ddot{\gamma}_1 e^{-\gamma_3} + \dot{\gamma}_1\dot{\gamma}_3 e^{-\gamma_3})\right]^2\\ &\left.\cdot\left\{\left[\frac{1}{\lambda_1\lambda_2}\dot{\gamma}_3(t)\right]^2 + \left[\frac{\lambda_1\sqrt{2}}{2}(-e^{-\gamma_3}\dot{\gamma}_1(t) + e^{\gamma_3}\dot{\gamma}_2(t))\right]^2\right\}^{-3}\right\}^{1/2},\\ &\text{if } \omega(\dot{\gamma}(t)) = 0,\ \frac{d}{dt}(\omega(\dot{\gamma}(t))) = 0,\end{aligned}$$

$$\begin{aligned}\lim_{L\longrightarrow\infty}\frac{\kappa_\gamma^L}{\sqrt{L}} = &\frac{|(d/dt)(\omega(\dot{\gamma}(t)))|}{[(1/\lambda_1\lambda_2)\dot{\gamma}_3(t)]^2 + \left[\left(\lambda_1\sqrt{2}/2\right)(-e^{-\gamma_3}\dot{\gamma}_1(t) + e^{\gamma_3}\dot{\gamma}_2(t))\right]^2},\\ &\text{if } \omega(\dot{\gamma}(t)) = 0,\ \frac{d}{dt}(\omega(\dot{\gamma}(t))) \neq 0.\end{aligned} \tag{25}$$

*Proof.* When $\omega(\dot{\gamma}(t)) \neq 0$, we have

$$\begin{aligned}\left\langle\nabla_{\dot{\gamma}}^L\dot{\gamma}, \nabla_{\dot{\gamma}}^L\dot{\gamma}\right\rangle \sim &\left[\frac{\lambda_1^2\lambda_2^4}{2}(-e^{-\gamma_3}\dot{\gamma}_1(t) + e^{\gamma_3}\dot{\gamma}_2(t))^2 + (\dot{\gamma}_3(t))^2\right]\\ &\cdot(\omega(\dot{\gamma}(t)))^2 L^2 \quad \text{as } L\longrightarrow+\infty,\end{aligned}$$

$$\|\dot{\gamma}\|_L^2 \sim L(\omega(\dot{\gamma}(t)))^2,$$

$$\left\langle\nabla_{\dot{\gamma}}^L\dot{\gamma}, \dot{\gamma}\right\rangle_L^2 \sim O(L^2) \quad \text{as } L\longrightarrow+\infty. \tag{26}$$

Therefore, we have

$$\begin{aligned}\frac{\left\|\nabla_{\dot{\gamma}}^L\dot{\gamma}\right\|_L^2}{\|\dot{\gamma}\|_L^4} &\longrightarrow \frac{(\lambda_1^2\lambda_2^4/2)(-e^{-\gamma_3}\dot{\gamma}_1(t) + e^{\gamma_3}\dot{\gamma}_2(t))^2 + (\dot{\gamma}_3(t))^2}{(\omega(\dot{\gamma}(t)))^2}, \quad \text{as } L\longrightarrow+\infty,\\ \frac{\left\langle\nabla_{\dot{\gamma}}^L\dot{\gamma}, \dot{\gamma}\right\rangle_L^2}{\|\dot{\gamma}\|_L^6} &\longrightarrow 0, \quad \text{as } L\longrightarrow+\infty.\end{aligned} \tag{27}$$

If $\omega(\dot{\gamma}(t)) \neq 0$, by (14), we have

$$\kappa_\gamma^\infty = \frac{\sqrt{(\lambda_1^2\lambda_2^4/2)(-e^{-\gamma_3}\dot{\gamma}_1(t) + e^{\gamma_3}\dot{\gamma}_2(t))^2 + (\dot{\gamma}_3(t))^2}}{|\omega(\dot{\gamma}(t))|}. \tag{28}$$

By (16) and $(d/dt)(\omega(\dot{\gamma}(t))) = 0$, we have

$$\begin{aligned}\kappa_\gamma^\infty = &\left\{\left\{\left[\frac{1}{\lambda_1\lambda_2}\ddot{\gamma}_3(t)\right]^2 + \left[\frac{\lambda_1\sqrt{2}}{2}(\ddot{\gamma}_2 e^{\gamma_3} + \dot{\gamma}_2\dot{\gamma}_3 e^{\gamma_3} - \ddot{\gamma}_1 e^{-\gamma_3} + \dot{\gamma}_1\dot{\gamma}_3 e^{-\gamma_3})\right]^2\right\}\right.\\ &\cdot\left\{\left[\frac{1}{\lambda_1\lambda_2}\dot{\gamma}_3(t)\right]^2 + \left[\frac{\lambda_1\sqrt{2}}{2}(-e^{-\gamma_3}\dot{\gamma}_1(t) + e^{\gamma_3}\dot{\gamma}_2(t))\right]^2\right\}^{-2}\\ &-\left[\frac{1}{\lambda_1\lambda_2}\dot{\gamma}_3(t)\left(\frac{1}{\lambda_1\lambda_2}\ddot{\gamma}_3(t)\right) + \frac{\lambda_1^2}{2}(-e^{-\gamma_3}\dot{\gamma}_1(t) + e^{\gamma_3}\dot{\gamma}_2(t))\right.\\ &\left.\cdot(\ddot{\gamma}_2 e^{\gamma_3} + \dot{\gamma}_2\dot{\gamma}_3 e^{\gamma_3} - \ddot{\gamma}_1 e^{-\gamma_3} + \dot{\gamma}_1\dot{\gamma}_3 e^{-\gamma_3})\right]^2\\ &\left.\cdot\left\{\left[\frac{1}{\lambda_1\lambda_2}\dot{\gamma}_3(t)\right]^2 + \left[\frac{\lambda_1\sqrt{2}}{2}(-e^{-\gamma_3}\dot{\gamma}_1(t) + e^{\gamma_3}\dot{\gamma}_2(t))\right]^2\right\}^{-3}\right\}^{1/2}.\end{aligned} \tag{29}$$

When $\omega(\dot{\gamma}(t)) = 0$ and $(d/dt)(\omega(\dot{\gamma}(t))) \neq 0$, we have

$$\left\|\nabla_{\dot{\gamma}}^L\gamma\right\|_L^2 \sim L\left[\frac{d}{dt}(\omega(\dot{\gamma}(t)))\right]^2, \quad \text{as } L\longrightarrow+\infty,$$

$$\|\dot{\gamma}\|_L^2 = \left[\frac{1}{\lambda_1\lambda_2}\dot{\gamma}_3(t)\right]^2 + \left[\frac{\lambda_1\sqrt{2}}{2}(-e^{-\gamma_3}\dot{\gamma}_1(t) + e^{\gamma_3}\dot{\gamma}_2(t))\right]^2,$$

$$\left\langle\nabla_{\dot{\gamma}}^L\dot{\gamma}, \dot{\gamma}\right\rangle_L^2 = O(1) \quad \text{as } L\longrightarrow+\infty. \tag{30}$$

If $\omega(\dot{\gamma}(t)) = 0$ and $(d/dt)(\omega(\dot{\gamma}(t))) \neq 0$, by (14), we get

$$\lim_{L\longrightarrow\infty}\frac{\kappa_\gamma^L}{\sqrt{L}} = \frac{|(d/dt)(\omega(\dot{\gamma}(t)))|}{[(1/\lambda_1\lambda_2)\dot{\gamma}_3(t)]^2 + \left[\left(\lambda_1\sqrt{2}/2\right)(-e^{-\gamma_3}\dot{\gamma}_1(t) + e^{\gamma_3}\dot{\gamma}_2(t))\right]^2}. \tag{31}$$

□ □

## 3. The Sub-Riemannian Limit of Geodesic Curvature of Curves on Surfaces in $(E(1,1), g_L(\lambda_1,\lambda_2))$

In this section, we will compute the sub-Riemannian limit of the geodesic curvature of curves on surfaces in $(E(1,1), g_L(\lambda_1,\lambda_2))$. We will say that a surface $\Sigma \subset (E(1,1), g_L(\lambda_1,\lambda_2))$ is regular if $\Sigma$ is a Euclidean $C^2$-smooth compact and oriented surface. In particular, we will assume that there exists a Euclidean $C^2$-smooth function $u : E(1,1) \longrightarrow \mathbb{R}$ such that

$$\Sigma = \{(x,y,z) \in E(1,1) \colon u(x,y,z) = 0\}, \tag{32}$$

and $u_x\partial_x + u_y\partial_y + u_z\partial_z \neq 0$. Let $\nabla_H u = X_1(u)X_1 + X_2(u)X_2$. A point $(x,y,z) \in \Sigma$ is called characteristic if $\nabla_H u(x,y,z) = 0$. We define the characteristic set as follows:

$$C(\Sigma) := \{(x,y,z) \in \Sigma \mid \nabla_H u(x,y,z) = 0\}. \tag{33}$$

Note that the computations in the present paper will be local and away from characteristic points of $\Sigma$. We define $p$

$:= X_1 u$, $q := X_2 u$, $r := \widetilde{X_3} u$, and

$$\begin{aligned} l &:= \sqrt{p^2+q^2}, \\ l_L &:= \sqrt{p^2+q^2+r^2}, \\ \bar{p} &:= \frac{p}{l}, \\ \bar{q} &:= \frac{q}{l}, \\ \bar{p_L} &:= \frac{p}{l_L}, \\ \bar{q_L} &:= \frac{q}{l_L}, \\ \bar{r_L} &:= \frac{r}{l_L}. \end{aligned} \tag{34}$$

In particular, $\bar{p}^2 + \bar{q}^2 = 1$. We remark that these functions are well defined at every noncharacteristic point. Let

$$\begin{aligned} \nu_L &= \bar{p_L} X_1 + \bar{q_L} X_2 + \bar{r_L} \widetilde{X_3}, \\ e_1 &= \bar{q} X_1 - \bar{p} X_2, \\ e_2 &= \bar{r_L} \bar{p} X_1 + \bar{r_L} \bar{q} X_2 - \frac{l}{l_L} \widetilde{X_3}, \end{aligned} \tag{35}$$

then $\nu_L$ is the Riemannian unit normal vector to $\Sigma$, and $e_1$, $e_2$ are the orthonormal basis of $\Sigma$. On $T\Sigma$, we define a linear transformation $J_L : T\Sigma \longrightarrow T\Sigma$ such that

$$\begin{aligned} J_L(e_1) &:= e_2, \\ J_L(e_2) &:= -e_1. \end{aligned} \tag{36}$$

For every $U, V \in T\Sigma$, we define $\nabla_U^{\Sigma,L} V = \pi \nabla_U^L V$, where $\pi : TE(1,1) \longrightarrow T\Sigma$ is the projection. Then, $\nabla^{\Sigma,L}$ is the Levi-Civita connection on $\Sigma$ with respect to the metric $g_L(\lambda_1, \lambda_2)$. By (21), (34), and

$$\nabla_{\dot{\gamma}}^{\Sigma,L} \dot{\gamma} = \left\langle \nabla_{\dot{\gamma}}^L \dot{\gamma}, e_1 \right\rangle_L e_1 + \left\langle \nabla_{\dot{\gamma}}^L \dot{\gamma}, e_2 \right\rangle_L e_2, \tag{37}$$

we have

$$\begin{aligned} \nabla_{\dot{\gamma}}^{\Sigma,L} \dot{\gamma} = {} & \left\{ \bar{q} \left[ \frac{1}{\lambda_1 \lambda_2} \ddot{\gamma}_3(t) + \frac{\lambda_1^3 \sqrt{2} + \lambda_1 \lambda_2^2 \sqrt{2} L}{2} (-e^{-\gamma_3} \dot{\gamma}_1(t) + e^{\gamma_3} \dot{\gamma}_2(t)) \omega(\dot{\gamma}(t)) \right] \right. \\ & \left. - \bar{p} \left[ \frac{\lambda_1 \sqrt{2}}{2} (\ddot{\gamma}_2 e^{\gamma_3} + \dot{\gamma}_2 \dot{\gamma}_3 e^{\gamma_3} - \ddot{\gamma}_1 e^{-\gamma_3} + \dot{\gamma}_1 \dot{\gamma}_3 e^{-\gamma_3}) - L \dot{\gamma}_3(t) \omega(\dot{\gamma}(t)) \right] \right\} e_1 \\ & + \left\{ \bar{r}_L \bar{p} \left[ \frac{1}{\lambda_1 \lambda_2} \ddot{\gamma}_3(t) + \frac{\lambda_1^3 \sqrt{2} + \lambda_1 \lambda_2^2 \sqrt{2} L}{2} (-e^{-\gamma_3} \dot{\gamma}_1(t) + e^{\gamma_3} \dot{\gamma}_2(t)) \omega(\dot{\gamma}(t)) \right] \right. \\ & + \bar{r}_L \bar{q} \left[ \frac{\lambda_1 \sqrt{2}}{2} (\ddot{\gamma}_2 e^{\gamma_3} + \dot{\gamma}_2 \dot{\gamma}_3 e^{\gamma_3} - \ddot{\gamma}_1 e^{-\gamma_3} + \dot{\gamma}_1 \dot{\gamma}_3 e^{-\gamma_3}) - L \dot{\gamma}_3(t) \omega(\dot{\gamma}(t)) \right] \\ & \left. - \frac{l}{l_L} L^{1/2} \left[ \frac{d}{dt} (\omega(\dot{\gamma}(t))) - \frac{\lambda_1^2 \sqrt{2}}{2 \lambda_2 L} (-e^{-\gamma_3} \dot{\gamma}_1(t) + e^{\gamma_3} \dot{\gamma}_2(t)) \dot{\gamma}_3(t) \right] \right\} e_2. \end{aligned} \tag{38}$$

Moreover, if $\omega(\dot{\gamma}(t)) = 0$, then

$$\begin{aligned} \nabla_{\dot{\gamma}}^{\Sigma,L} \dot{\gamma} = {} & \left\{ \bar{q} \left[ \frac{1}{\lambda_1 \lambda_2} \ddot{\gamma}_3(t) \right] - \bar{p} \left[ \frac{\lambda_1 \sqrt{2}}{2} (\ddot{\gamma}_2 e^{\gamma_3} + \dot{\gamma}_2 \dot{\gamma}_3 e^{\gamma_3} - \ddot{\gamma}_1 e^{-\gamma_3} + \dot{\gamma}_1 \dot{\gamma}_3 e^{-\gamma_3}) \right] \right\} e_1 \\ & + \left\{ \bar{r}_L \bar{p} \left[ \frac{1}{\lambda_1 \lambda_2} \ddot{\gamma}_3(t) \right] + \bar{r}_L \bar{q} \left[ \frac{\lambda_1 \sqrt{2}}{2} (\ddot{\gamma}_2 e^{\gamma_3} + \dot{\gamma}_2 \dot{\gamma}_3 e^{\gamma_3} - \ddot{\gamma}_1 e^{-\gamma_3} + \dot{\gamma}_1 \dot{\gamma}_3 e^{-\gamma_3}) \right] \right. \\ & \left. - \frac{l}{l_L} L^{1/2} \left[ \frac{d}{dt} (\omega(\dot{\gamma}(t))) - \frac{\lambda_1^2 \sqrt{2}}{2 \lambda_2 L} (-e^{-\gamma_3} \dot{\gamma}_1(t) + e^{\gamma_3} \dot{\gamma}_2(t)) \dot{\gamma}_3(t) \right] \right\} e_2. \end{aligned} \tag{39}$$

*Definition 7.* Let $\Sigma \subset (E(1,1), g_L(\lambda_1, \lambda_2))$ be a regular surface, $\gamma : [a,b] \longrightarrow \Sigma$ be a Euclidean $C^2$-smooth regular curve. The geodesic curvature $\kappa_{\gamma,\Sigma}^L$ of $\gamma$ at $\gamma(t)$ is defined as follows:

$$\kappa_{\gamma,\Sigma}^L := \sqrt{\frac{\left\| \nabla_{\dot{\gamma}}^{\Sigma,L} \dot{\gamma} \right\|_{\Sigma,L}^2}{\|\dot{\gamma}\|_{\Sigma,L}^4} - \frac{\left\langle \nabla_{\dot{\gamma}}^{\Sigma,L} \dot{\gamma}, \dot{\gamma} \right\rangle_{\Sigma,L}^2}{\|\dot{\gamma}\|_{\Sigma,L}^6}}. \tag{40}$$

*Definition 8.* Let $\Sigma \subset (E(1,1), g_L(\lambda_1, \lambda_2))$ be a regular surface, $\gamma : [a,b] \longrightarrow \Sigma$ be a Euclidean $C^2$-smooth regular curve. We define the intrinsic geodesic curvature $\kappa_{\gamma,\Sigma}^\infty$ of $\gamma$ at $\gamma(t)$ to be

$$\kappa_{\gamma,\Sigma}^\infty := \lim_{L \longrightarrow +\infty} \kappa_{\gamma,\Sigma}^L, \tag{41}$$

if the limit exists.

**Proposition 9.** *Let $\Sigma \subset (E(1,1), g_L(\lambda_1, \lambda_2))$ be a regular surface and $\gamma : [a,b] \longrightarrow \Sigma$ be a Euclidean $C^2$-smooth regular curve. Then, we have*

$$\kappa_{\gamma,\Sigma}^\infty = \frac{\sqrt{(\lambda_1^2 \lambda_2^4 / 2) \bar{q}^2 (-e^{-\gamma_3} \dot{\gamma}_1(t) + e^{\gamma_3} \dot{\gamma}_2(t))^2 + \bar{p}^2 (\dot{\gamma}_3(t))^2}}{|\omega(\dot{\gamma}(t))|}, \quad \text{if } \omega(\dot{\gamma}(t)) \neq 0,$$

$$\kappa_{\gamma,\Sigma}^\infty = 0, \quad \text{if } \omega(\dot{\gamma}(t)) = 0, \ \frac{d}{dt}(\omega(\dot{\gamma}(t))) = 0,$$

$$\begin{aligned} \lim_{L \longrightarrow +\infty} \frac{\kappa_{\gamma,\Sigma}^L}{\sqrt{L}} &= \frac{|(d/dt)(\omega(\dot{\gamma}(t)))|}{\left[ (1/\lambda_1 \lambda_2) \dot{\gamma}_3(t) \bar{q} - \left( \lambda_1 \sqrt{2}/2 \right) (-e^{-\gamma_3} \dot{\gamma}_1(t) + e^{\gamma_3} \dot{\gamma}_2(t)) \bar{p} \right]^2}, \quad \text{if } \omega(\dot{\gamma}(t)) \\ &= 0, \ \frac{d}{dt}(\omega(\dot{\gamma}(t))) \neq 0. \end{aligned} \tag{42}$$

*Proof.* By (17) and $\dot{\gamma} \in T\Sigma$, we have

$$\dot{\gamma}(t) = \frac{1}{\lambda_1 \lambda_2} \dot{\gamma}_3(t) X_1 + \frac{\lambda_1 \sqrt{2}}{2} (-e^{-\gamma_3} \dot{\gamma}_1(t) + e^{\gamma_3} \dot{\gamma}_2(t)) X_2 + \omega(\dot{\gamma}(t)) X_3. \tag{43}$$

On the other hand,

$$\begin{aligned}\dot{\gamma}(t) &= ae_1 + be_2 = a(\bar{q}X_1 - \bar{p}X_2) + b\left(\bar{r}_L\bar{p}X_1 + \bar{r}_L\bar{q}X_2 - \frac{l}{l_L}\widetilde{X_3}\right)\\ &= (a\bar{q} + b\bar{r}_L\bar{p})X_1 + (-a\bar{p} + b\bar{r}_L\bar{q})X_2 - \frac{bl}{l_L}L^{-1/2}X_3.\end{aligned} \tag{44}$$

Comparing the above equations, we get

$$\begin{cases} a\bar{q} + b\bar{r}_L\bar{p} = \dfrac{1}{\lambda_1\lambda_2}\dot{\gamma}_3(t),\\ -a\bar{p} + b\bar{r}_L\bar{q} = \dfrac{\lambda_1\sqrt{2}}{2}(-e^{-\gamma_3}\dot{\gamma}_1(t) + e^{\gamma_3}\dot{\gamma}_2(t)),\\ -\dfrac{bl}{l_L}L^{-1/2} = \omega(\dot{\gamma}(t)),\end{cases} \tag{45}$$

from which

$$\begin{cases} a = \dfrac{1}{\lambda_1\lambda_2}\dot{\gamma}_3(t)\bar{q} - \dfrac{\lambda_1\sqrt{2}}{2}(-e^{-\gamma_3}\dot{\gamma}_1(t) + e^{\gamma_3}\dot{\gamma}_2(t))\bar{p},\\ b = -\dfrac{l_L}{l}L^{1/2}\omega(\dot{\gamma}(t)).\end{cases} \tag{46}$$

This proves the following:

$$\dot{\gamma} = \left[\frac{1}{\lambda_1\lambda_2}\dot{\gamma}_3(t)\bar{q} - \frac{\lambda_1\sqrt{2}}{2}(-e^{-\gamma_3}\dot{\gamma}_1(t) + e^{\gamma_3}\dot{\gamma}_2(t))\bar{p}\right]e_1 - \frac{l_L}{l}L^{1/2}\omega(\dot{\gamma}(t))e_2. \tag{47}$$

By (37), we have

$$\begin{aligned}\left\|\nabla^{\Sigma,L}_{\dot{\gamma}}\dot{\gamma}\right\|^2_{\Sigma,L} &= \left\{\bar{q}\left[\frac{1}{\lambda_1\lambda_2}\ddot{\gamma}_3(t) + \frac{\lambda_1^3\sqrt{2} + \lambda_1\lambda_2^2\sqrt{2}L}{2}\right.\right.\\ &\quad\cdot(-e^{-\gamma_3}\dot{\gamma}_1(t) + e^{\gamma_3}\dot{\gamma}_2(t))\omega(\dot{\gamma}(t))] - \bar{p}\\ &\quad\cdot\left.\left[\frac{\lambda_1\sqrt{2}}{2}(\ddot{\gamma}_2e^{\gamma_3} + \dot{\gamma}_2\dot{\gamma}_3e^{\gamma_3} - \ddot{\gamma}_1e^{-\gamma_3} + \dot{\gamma}_1\dot{\gamma}_3e^{-\gamma_3}) - L\dot{\gamma}_3(t)\omega(\dot{\gamma}(t))\right]\right\}^2\\ &\quad+\left\{\bar{r}_L\bar{p}\left[\frac{1}{\lambda_1\lambda_2}\ddot{\gamma}_3(t) + \frac{\lambda_1^3\sqrt{2} + \lambda_1\lambda_2^2\sqrt{2}L}{2}(-e^{-\gamma_3}\dot{\gamma}_1(t) + e^{\gamma_3}\dot{\gamma}_2(t))\omega(\dot{\gamma}(t))\right]\right.\\ &\quad+\bar{r}_L\bar{q}\left[\frac{\lambda_1\sqrt{2}}{2}(\ddot{\gamma}_2e^{\gamma_3} + \dot{\gamma}_2\dot{\gamma}_3e^{\gamma_3} - \ddot{\gamma}_1e^{-\gamma_3} + \dot{\gamma}_1\dot{\gamma}_3e^{-\gamma_3}) - L\dot{\gamma}_3(t)\omega(\dot{\gamma}(t))\right]\\ &\quad-\left.\frac{l}{l_L}L^{1/2}\left[\frac{d}{dt}(\omega(\dot{\gamma}(t))) - \frac{\lambda_1^2\sqrt{2}}{2\lambda_2L}(-e^{-\gamma_3}\dot{\gamma}_1(t) + e^{\gamma_3}\dot{\gamma}_2(t))\dot{\gamma}_3(t)\right]\right\}^2\\ &\sim\left[\frac{\lambda_1^2\lambda_2^4}{2}\bar{q}^2(-e^{-\gamma_3}\dot{\gamma}_1(t) + e^{\gamma_3}\dot{\gamma}_2(t))^2 + \bar{p}^2(\dot{\gamma}_3(t))^2\right]L^2(\omega(\dot{\gamma}(t)))^2,\\ &\quad\text{as } L\longrightarrow+\infty.\end{aligned} \tag{48}$$

Similarly, we get that when $(\omega(\dot{\gamma}(t))) \neq 0$,

$$\begin{aligned}\|\dot{\gamma}\|_{\Sigma,L} &= \left\{\left[\frac{1}{\lambda_1\lambda_2}\dot{\gamma}_3(t)\bar{q} - \frac{\lambda_1\sqrt{2}}{2}(-e^{-\gamma_3}\dot{\gamma}_1(t) + e^{\gamma_3}\dot{\gamma}_2(t))\bar{p}\right]^2\right.\\ &\quad\left.+L\left(\frac{l_L}{l}\omega(\dot{\gamma}(t))\right)^2\right\}^{1/2} \sim L^{1/2}|\omega(\dot{\gamma}(t))|, \quad \text{as } L\longrightarrow+\infty.\end{aligned} \tag{49}$$

By (37) and (46), we have

$$\begin{aligned}\left\langle\nabla^{\Sigma,L}_{\dot{\gamma}}\dot{\gamma},\dot{\gamma}\right\rangle_{\Sigma,L} &\sim \left[\frac{1}{\lambda_1\lambda_2}\dot{\gamma}_3(t)\bar{q} - \frac{\lambda_1\sqrt{2}}{2}(-e^{-\gamma_3}\dot{\gamma}_1(t) + e^{\gamma_3}\dot{\gamma}_2(t))\bar{p}\right]\\ &\quad\cdot\left[\bar{q}\frac{\lambda_1\lambda_2^2\sqrt{2}}{2}(-e^{-\gamma_3}\dot{\gamma}_1(t) + e^{\gamma_3}\dot{\gamma}_2(t)) + \bar{p}\dot{\gamma}_3(t)\right]\omega(\dot{\gamma}(t))L\\ &\quad+\frac{d}{dt}(\omega(\dot{\gamma}(t)))\omega(\dot{\gamma}(t))L \sim N_0L, \quad \text{as } L\longrightarrow+\infty,\end{aligned} \tag{50}$$

where $N_0$ does not depend on $L$. By (39), we get

$$k^{\infty}_{\gamma,\Sigma} = \lim_{L\longrightarrow+\infty}k^L_{\gamma,\Sigma} = \frac{\sqrt{(\lambda_1^2\lambda_2^4/2)\bar{q}^2(-e^{-\gamma_3}\dot{\gamma}_1(t) + e^{\gamma_3}\dot{\gamma}_2(t))^2 + \bar{p}^2(\dot{\gamma}_3(t))^2}}{|\omega(\dot{\gamma}(t))|}, \quad \text{if } \omega(\dot{\gamma}(t)) \neq 0. \tag{51}$$

When $\omega(\dot{\gamma}(t)) = 0$ and $(d/dt)\omega(\dot{\gamma}(t)) = 0$, we have

$$\begin{aligned}\left\|\nabla^{\Sigma,L}_{\dot{\gamma}}\dot{\gamma}\right\|^2_{\Sigma,L} &= \left\{\bar{q}\left[\frac{1}{\lambda_1\lambda_2}\ddot{\gamma}_3(t)\right] - \bar{p}\left[\frac{\lambda_1\sqrt{2}}{2}(\ddot{\gamma}_2e^{\gamma_3} + \dot{\gamma}_2\dot{\gamma}_3e^{\gamma_3}\right.\right.\\ &\quad\left.\left.-\ddot{\gamma}_1e^{-\gamma_3} + \dot{\gamma}_1\dot{\gamma}_3e^{-\gamma_3})\right]\right\}^2 + \left\{\bar{r}_L\bar{p}\left[\frac{1}{\lambda_1\lambda_2}\ddot{\gamma}_3(t)\right]\right.\\ &\quad+\bar{r}_L\bar{q}\left[\frac{\lambda_1\sqrt{2}}{2}(\ddot{\gamma}_2e^{\gamma_3} + \dot{\gamma}_2\dot{\gamma}_3e^{\gamma_3} - \ddot{\gamma}_1e^{-\gamma_3} + \dot{\gamma}_1\dot{\gamma}_3e^{-\gamma_3})\right]\\ &\quad+\left.\frac{l}{l_L}L^{1/2}\left[\frac{\lambda_1^2\sqrt{2}}{2\lambda_2L}(-e^{-\gamma_3}\dot{\gamma}_1(t) + e^{\gamma_3}\dot{\gamma}_2(t))\dot{\gamma}_3(t)\right]\right\}^2 \sim \left\{\bar{q}\left[\frac{1}{\lambda_1\lambda_2}\ddot{\gamma}_3(t)\right]\right.\\ &\quad\left.-\bar{p}\left[\frac{\lambda_1\sqrt{2}}{2}(\ddot{\gamma}_2e^{\gamma_3} + \dot{\gamma}_2\dot{\gamma}_3e^{\gamma_3} - \ddot{\gamma}_1e^{-\gamma_3} + \dot{\gamma}_1\dot{\gamma}_3e^{-\gamma_3})\right]\right\}^2,\end{aligned} \tag{52}$$

as $L\longrightarrow+\infty$ and

$$\|\dot{\gamma}\|^2_{\Sigma,L} = \left[\frac{1}{\lambda_1\lambda_2}\dot{\gamma}_3(t)\bar{q} - \frac{\lambda_1\sqrt{2}}{2}(-e^{-\gamma_3}\dot{\gamma}_1(t) + e^{\gamma_3}\dot{\gamma}_2(t))\bar{p}\right]^2, \tag{53}$$

$$\begin{aligned}\left\langle\nabla^{\Sigma,L}_{\dot{\gamma}}\dot{\gamma},\dot{\gamma}\right\rangle_{\Sigma,L} &= \left[\frac{1}{\lambda_1\lambda_2}\dot{\gamma}_3(t)\bar{q} - \frac{\lambda_1\sqrt{2}}{2}(-e^{-\gamma_3}\dot{\gamma}_1(t) + e^{\gamma_3}\dot{\gamma}_2(t))\bar{p}\right]\\ &\quad\cdot\left[\bar{q}\frac{1}{\lambda_1\lambda_2}\ddot{\gamma}_3(t) - \bar{p}\frac{\lambda_1\sqrt{2}}{2}(\ddot{\gamma}_2e^{\gamma_3} + \dot{\gamma}_2\dot{\gamma}_3e^{\gamma_3} - \ddot{\gamma}_1e^{-\gamma_3} + \dot{\gamma}_1\dot{\gamma}_3e^{-\gamma_3})\right]\\ &:= AB.\end{aligned} \tag{54}$$

By (51), (52), (53), and (39), we get

$$\kappa_{\gamma,\Sigma}^{\infty} = \sqrt{\frac{B^2}{A^4} - \frac{A^2B^2}{A^6}} = 0. \tag{55}$$

When $\omega(\dot{\gamma}(t)) = 0$ and $(d/dt)\omega(\dot{\gamma}(t)) \neq 0$, we have

$$\left\|\nabla_{\dot{\gamma}}^{\Sigma,L}\dot{\gamma}\right\|_{\Sigma,L}^2 \sim L\left[\frac{d}{dt}(\omega(\dot{\gamma}(t)))\right]^2, \quad \text{as } L \longrightarrow +\infty,$$
$$\|\dot{\gamma}\|_{\Sigma,L} = \left|\frac{1}{\lambda_1\lambda_2}\dot{\gamma}_3(t)\bar{q} - \frac{\lambda_1\sqrt{2}}{2}(-e^{-\gamma_3}\dot{\gamma}_1(t) + e^{\gamma_3}\dot{\gamma}_2(t))\bar{p}\right|,$$
$$\left\langle\nabla_{\dot{\gamma}}^{\Sigma,L}\dot{\gamma}, \dot{\gamma}\right\rangle_{\Sigma,L} = O(1). \tag{56}$$

So, we get

$$\lim_{L\longrightarrow+\infty}\frac{\kappa_{\gamma,\Sigma}^{L}}{\sqrt{L}} = \frac{|(d/dt)(\omega(\dot{\gamma}(t)))|}{\left[(1/\lambda_1\lambda_2)\dot{\gamma}_3(t)\bar{q} - \left(\lambda_1\sqrt{2}/2\right)(-e^{-\gamma_3}\dot{\gamma}_1(t) + e^{\gamma_3}\dot{\gamma}_2(t))\bar{p}\right]^2}. \tag{57}$$

□ □

*Definition 10.* Let $\Sigma \subset (E(1,1), g_L(\lambda_1,\lambda_2))$ be a regular surface. Let $\gamma : [a,b] \longrightarrow \Sigma$ be a Euclidean $C^2$-smooth regular curve. The signed geodesic curvature $\kappa_{\gamma,\Sigma}^{L,s}$ of $\gamma$ at $\gamma(t)$ is defined as follows:

$$\kappa_{\gamma,\Sigma}^{L,s} := \frac{\left\langle\nabla_{\dot{\gamma}}^{\Sigma,L}\dot{\gamma}, J_L(\dot{\gamma})\right\rangle_{\Sigma,L}}{\|\dot{\gamma}\|_{\Sigma,L}^3}, \tag{58}$$

where $J_L$ is defined by (35).

*Definition 11.* Let $\Sigma \subset (E(1,1), g_L(\lambda_1,\lambda_2))$ be a regular surface. Let $\gamma : [a,b] \longrightarrow \Sigma$ be a Euclidean $C^2$-smooth regular curve. We define the intrinsic geodesic curvature $\kappa_{\gamma,\Sigma}^{\infty}$ of $\gamma$ at the noncharacteristic point $\gamma(t)$ to be

$$\kappa_{\gamma,\Sigma}^{\infty,s} := \lim_{L\longrightarrow+\infty}\kappa_{\gamma,\Sigma}^{L,s}, \tag{59}$$

if the limit exists.

**Proposition 12.** *Let $\Sigma \subset (E(1,1), g_L(\lambda_1,\lambda_2))$ be a regular surface. Let $\gamma : [a,b] \longrightarrow \Sigma$ be a Euclidean $C^2$-smooth regular curve. Then, we have*

$$\kappa_{\gamma,\Sigma}^{\infty,s} = \frac{\bar{q}\left(\lambda_1\lambda_2^2\sqrt{2}/2\right)(-e^{-\gamma_3}\dot{\gamma}_1(t) + e^{\gamma_3}\dot{\gamma}_2(t)) + \bar{p}\dot{\gamma}_3(t)}{|\omega(\dot{\gamma}(t))|}, \quad \text{if } \omega(\dot{\gamma}(t)) \neq 0,$$

$$\kappa_{\gamma,\Sigma}^{\infty,s} = 0, \quad \text{if } \omega(\dot{\gamma}(t)) = 0,\ \frac{d}{dt}(\omega(\dot{\gamma}(t))) = 0,$$

$$\lim_{L\longrightarrow+\infty}\frac{k_{\gamma,\Sigma}^{L,s}}{\sqrt{L}} = \frac{\left[-(1/\lambda_1\lambda_2)\dot{\gamma}_3(t)\bar{q} + \left(\lambda_1\sqrt{2}/2\right)(-e^{-\gamma_3}\dot{\gamma}_1(t) + e^{\gamma_3}\dot{\gamma}_2(t))\bar{p}\right](d/dt)(\omega(\dot{\gamma}(t)))}{\left|(1/\lambda_1\lambda_2)\dot{\gamma}_3(t)\bar{q} - \left(\lambda_1\sqrt{2}/2\right)(-e^{-\gamma_3}\dot{\gamma}_1(t) + e^{\gamma_3}\dot{\gamma}_2(t))\bar{p}\right|^3},$$
$$\text{if } \omega(\dot{\gamma}(t)) = 0,\ \frac{d}{dt}(\omega(\dot{\gamma}(t))) \neq 0. \tag{60}$$

*Proof.* By (35)(46), we get

$$\begin{aligned} J_L(\dot{\gamma}) &= \left[\frac{1}{\lambda_1\lambda_2}\dot{\gamma}_3(t)\bar{q} - \frac{\lambda_1\sqrt{2}}{2}(-e^{-\gamma_3}\dot{\gamma}_1(t) + e^{\gamma_3}\dot{\gamma}_2(t))\bar{p}\right]J_L(e_1) \\ &\quad - \frac{l_L}{l}L^{1/2}\omega(\dot{\gamma}(t))J_L(e_2) \\ &= \frac{l_L}{l}L^{1/2}\omega(\dot{\gamma}(t))e_1 + \left[\frac{1}{\lambda_1\lambda_2}\dot{\gamma}_3(t)\bar{q} - \frac{\lambda_1\sqrt{2}}{2}(-e^{-\gamma_3}\dot{\gamma}_1(t) + e^{\gamma_3}\dot{\gamma}_2(t))\bar{p}\right]e_2. \end{aligned} \tag{61}$$

By (37) and the above equation, we have

$$\begin{aligned} \left\langle\nabla_{\dot{\gamma}}^{\Sigma,L}\dot{\gamma}, J_L(\dot{\gamma})\right\rangle &= \frac{l_L}{l}L^{1/2}\omega(\dot{\gamma}(t)) \\ &\quad \cdot\left\{\bar{q}\left[\frac{1}{\lambda_1\lambda_2}\ddot{\gamma}_3(t) + \frac{\lambda_1^3\sqrt{2} + \lambda_1\lambda_2^2\sqrt{2}L}{2}(-e^{-\gamma_3}\dot{\gamma}_1(t) + e^{\gamma_3}\dot{\gamma}_2(t))\omega(\dot{\gamma}(t))\right]\right. \\ &\quad \left. - \bar{p}\left[\frac{\lambda_1\sqrt{2}}{2}(\ddot{\gamma}_2e^{\gamma_3} + \dot{\gamma}_2\dot{\gamma}_3e^{\gamma_3} - \ddot{\gamma}_1e^{-\gamma_3} + \dot{\gamma}_1\dot{\gamma}_3e^{-\gamma_3}) - L\dot{\gamma}_3(t)\omega(\dot{\gamma}(t))\right]\right\} \\ &\quad + \left[\frac{1}{\lambda_1\lambda_2}\dot{\gamma}_3(t)\bar{q} - \frac{\lambda_1\sqrt{2}}{2}(-e^{-\gamma_3}\dot{\gamma}_1(t) + e^{\gamma_3}\dot{\gamma}_2(t))\bar{p}\right] \\ &\quad \cdot\left\{\bar{r}_L\bar{p}\left[\frac{1}{\lambda_1\lambda_2}\ddot{\gamma}_3(t) + \frac{\lambda_1^3\sqrt{2} + \lambda_1\lambda_2^2\sqrt{2}L}{2}(-e^{-\gamma_3}\dot{\gamma}_1(t) + e^{\gamma_3}\dot{\gamma}_2(t))\omega(\dot{\gamma}(t))\right]\right. \\ &\quad + \bar{r}_L\bar{q}\left[\frac{\lambda_1\sqrt{2}}{2}(\ddot{\gamma}_2e^{\gamma_3} + \dot{\gamma}_2\dot{\gamma}_3e^{\gamma_3} - \ddot{\gamma}_1e^{-\gamma_3} + \dot{\gamma}_1\dot{\gamma}_3e^{-\gamma_3}) - L\dot{\gamma}_3(t)\omega(\dot{\gamma}(t))\right] \\ &\quad \left. - \frac{l}{l_L}L^{1/2}\left[\frac{d}{dt}(\omega(\dot{\gamma}(t))) - \frac{\lambda_1^2\sqrt{2}}{2\lambda_2L}(-e^{-\gamma_3}\dot{\gamma}_1(t) + e^{\gamma_3}\dot{\gamma}_2(t))\dot{\gamma}_3(t)\right]\right\} \\ &\quad \sim\left[\bar{q}\frac{\lambda_1\lambda_2^2\sqrt{2}}{2}(-e^{-\gamma_3}\dot{\gamma}_1(t) + e^{\gamma_3}\dot{\gamma}_2(t)) + \bar{p}\dot{\gamma}_3(t)\right] \\ &\quad \cdot(\omega(\dot{\gamma}(t)))^2L^{3/2}, \quad \text{as } L \longrightarrow +\infty. \end{aligned} \tag{62}$$

So, we get

$$\begin{aligned} \kappa_{\gamma,\Sigma}^{L,s} &= \frac{\left\langle\nabla_{\dot{\gamma}}^{\Sigma,L}\dot{\gamma}, J_L(\dot{\gamma})\right\rangle_{\Sigma,L}}{\|\dot{\gamma}\|_{\Sigma,L}^3} \\ &= \frac{L^{3/2}\left[\bar{q}\left(\lambda_1\lambda_2^2\sqrt{2}/2\right)(-e^{-\gamma_3}\dot{\gamma}_1(t) + e^{\gamma_3}\dot{\gamma}_2(t)) + \bar{p}\dot{\gamma}_3(t)\right](\omega(\dot{\gamma}(t)))^2}{L^{3/2}|\omega(\dot{\gamma}(t))|^3}. \end{aligned} \tag{63}$$

Furthermore,

$$\kappa_{\gamma,\Sigma}^{\infty,s} = \lim_{L\longrightarrow+\infty}k_{\gamma,\Sigma}^{L,s} = \frac{\bar{q}\left(\lambda_1\lambda_2^2\sqrt{2}/2\right)(-e^{-\gamma_3}\dot{\gamma}_1(t) + e^{\gamma_3}\dot{\gamma}_2(t)) + \bar{p}\dot{\gamma}_3(t)}{|\omega(\dot{\gamma}(t))|}. \tag{64}$$

When $\omega(\dot{\gamma}(t)) = 0$ and $(d/dt)\omega(\dot{\gamma}(t)) = 0$, we get

$$\begin{aligned}\left\langle \nabla^{\Sigma,L}_{\dot{\gamma}}\dot{\gamma}, J_L(\dot{\gamma})\right\rangle_{L,\Sigma} &= \left[\frac{1}{\lambda_1\lambda_2}\dot{\gamma}_3(t)\bar{q} - \frac{\lambda_1\sqrt{2}}{2}(-e^{-\gamma_3}\dot{\gamma}_1(t) + e^{\gamma_3}\dot{\gamma}_2(t))\bar{p}\right] \\ &\cdot \left\{\bar{r}_L\bar{p}\left[\frac{1}{\lambda_1\lambda_2}\ddot{\gamma}_3(t)\right]\right. \\ &+ \bar{r}_L\bar{q}\left[\frac{\lambda_1\sqrt{2}}{2}(\ddot{\gamma}_2 e^{\gamma_3} + \dot{\gamma}_2\dot{\gamma}_3 e^{\gamma_3} - \ddot{\gamma}_1 e^{-\gamma_3} + \dot{\gamma}_1\dot{\gamma}_3 e^{-\gamma_3})\right] \\ &\left. + \frac{l}{l_L}L^{1/2}\left[\frac{\lambda_1^2\sqrt{2}}{2\lambda_2 L}(-e^{-\gamma_3}\dot{\gamma}_1(t) + e^{\gamma_3}\dot{\gamma}_2(t))\dot{\gamma}_3(t)\right]\right\} \\ &\sim N_1 L^{-1/2}, \quad \text{as } L \longrightarrow +\infty,\end{aligned} \tag{65}$$

where $N_1$ does not depend on $L$. So, $\kappa^{\infty,s}_{\gamma,\Sigma} = 0$. When $\omega(\dot{\gamma}(t)) = 0$ and $(d/dt)(\omega(\dot{\gamma}(t))) \neq 0$, we have

$$\begin{aligned}\left\langle \nabla^{\Sigma,L}_{\dot{\gamma}}\dot{\gamma}, J_L(\dot{\gamma})\right\rangle_{L,\Sigma} &\sim L^{1/2}\left[-\frac{1}{\lambda_1\lambda_2}\dot{\gamma}_3(t)\bar{q} + \frac{\lambda_1\sqrt{2}}{2}(-e^{-\gamma_3}\dot{\gamma}_1(t) + e^{\gamma_3}\dot{\gamma}_2(t))\bar{p}\right]\frac{d}{dt} \\ &\cdot (\omega(\dot{\gamma}(t))), \quad \text{as } L \longrightarrow +\infty.\end{aligned} \tag{66}$$

We get

$$\begin{aligned}\kappa^{\infty,s}_{\gamma,\Sigma} &= \lim_{L\longrightarrow+\infty}\frac{\kappa^{L,s}_{\gamma,\Sigma}}{\sqrt{L}} \\ &= \lim_{L\longrightarrow+\infty}\frac{\left[-(1/\lambda_1\lambda_2)\dot{\gamma}_3(t)\bar{q} + \left(\lambda_1\sqrt{2}/2\right)(-e^{-\gamma_3}\dot{\gamma}_1(t) + e^{\gamma_3}\dot{\gamma}_2(t))\bar{p}\right](d/dt)(\omega(\dot{\gamma}(t)))L^{1/2}}{\left|(1/\lambda_1\lambda_2)\dot{\gamma}_3(t)\bar{q} - \left(\lambda_1\sqrt{2}/2\right)(-e^{-\gamma_3}\dot{\gamma}_1(t) + e^{\gamma_3}\dot{\gamma}_2(t))\bar{p}\right|^3\sqrt{L}} \\ &= \frac{\left[-(1/\lambda_1\lambda_2)\dot{\gamma}_3(t)\bar{q} + \left(\lambda_1\sqrt{2}/2\right)(-e^{-\gamma_3}\dot{\gamma}_1(t) + e^{\gamma_3}\dot{\gamma}_2(t))\bar{p}\right](d/dt)(\omega(\dot{\gamma}(t)))}{\left|(1/\lambda_1\lambda_2)\dot{\gamma}_3(t)\bar{q} - \left(\lambda_1\sqrt{2}/2\right)(-e^{-\gamma_3}\dot{\gamma}_1(t) + e^{\gamma_3}\dot{\gamma}_2(t))\bar{p}\right|^3}.\end{aligned} \tag{67}$$

□ □

## 4. The Sub-Riemannian Limit of the Riemannian Gaussian Curvature of Surfaces in $(E(1,1), g_L(\lambda_1,\lambda_2))$

In the following, we compute the sub-Riemannian limit of the Riemannian Gaussian curvature of surfaces in $(E(1,1), g_L(\lambda_1,\lambda_2))$. We define the second fundamental form $II^L$ of the embedding of $\Sigma$ into $(E(1,1), g_L(\lambda_1,\lambda_2))$ by

$$II^L = \begin{pmatrix}\left\langle \nabla^L_{e_1}\nu_L, e_1\right\rangle_L & \left\langle \nabla^L_{e_1}\nu_L, e_2\right\rangle_L \\ \left\langle \nabla^L_{e_2}\nu_L, e_1\right\rangle_L & \left\langle \nabla^L_{e_2}\nu_L, e_2\right\rangle_L\end{pmatrix}. \tag{68}$$

We have the following theorem similar to Theorem 4.3 in [10].

**Theorem 13.** *The second fundamental form $II_L$ of the embedding of $\Sigma$ into $(E(1,1), g_L(\lambda_1,\lambda_2))$ is given by*

$$II^L = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22}\end{pmatrix}, \tag{69}$$

*where*

$$\begin{aligned} h_{11} &= \frac{l}{l_L}(X_1(\bar{p}) + X_2(\bar{q})) - \lambda_1^2\bar{r}_L\bar{p}\bar{q}L^{-1/2}, \\ h_{12} = h_{21} &= -\frac{l_L}{l}\langle e_1, \nabla_H(\bar{r}_L)\rangle_L - \frac{\sqrt{L}}{2}\lambda_2^2 + \frac{\lambda_1^2}{2\sqrt{L}}\left(\bar{q}_L^2 - \bar{p}_L^2\right) + \frac{\bar{r}_L^2\lambda_1^2}{2\sqrt{L}}\left(\bar{q}^2 - \bar{p}^2\right), \\ h_{22} &= -\frac{l^2}{l_L^2}\left\langle e_2, \nabla_H\left(\frac{r}{l}\right)\right\rangle_L + \widetilde{X_3}(\bar{r}_L) + \frac{\bar{r}_L\bar{p}_L\bar{q}_L}{\sqrt{L}}\lambda_1^2 + \frac{\bar{r}_L^3\bar{p}\bar{q}}{\sqrt{L}}\lambda_1^2.\end{aligned} \tag{70}$$

*Proof.* Since $\langle e_1, \nu_L\rangle_L = 0$ and $\langle e_2, \nu_L\rangle_L = 0$, we have

$$\begin{aligned}\left\langle \nabla^L_{e_1}\nu_L, e_1\right\rangle_L &= -\left\langle \nabla^L_{e_1}e_1, \nu_L\right\rangle_L, \\ \left\langle \nabla^L_{e_2}\nu_L, e_2\right\rangle_L &= -\left\langle \nabla^L_{e_2}e_2, \nu_L\right\rangle_L.\end{aligned} \tag{71}$$

Using the definition of the connection, the identities in (8) and grouping terms, we have

$$\begin{aligned}\nabla^L_{e_1}e_1 &= \nabla^L_{\bar{q}X_1 - \bar{p}X_2}\bar{q}X_1 - \bar{p}X_2 = \bar{q}\left(X_1\bar{q}X_1 - X_1\bar{p}X_2 - \bar{p}\nabla^L_{X_1}X_2\right) \\ &\quad - \bar{p}\left(X_2\bar{q}X_1 - X_2\bar{p}X_2 + \bar{q}\nabla^L_{X_2}X_1\right) \\ &= (\bar{q}X_1\bar{q} - \bar{p}X_2\bar{q})X_1 - (\bar{q}X_1\bar{p} - \bar{p}X_2\bar{p})X_2 + \frac{\bar{p}\bar{q}\lambda_1^2}{L}X_3,\end{aligned} \tag{72}$$

since $\bar{p}^2 + \bar{q}^2 = 1$, we have $\bar{p}X_i\bar{p} + \bar{q}X_i\bar{q} = 0$, $i = 1, 2, 3$. We have

$$\begin{aligned} h_{11} &= -\left\langle \nabla^L_{e_1}e_1, \nu_L\right\rangle_L = -\left[\bar{p}_L(\bar{q}X_1\bar{q} - \bar{p}X_2\bar{q}) - \bar{q}_L(\bar{q}X_1\bar{p} - \bar{p}X_2\bar{p}) + \bar{r}_L\frac{\bar{p}\bar{q}\lambda_1^2}{\sqrt{L}}\right] \\ &= \frac{l}{l_L}(X_1(\bar{p}) + X_2(\bar{q})) - \lambda_1^2\bar{r}_L\bar{p}\bar{q}L^{-1/2}.\end{aligned} \tag{73}$$

To compute $h_{12}$ and $h_{21}$, using the definition of the connection, we obtain

$$\begin{aligned}
\nabla^L_{e_1} e_2 &= \nabla^L_{\bar{q}X_1-\bar{p}X_2}\bar{r}_L\bar{p}X_1 + \bar{r}_L\bar{q}X_2 - \frac{l}{l_L}L^{-1/2}X_3 \\
&= \bar{q}\Big(X_1(\bar{r}_L\bar{p})X_1 + X_1(\bar{r}_L\bar{q})X_2 + \bar{r}_L\bar{q}\nabla^L_{X_1}X_2 \\
&\quad - L^{-1/2}X_1\left(\frac{l}{l_L}\right)X_3 - \frac{l}{l_L}L^{-1/2}\nabla^L_{X_1}X_3\Big) \\
&\quad - \bar{p}\Big(X_2(\bar{r}_L\bar{p})X_1 + \bar{r}_L\bar{p}\nabla^L_{X_2}X_1 + X_2(\bar{r}_L\bar{q})X_2 \\
&\quad - L^{-1/2}X_2\left(\frac{l}{l_L}\right)X_3 - \frac{l}{l_L}L^{-1/2}\nabla^L_{X_2}X_3\Big) \\
&= \left(\bar{q}X_1(\bar{r}_L\bar{p}) - \bar{p}X_2(\bar{r}_L\bar{p}) + \bar{p}\frac{l}{l_L}L^{-1/2}\frac{\lambda_1^2+\lambda_2^2L}{2}\right)X_1 \\
&\quad + \left(\bar{q}X_1(\bar{r}_L\bar{q}) - \bar{p}X_2(\bar{r}_L\bar{q}) - \bar{q}\frac{l}{l_L}L^{-1/2}\frac{\lambda_1^2-\lambda_2^2L}{2}\right)X_2 \\
&\quad + \left[\bar{p}X_2L^{-1/2}\left(\frac{l}{l_L}\right) - \bar{q}X_1L^{-1/2}\left(\frac{l}{l_L}\right) + \frac{\bar{r}_L}{2}\left(\lambda_2^2 + \frac{\lambda_1^2}{L}(\bar{p}^2-\bar{q}^2)\right)\right]X_3.
\end{aligned} \tag{74}$$

Next, we compute the inner product of this with $\nu_L$. Using the product rule and the identity $\bar{q}_L\bar{p} = \bar{p}_L\bar{q}$, we obtain

$$\begin{aligned}
\left\langle \nabla^L_{e_1} e_2, \nu_L\right\rangle_L &= \bar{p}_L\left(\bar{q}X_1(\bar{r}_L\bar{p}) - \bar{p}X_2(\bar{r}_L\bar{p}) + \bar{p}\frac{l}{l_L}L^{-1/2}\frac{\lambda_1^2+\lambda_2^2L}{2}\right) \\
&\quad + \bar{q}_L\left(\bar{q}X_1(\bar{r}_L\bar{q}) - \bar{p}X_2(\bar{r}_L\bar{q}) - \bar{q}\frac{l}{l_L}L^{-1/2}\frac{\lambda_1^2-\lambda_2^2L}{2}\right) \\
&\quad + \bar{r}_L\left[\bar{p}X_2\left(\frac{l}{l_L}\right) - \bar{q}X_1\left(\frac{l}{l_L}\right) + \frac{\bar{r}_L}{2}\left(\lambda_2^2\sqrt{L} + \frac{\lambda_1^2}{\sqrt{L}}(\bar{p}^2-\bar{q}^2)\right)\right] \\
&= \bar{p}_L\bar{q}(\bar{p}X_1\bar{r}_L + \bar{r}_LX_1\bar{p}) - \bar{p}_L\bar{p}(\bar{p}X_2\bar{r}_L + \bar{r}_LX_2\bar{p}) \\
&\quad + \bar{q}_L\bar{q}(\bar{q}X_1\bar{r}_L + \bar{r}_LX_1\bar{q}) - \bar{q}_L\bar{p}(\bar{q}X_2\bar{r}_L + \bar{r}_LX_2\bar{q}) \\
&\quad + \frac{\sqrt{L}}{2}\lambda_2^2(\bar{p}_L^2 + \bar{q}_L^2 + \bar{r}_L^2) + \bar{r}_L\bar{p}X_2\left(\frac{l}{l_L}\right) - \bar{r}_L\bar{q}X_1\left(\frac{l}{l_L}\right) \\
&\quad + \frac{\lambda_1^2}{\sqrt{L}}(\bar{p}_L^2 - \bar{q}_L^2) + \frac{\bar{r}_L^2}{2}\frac{\lambda_1^2}{\sqrt{L}}(\bar{p}^2-\bar{q}^2).
\end{aligned} \tag{75}$$

The identities $\bar{p}_L{}^2 + \bar{q}_L{}^2 + \bar{r}_L{}^2 = 1$ and $\bar{p}^2 + \bar{q}^2 = 1$ yield

$$\begin{aligned}
\left\langle \nabla^L_{e_1} e_2, \nu_L\right\rangle_L &= \frac{l}{l_L}\bar{q}X_1\bar{r}_L - \frac{l}{l_L}\bar{p}X_2\bar{r}_L + \bar{r}_L\bar{p}X_2\left(\frac{l}{l_L}\right) - \bar{r}_L\bar{q}X_1\left(\frac{l}{l_L}\right) \\
&\quad + \frac{\sqrt{L}}{2}\lambda_2^2 + \frac{\lambda_1^2}{\sqrt{L}}(\bar{p}_L^2 - \bar{q}_L^2) + \frac{\bar{r}_L^2}{2}\frac{\lambda_1^2}{\sqrt{L}}(\bar{p}^2-\bar{q}^2) \\
&= \frac{l}{l_L}\langle e_1, \nabla_H\bar{r}_L\rangle_L - \bar{r}_L\left\langle e_1, \nabla_H\left(\frac{l}{l_L}\right)\right\rangle_L + \frac{\sqrt{L}}{2}\lambda_2^2 \\
&\quad + \frac{\lambda_1^2}{\sqrt{L}}(\bar{p}_L^2 - \bar{q}_L^2) + \frac{\bar{r}_L^2}{2}\frac{\lambda_1^2}{\sqrt{L}}(\bar{p}^2-\bar{q}^2).
\end{aligned} \tag{76}$$

Finally, we use the identity $(l/l_L - l_L/l)\nabla_H\bar{r}_L = \bar{r}_L\nabla_H(l/l_L)$, we have

$$\left\langle \nabla^L_{e_1} e_2, \nu_L\right\rangle_L = \frac{l_L}{l}\langle e_1, \nabla_H(\bar{r}_L)\rangle_L + \frac{\sqrt{L}}{2}\lambda_2^2 + \frac{\lambda_1^2}{\sqrt{L}}(\bar{p}_L^2 - \bar{q}_L^2) + \frac{\bar{r}_L^2}{2}\frac{\lambda_1^2}{\sqrt{L}}(\bar{p}^2-\bar{q}^2). \tag{77}$$

Therefore, we have

$$\begin{aligned}
h_{21} = h_{12} = -\left\langle \nabla^L_{e_1} e_2, \nu_L\right\rangle_L &= -\frac{l_L}{l}\langle e_1, \nabla_H(\bar{r}_L)\rangle_L - \frac{\sqrt{L}}{2}\lambda_2^2 \\
&\quad + \frac{\lambda_1^2}{2\sqrt{L}}(\bar{q}_L^2 - \bar{p}_L^2) + \frac{\bar{r}_L^2\lambda_1^2}{2\sqrt{L}}(\bar{q}^2-\bar{p}^2).
\end{aligned} \tag{78}$$

Since $\langle \nabla_{e_2}\nu_L, e_2\rangle_L = -\langle \nabla_{e_2}e_2, \nu_L\rangle_L$, using the definition of connection, the identities in (9) and grouping terms, we have

$$\begin{aligned}
\nabla^L_{e_2} e_2 &= \nabla^L_{\bar{r}_L\bar{p}X_1+\bar{r}_L\bar{q}X_2-l/l_L\widetilde{X_3}}\bar{r}_L\bar{p}X_1 + \bar{r}_L\bar{q}X_2 - \frac{l}{l_L}\widetilde{X_3} \\
&= \bar{r}_L\bar{p}\Big(X_1(\bar{r}_L\bar{p})X_1 + X_1(\bar{r}_L\bar{q})X_2 + \bar{r}_L\bar{q}\nabla^L_{X_1}X_2 \\
&\quad - L^{-1/2}X_1\left(\frac{l}{l_L}\right)X_3 - L^{-1/2}\frac{l}{l_L}\nabla^L_{X_1}X_3\Big) \\
&\quad + \bar{r}_L\bar{q}\Big(X_2(\bar{r}_L\bar{p})X_1 + X_2(\bar{r}_L\bar{q})X_2 + \bar{r}_L\bar{p}\nabla^L_{X_2}X_1 \\
&\quad - L^{-1/2}X_2\left(\frac{l}{l_L}\right)X_3 - L^{-1/2}\frac{l}{l_L}\nabla^L_{X_2}X_3\Big) \\
&\quad - \frac{l}{l_L}L^{-1/2}\Big(X_3(\bar{r}_L\bar{p})X_1 + \bar{r}_L\bar{p}\nabla^L_{X_3}X_1 + X_3(\bar{r}_L\bar{q})X_2 \\
&\quad + \bar{r}_L\bar{q}\nabla^L_{X_3}X_2 - L^{-1/2}X_3\left(\frac{l}{l_L}\right)X_3\Big) \\
&= \Big(\bar{r}_L\bar{p}X_1\bar{r}_L\bar{p} + \bar{r}_L\bar{q}X_2\bar{r}_L\bar{p} - L^{-1/2}\frac{l}{l_L}X_3(\bar{r}_L\bar{p}) \\
&\quad - \bar{r}_L\bar{q}L^{-1/2}\frac{l}{l_L}(\lambda_1^2+\lambda_2^2L)\Big)X_1 + (\bar{r}_L\bar{p}X_1\bar{r}_L\bar{q} + \bar{r}_L\bar{q}X_2\bar{r}_L\bar{q} \\
&\quad - L^{-1/2}\frac{l}{l_L}X_3(\bar{r}_L\bar{q}) - \bar{r}_L\bar{p}L^{-1/2}\frac{l}{l_L}(-\lambda_2^2L)\Big)X_2 \\
&\quad + \left(-\bar{r}_L\bar{p}L^{-1/2}X_1\frac{l}{l_L} - \bar{r}_L\bar{q}X_2L^{-1/2}X_2\frac{l}{l_L} + L^{-1}\frac{l}{l_L}X_3\left(\frac{l}{l_L}\right) - \bar{r}_L^2\bar{p}\bar{q}\frac{\lambda_1^2}{L}\right)X_3.
\end{aligned} \tag{79}$$

Taking the inner product with $\nu_L$ yields

$$\begin{aligned}
\left\langle \nabla^L_{e_2} e_2, \nu_L\right\rangle_L &= \bar{p}_L\Big(\bar{r}_L\bar{p}X_1\bar{r}_L\bar{p} + \bar{r}_L\bar{q}X_2\bar{r}_L\bar{p} - L^{-1/2}\frac{l}{l_L}X_3(\bar{r}_L\bar{p}) \\
&\quad - \bar{r}_L\bar{q}L^{-1/2}\frac{l}{l_L}(\lambda_1^2+\lambda_2^2L)\Big) \\
&\quad + \bar{q}_L\Big(\bar{r}_L\bar{p}X_1\bar{r}_L\bar{q} + \bar{r}_L\bar{q}X_2\bar{r}_L\bar{q} - L^{-1/2}\frac{l}{l_L}X_3(\bar{r}_L\bar{q}) \\
&\quad - \bar{r}_L\bar{p}L^{-1/2}\frac{l}{l_L}(-\lambda_2^2L)\Big) + \bar{r}_L\Big(-\bar{r}_L\bar{p}X_1\frac{l}{l_L} - \bar{r}_L\bar{q}X_2X_2\frac{l}{l_L} \\
&\quad + L^{-1/2}\frac{l}{l_L}X_3\left(\frac{l}{l_L}\right) - \bar{r}_L^2\bar{p}\bar{q}\frac{\lambda_1^2}{\sqrt{L}}\Big).
\end{aligned} \tag{80}$$

To simplify this, first use the product rule for the terms involving $X_i(\bar{p}\bar{r}_L)$ and $X_i(\bar{q}\bar{r}_L)$ together with the identities $\bar{p}X_i\bar{p} + \bar{q}X_i\bar{q} = 0$, $\bar{r}_L\nabla_H(l/l_L) = (l/l_L - l_L/l)\nabla_H\bar{r}_L$, and $\bar{p}^2 + \bar{q}^2 = 1$. Under these simplifications, terms involving $X_i(\bar{p})$ and $X_i(\bar{q})$ cancel and one is left with terms involving components

of $\nabla\bar{r}_L$:

$$\left\langle \nabla^L_{e_2} e_2, \nu_L \right\rangle_L = \bar{p}_L\left(\bar{p}_L + \bar{p}\frac{r}{l}\right)X_1(\bar{r}_L) + \bar{q}_L\left(\bar{q}_L + \bar{q}\frac{r}{l}\right)X_2(\bar{r}_L) - \widetilde{X_3}(\bar{r}_L) - \frac{\bar{r}_L\bar{p}_L\bar{q}_L}{\sqrt{L}}\lambda_1^2 - \frac{\bar{r}_L^3\bar{p}\bar{q}}{\sqrt{L}}\lambda_1^2. \tag{81}$$

We conclude by rewriting the expression $X_i(\bar{r}_L)$ in terms of $X_i(r/l)$. Therefore, we have

$$h_{22} = -\left\langle \nabla^L_{e_2} e_2, \nu_L \right\rangle_L = -\frac{l^2}{l_L^2}\left\langle e_2, \nabla_H\left(\frac{r}{l}\right)\right\rangle_L + \widetilde{X_3}(\bar{r}_L) + \frac{\bar{r}_L\bar{p}_L\bar{q}_L}{\sqrt{L}}\lambda_1^2 + \frac{\bar{r}_L^3\bar{p}\bar{q}}{\sqrt{L}}\lambda_1^2. \tag{82}$$

□ □

The Riemannian mean curvature $\mathscr{H}_L$ of $\Sigma$ is defined by

$$\mathscr{H}_L := tr(II^L) = \frac{l}{l_L}(X_1(\bar{p}) + X_2(\bar{q})) - \lambda_1^2\bar{r}_L\bar{p}\bar{q}L^{-1/2} - \frac{l^2}{l_L^2}\left\langle e_2, \nabla_H\left(\frac{r}{l}\right)\right\rangle_L + \widetilde{X_3}(\bar{r}_L) + \frac{\bar{r}_L\bar{p}_L\bar{q}_L}{\sqrt{L}}\lambda_1^2 + \frac{\bar{r}_L^3\bar{p}\bar{q}}{\sqrt{L}}\lambda_1^2. \tag{83}$$

Define the curvature of a connection $\nabla$ by

$$R(X, Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z. \tag{84}$$

Let

$$\begin{aligned} \mathscr{K}^{\Sigma,L}(e_1, e_2) &= -\left\langle R^{\Sigma,L}(e_1, e_2)e_1, e_2\right\rangle_{\Sigma,L}, \\ \mathscr{K}^{L}(e_1, e_2) &= -\left\langle R^{L}(e_1, e_2)e_1, e_2\right\rangle_{L}. \end{aligned} \tag{85}$$

By the Gauss equation, we have

$$\mathscr{K}^{\Sigma,L}(e_1, e_2) = \mathscr{K}^{L}(e_1, e_2) + \det\left(II^L\right). \tag{86}$$

**Proposition 14.** *Away from characteristic point, the horizontal mean curvature $\mathscr{H}_\infty$ of $\Sigma \in (E(1,1), g_L(\lambda_1, \lambda_2))$ is given by*

$$\mathscr{H}_\infty = \lim_{L\longrightarrow\infty}\mathscr{H}_L = X_1(\bar{p}) + X_2(\bar{q}). \tag{87}$$

*Proof.* By

$$\begin{aligned} \frac{l^2}{l_L^2}\left\langle e_2, \nabla_H\left(\frac{r}{l}\right)\right\rangle_L &= \frac{\bar{p}r}{l}X_1(\bar{r}_L) + \frac{\bar{q}r}{l}X_2(\bar{r}_L) = O\left(L^{-1}\right), \\ \frac{l}{l_L}(X_1(\bar{p}) + X_2(\bar{q})) &\longrightarrow X_1(\bar{p}) + X_2(\bar{q}), \\ \widetilde{X_3}(\bar{r}_L) &\longrightarrow 0, \\ \lambda_1^2\bar{r}_L\bar{p}\bar{q}L^{-1/2} &\longrightarrow 0, \\ \frac{\bar{r}_L\bar{p}_L\bar{q}_L}{\sqrt{L}}\lambda_1^2 + \frac{\bar{r}_L^3\bar{p}\bar{q}}{\sqrt{L}}\lambda_1^2 &\longrightarrow 0, \end{aligned} \tag{88}$$

we get (86) □ □

By Proposition 1 and (83), we have the following lemma.

**Lemma 15.** *Let $(E(1,1), g_L(\lambda_1, \lambda_2))$ be the group of rigid motions of the Minkowski plane with a general left-invariant metric. Then, we have*

$$\begin{aligned} R^L(X_1, X_2)X_1 &= \frac{-\lambda_1^4 + 2\lambda_1^2\lambda_2^2L + 3\lambda_2^4L^2}{4L}X_2, \\ R^L(X_1, X_2)X_2 &= \frac{\lambda_1^4 - 2\lambda_1^2\lambda_2^2L - 3\lambda_2^4L^2}{4L}X_1, \\ R^L(X_1, X_2)X_3 &= 0, \\ R^L(X_1, X_3)X_1 &= \frac{3\lambda_1^4 + 2\lambda_1^2\lambda_2^2L - \lambda_2^4L^2}{4L}X_3, \\ R^L(X_1, X_3)X_2 &= 0, \\ R^L(X_1, X_3)X_3 &= \frac{-3\lambda_1^4 - 2\lambda_1^2\lambda_2^2L + \lambda_2^4L^2}{4}X_1, \\ R^L(X_2, X_3)X_1 &= 0, \\ R^L(X_2, X_3)X_2 &= \frac{-\lambda_1^4 - 2\lambda_1^2\lambda_2^2L - \lambda_2^4L^2}{4L}X_3, \\ R^L(X_2, X_3)X_3 &= \frac{\lambda_1^4 + 2\lambda_1^2\lambda_2^2L + \lambda_2^4L^2}{4}X_2. \end{aligned} \tag{89}$$

**Proposition 16.** *Away from characteristic points, we have*

$$\mathscr{K}^{\Sigma,L}(e_1, e_2) = -\left\langle e_1, \nabla_H\left(\frac{X_3u}{|\nabla_H u|}\right)\right\rangle_L - \lambda_2{}^4\frac{(X_3u)^2}{l^2}, \quad as\, L \longrightarrow \infty. \tag{90}$$

*Proof.* We compute

$$\begin{aligned}
R^L(e_1,e_2)e_1 &= R^L\left(\bar{q}X_1-\bar{p}X_2, \bar{r}_L\bar{p}X_1+\bar{r}_L\bar{q}X_2-\frac{l}{l_L\sqrt{L}}X_3\right)(\bar{q}X_1-\bar{p}X_2)\\
&= \bar{r}_L\bar{p}\bar{q}^2R^L(X_1,X_1)X_1+\bar{r}_L\bar{q}^3R^L(X_1,X_2)X_1-\frac{l\bar{q}^2}{l_L\sqrt{L}}R^L(X_1,X_3)X_1\\
&\quad-\bar{r}_L\bar{p}^2\bar{q}R^L(X_2,X_1)X_1-\bar{r}_L\bar{p}\bar{q}^2R^L(X_2,X_2)X_1\\
&\quad+\frac{l\bar{p}\bar{q}}{l_L\sqrt{L}}R^L(X_2,X_3)X_1-\bar{r}_L\bar{p}^2\bar{q}R^L(X_1,X_1)X_2\\
&\quad-\bar{r}_L\bar{p}\bar{q}^2R^L(X_1,X_2)X_2+\frac{l\bar{p}\bar{q}}{l_L\sqrt{L}}R^L(X_1,X_3)X_2\\
&\quad+\bar{r}_L\bar{p}^3R^L(X_2,X_1)X_2+\bar{r}_L\bar{p}^2\bar{q}R^L(X_2,X_2)X_2-\frac{l\bar{p}^2}{l_L\sqrt{L}}R^L(X_2,X_3)X_2\\
&= \bar{r}_L\bar{q}^3R^L(X_1,X_2)X_1-\bar{q}^2\frac{l}{l_L\sqrt{L}}R^L(X_1,X_3)X_1-\bar{r}_L\bar{p}^2\bar{q}R^L(X_2,X_1)X_1\\
&\quad-\bar{r}_L\bar{q}^2\bar{p}R^L(X_1,X_2)X_2+\bar{r}_L\bar{p}^3R^L(X_2,X_1)X_2\\
&\quad-\bar{p}^2\frac{l}{l_L\sqrt{L}}R^L(X_2,X_3)X_2\\
&= -\bar{r}_L\bar{p}\frac{\lambda_1^4-2\lambda_1^2\lambda_2^2L-3\lambda_2^4L^2}{4L}X_1+\bar{r}_L\bar{q}\frac{-\lambda_1^4+2\lambda_1^2\lambda_2^2L+3\lambda_2^4L^2}{4L}X_2\\
&\quad-\frac{l}{l_L\sqrt{L}}\left(\bar{q}^2\frac{3\lambda_1^4+2\lambda_1^2\lambda_2^2L-\lambda_2^4L^2}{4L}+\bar{p}^2\frac{-\lambda_1^4-2\lambda_1^2\lambda_2^2L-\lambda_2^4L^2}{4L}\right)X_3,
\end{aligned}\tag{91}$$

$$\begin{aligned}
\mathscr{K}^L(e_1,e_2) &= -\langle R^L(e_1,e_2)e_1,e_2\rangle_L = -\bar{r}_L{}^2\left(\frac{\lambda_1^2\lambda_2^2}{2}-\frac{\lambda_1^4}{4L}+\frac{3\lambda_2^4L}{4}\right)\\
&\quad-\left(\frac{l}{l_L}\bar{q}\right)^2\left(\frac{\lambda_1^2\lambda_2^2}{2}-\frac{\lambda_2^4L}{4}+\frac{3\lambda_1^4}{4L}\right)\\
&\quad+\left(\frac{l}{l_L}\bar{p}\right)^2\left(\frac{\lambda_1^2\lambda_2^2}{2}+\frac{\lambda_1^4}{4L}+\frac{\lambda_2^4L}{4}\right)\sim\left(\frac{l}{l_L}\right)^2\frac{\lambda_2^4L}{4}\\
&\quad-\frac{3\lambda_2^4}{4}\frac{(X_3u)^2}{l^2}-\lambda_1^2\lambda_2^2\bar{q}^2+\frac{\lambda_1^2\lambda_2^2}{2},\quad \text{as } L\longrightarrow\infty.
\end{aligned}\tag{92}$$

By Theorem 13 and $\nabla_H(\bar{r}_L)=L^{-1/2}\nabla_H(X_3u/|\nabla_H u|)+O(L^{-1})$ as $L\longrightarrow+\infty$, we get

$$\begin{aligned}
\det\left(II^L\right) &= h_{11}h_{22}-h_{12}^2=-\frac{\lambda_2^4L}{4}-\left\langle e_1,\nabla_H\left(\frac{X_3u}{|\nabla_H u|}\right)\right\rangle_L\\
&\quad+\frac{\lambda_1^2\lambda_2^2}{2}\left(\bar{q}^2-\bar{p}^2\right)+O\left(L^{-1/2}\right)\quad \text{as } L\longrightarrow\infty.
\end{aligned}\tag{93}$$

By (85), (91), and (92), we get the desired equation.□ □

## 5. A Gauss-Bonnet Theorem in $(E(1,1), g_L(\lambda_1,\lambda_2))$

In this section, we will prove the Gauss-Bonnet theorem in the group of rigid motions of the Minkowski plane $(E(1,1), g_L(\lambda_1,\lambda_2))$. Let us first consider the case of a regular curve $\gamma:[a,b]\longrightarrow(E(1,1),g_L(\lambda_1,\lambda_2))$, and we define the Riemannian length measure as follows:

$$ds_L=\|\dot{\gamma}\|_L dt.\tag{94}$$

**Lemma 17.** *Let $\gamma:[a,b]\longrightarrow(E(1,1),g_L(\lambda_1,\lambda_2))$ be a Euclidean $C^2$-smooth regular curve. Let*

$$\begin{aligned}
ds &:= |\omega(\dot{\gamma}(t))|dt,\\
d\bar{s} &:= \frac{1}{2}\frac{1}{|\omega(\dot{\gamma}(t))|}\left\{\left[\frac{1}{\lambda_1\lambda_2}\dot{\gamma}_3(t)\right]^2+\left[\frac{\lambda_1\sqrt{2}}{2}\left(-e^{\gamma_3}\dot{\gamma}_1(t)+e^{\gamma_3}\dot{\gamma}_2(t)\right)\right]^2\right\}dt.
\end{aligned}\tag{95}$$

*Then, we have*

$$\lim_{L\longrightarrow\infty}\frac{1}{\sqrt{L}}\int_\gamma ds_L=\int_a^b ds.\tag{96}$$

*When $\omega(\dot{\gamma}(t))\neq 0$, we have*

$$\frac{1}{\sqrt{L}}ds_L=ds+d\bar{s}L^{-1}+O\left(L^{-2}\right),\quad \text{as } L\longrightarrow+\infty.\tag{97}$$

*When $\omega(\dot{\gamma}(t))=0$, we have*

$$\frac{1}{\sqrt{L}}ds_L=\frac{1}{\sqrt{L}}\sqrt{\left[\frac{1}{\lambda_1\lambda_2}\dot{\gamma}_3(t)\right]^2+\left[\frac{\lambda_1\sqrt{2}}{2}\left(-e^{\gamma_3}\dot{\gamma}_1(t)+e^{\gamma_3}\dot{\gamma}_2(t)\right)\right]^2}dt.\tag{98}$$

*Proof.* We know that

$$\|\dot{\gamma}(t)\|_L=\sqrt{\left[\frac{1}{\lambda_1\lambda_2}\dot{\gamma}_3(t)\right]^2+\left[\frac{\lambda_1\sqrt{2}}{2}\left(-e^{\gamma_3}\dot{\gamma}_1(t)+e^{\gamma_3}\dot{\gamma}_2(t)\right)\right]^2+L(\omega(\dot{\gamma}(t)))^2}.\tag{99}$$

Similar to the proof of Lemma 6.1 in [4], we can prove

$$\begin{aligned}
\lim_{L\longrightarrow\infty}\frac{1}{\sqrt{L}}\int_\gamma\|\dot{\gamma}(t)\|_L dt &= \int_a^b\lim_{L\longrightarrow\infty}\frac{1}{\sqrt{L}}\|\dot{\gamma}(t)\|_L dt\\
&= \int_a^b\lim_{L\longrightarrow\infty}\frac{1}{\sqrt{L}}\sqrt{\left[\frac{1}{\lambda_1\lambda_2}\dot{\gamma}_3(t)\right]^2+\left[\frac{\lambda_1\sqrt{2}}{2}\left(-e^{\gamma_3}\dot{\gamma}_1(t)+e^{\gamma_3}\dot{\gamma}_2(t)\right)\right]^2+L(\omega(\dot{\gamma}(t)))^2}dt\\
&= \int_a^b|\omega(\dot{\gamma}(t))|dt=\int_a^b ds,
\end{aligned}\tag{100}$$

is desired. When $\omega(\dot{\gamma}(t))\neq 0$, we have

$$\frac{1}{\sqrt{L}}ds_L=\sqrt{L^{-1}\left\{\left[\frac{1}{\lambda_1\lambda_2}\dot{\gamma}_3(t)\right]^2+\left[\frac{\lambda_1\sqrt{2}}{2}\left(-e^{\gamma_3}\dot{\gamma}_1(t)+e^{\gamma_3}\dot{\gamma}_2(t)\right)\right]^2\right\}+\omega(\dot{\gamma}(t))^2}dt.\tag{101}$$

Using the Taylor expansion, we can prove

$$\frac{1}{\sqrt{L}}ds_L=ds+d\bar{s}L^{-1}+O\left(L^{-2}\right),\quad \text{as } L\longrightarrow+\infty.\tag{102}$$

From the definition of $ds_L$ and $\omega(\dot{\gamma}(t)) = 0$, we get

$$\frac{1}{\sqrt{L}} ds_L = \frac{1}{\sqrt{L}} \sqrt{\left[\frac{1}{\lambda_1\lambda_2}\dot{\gamma}_3(t)\right]^2 + \left[\frac{\lambda_1\sqrt{2}}{2}\left(-e^{\gamma_3}\dot{\gamma}_1(t) + e^{\gamma_3}\dot{\gamma}_2(t)\right)\right]^2} dt. \tag{103}$$

□ □

**Proposition 18.** *Let $\Sigma \subset (E(1,1), g_L(\lambda_1, \lambda_2))$ be a Euclidean $C^2$-smooth surface and $\Sigma = \{u = 0\}$ and $d\sigma_{\Sigma,L}$ denote the surface measure on $\Sigma$ with respect to the Rimannian metric $g_L(\lambda_1, \lambda_2)$. Let*

$$\begin{aligned} d\sigma_\Sigma &:= (\bar{p}\omega_2 - \bar{q}\omega_1) \wedge \omega, \\ d\bar{\sigma}_\Sigma &:= \frac{X_3 u}{l}\omega_1 \wedge \omega_2 - \frac{(X_3 u)^2}{2l^2}(\bar{p}\omega_2 - \bar{q}\omega_1) \wedge \omega. \end{aligned} \tag{104}$$

*Then, we have*

$$\frac{1}{\sqrt{L}} d\sigma_{\Sigma,L} = d\sigma_\Sigma + d\bar{\sigma}_\Sigma L^{-1} + O\left(L^{-2}\right), \quad \text{as } L \longrightarrow +\infty. \tag{105}$$

*If $\Sigma = f(D)$ with $f = f(u_1, u_2) = (f_1, f_2, f_3)$: $D \subset \mathbb{R}^2 \longrightarrow (E(1,1), g_L(\lambda_1, \lambda_2))$, then*

$$\lim_{L \longrightarrow \infty} \frac{1}{\sqrt{L}} \int_\Sigma d\sigma_{\Sigma,L} = \int_D \left(\sigma_1^2 + \sigma_2^2\right)^{1/2} du_1 du_2, \tag{106}$$

*where*

$$\sigma_1 = (f_1)_{u_1}(f_2)_{u_2}\lambda_1\lambda_2 - (f_2)_{u_1}(f_1)_{u_2}\lambda_1\lambda_2,$$

$$\begin{aligned} \sigma_2 = \Bigg\{ &\left[(f_3)_{u_2}(f_1)_{u_1} - (f_3)_{u_1}(f_1)_{u_2}\right]\frac{\sqrt{2}}{2}e^{-z}\frac{1}{\lambda_1} + \Big[(f_3)_{u_2}(f_2)_{u_1} \\ &- (f_3)_{u_1}(f_2)_{u_2}\Big]\frac{\sqrt{2}}{2}e^{z}\frac{1}{\lambda_1}\Bigg\}^2. \end{aligned} \tag{107}$$

*Proof.* It is well known that $g_L(X_1, \cdot) = \omega_1$, $g_L(X_2, \cdot) = \omega_2$, and $g_L(X_3, \cdot) = L\omega$. We define $e_1{}^* := g_L(e_1, \cdot)$ and $e_2{}^* := g_L(e_2, \cdot)$; then, we have

$$\begin{aligned} e_1^* &= \bar{q}\omega_1 - \bar{p}\omega_2, \\ e_2^* &= \bar{r}_L\bar{p}\omega_1 + \bar{r}_L\bar{q}\omega_2 - \frac{l}{l_L}L^{1/2}\omega. \end{aligned} \tag{108}$$

Therefore, we have

$$\frac{1}{\sqrt{L}} d\sigma_{\Sigma,L} = \frac{1}{\sqrt{L}} e_1^* \wedge e_2^* = \frac{l}{l_L}(\bar{p}\omega_2 - \bar{q}\omega_1) \wedge \omega + \frac{1}{\sqrt{L}}\bar{r}_L\omega_1 \wedge \omega_2. \tag{109}$$

Recalling, we know that

$$\bar{r}_L = \frac{(X_3 u)L^{-1/2}}{\sqrt{p^2 + q^2 + L^{-1}(X_3 u)^2}}, \tag{110}$$

and with the Taylor expansion, we have

$$\frac{1}{l_L} = \frac{1}{l} - \frac{1}{2l^3}(X_3 u)L^{-1} + O\left(L^{-2}\right), \quad \text{as } L \longrightarrow +\infty, \tag{111}$$

and we get (104). By (4), we have

$$\begin{aligned} f_{u_1} &= (f_1)_{u_1}\partial_x + (f_2)_{u_1}\partial_y + (f_3)_{u_1}\partial_z \\ &= (f_3)_{u_1}\frac{1}{\lambda_1\lambda_2}X_1 + \left((f_1)_{u_1}\left(-\frac{\sqrt{2}}{2}e^{-z}\lambda_1\right) + (f_2)_{u_1}\frac{\sqrt{2}}{2}e^{z}\lambda_1\right)X_2 \\ &\quad + \sqrt{L}\left((f_1)_{u_1}\left(-\frac{\sqrt{2}}{2}e^{-z}\lambda_2\right) + (f_2)_{u_1}\left(-\frac{\sqrt{2}}{2}e^{z}\lambda_2\right)\right)\widetilde{X_3} \\ &= f_{u_1 1}X_1 + f_{u_1 2}X_2 + f_{u_1 3}\widetilde{X_3}, \\ f_{u_2} &= (f_1)_{u_2}\partial_x + (f_2)_{u_2}\partial_y + (f_3)_{u_2}\partial_z \\ &= (f_3)_{u_2}\frac{1}{\lambda_1\lambda_2}X_1 + \left((f_1)_{u_2}\left(-\frac{\sqrt{2}}{2}e^{-z}\lambda_1\right) + (f_2)_{u_2}\frac{\sqrt{2}}{2}e^{z}\lambda_1\right)X_2 \\ &\quad + \sqrt{L}\left((f_1)_{u_2}\left(-\frac{\sqrt{2}}{2}e^{-z}\lambda_2\right) + (f_2)_{u_2}\left(-\frac{\sqrt{2}}{2}e^{z}\lambda_2\right)\right)\widetilde{X_3} \\ &= f_{u_2 1}X_1 + f_{u_2 2}X_2 + f_{u_2 3}\widetilde{X_3}. \end{aligned} \tag{112}$$

Let

$$\begin{aligned} \bar{v}_L &= \begin{vmatrix} X_1 & X_2 & \widetilde{X_3} \\ f_{u_1 1} & f_{u_1 2} & f_{u_1 3} \\ f_{u_2 1} & f_{u_2 2} & f_{u_2 3} \end{vmatrix} \\ &= \sqrt{L}\left((f_1)_{u_1}(f_2)_{u_2}\lambda_1\lambda_2 - (f_2)_{u_1}(f_1)_{u_2}\lambda_1\lambda_2\right)X_1 \\ &\quad + \sqrt{L}\Bigg[\left((f_3)_{u_2}(f_1)_{u_1} - (f_3)_{u_1}(f_1)_{u_2}\right)\frac{\sqrt{2}}{2}e^{-z}\frac{1}{\lambda_1} \\ &\quad + \left((f_3)_{u_2}(f_2)_{u_1} - (f_3)_{u_1}(f_2)_{u_2}\right)\frac{\sqrt{2}}{2}e^{z}\frac{1}{\lambda_1}\Bigg]X_2 \\ &\quad + \Bigg[\left((f_3)_{u_2}(f_1)_{u_1} - (f_3)_{u_1}(f_1)_{u_2}\right)\frac{\sqrt{2}}{2}e^{-z}\frac{1}{\lambda_2} \\ &\quad + \left((f_3)_{u_1}(f_2)_{u_2} - (f_3)_{u_2}(f_2)_{u_1}\right)\frac{\sqrt{2}}{2}e^{z}\frac{1}{\lambda_2}\Bigg]\widetilde{X_3}. \end{aligned} \tag{113}$$

We know that $d\sigma_{\Sigma,L} = \sqrt{\det(g_{ij})}du_1 du_2$, $g_{ij} = g_L(f_{u_i},$

$f_{u_j}$), and

$$\det\left(g_{ij}\right) = \|\bar{v}_L\|_L^2 = L\left((f_1)_{u_1}(f_2)_{u_2}\lambda_1\lambda_2 - (f_2)_{u_1}(f_1)_{u_2}\lambda_1\lambda_2\right)^2 + L\left[\left((f_3)_{u_2}(f_1)_{u_1} - (f_3)_{u_1}(f_1)_{u_2}\right)\frac{\sqrt{2}}{2}e^{-z}\frac{1}{\lambda_1} + \left((f_3)_{u_2}(f_2)_{u_1} - (f_3)_{u_1}(f_2)_{u_2}\right)\frac{\sqrt{2}}{2}e^{z}\frac{1}{\lambda_1}\right]^2 + \left[\left((f_3)_{u_2}(f_1)_{u_1} - (f_3)_{u_1}(f_1)_{u_2}\right)\frac{\sqrt{2}}{2}e^{-z}\frac{1}{\lambda_2} + \left((f_3)_{u_1}(f_2)_{u_2} - (f_3)_{u_2}(f_2)_{u_1}\right)\frac{\sqrt{2}}{2}e^{z}\frac{1}{\lambda_2}\right]^2. \tag{114}$$

So by the dominated convergence theorem, we get (105). □ □

Similar to the proof of Theorem 4.3 in [6], we get a generalized Gauss-Bonnet theorem in $(E(1,1), g_L(\lambda_1,\lambda_2))$ as shown in the following theorem.

**Theorem 19.** *Let $\Sigma \subset (E(1,1), g_L(\lambda_1,\lambda_2))$ be a regular surface with finitely many boundary components $(\partial\Sigma)_i$, $i \in \{1,\cdots,n\}$, given by Euclidean $C^2$-smooth regular and closed curves $\gamma_i : [0,2\pi] \longrightarrow (\partial\Sigma)_i$. Suppose that the characteristic set $C(\Sigma)$ satisfies $\mathscr{H}^1(C(\Sigma)) = 0$, where $\mathscr{H}^1(C(\Sigma))$ denotes the Euclidean 1-dimensional Hausdorff measure of $C(\Sigma)$ and that $\|\nabla_H u\|_H^{-1}$ is locally summable with respect to the Euclidean 2-dimensional Hausdorff measure near the characteristic set $C(\Sigma)$. Then, we have*

$$\int_\Sigma A d\sigma_\Sigma + \sum_{i=1}^n \int_{\gamma_i} \kappa_{\gamma_i,\Sigma}^{\infty,s} ds = 0, \tag{115}$$

*where $A = -\langle e_1, \nabla_H(X_3 u/|\nabla_H u|)\rangle_L - \lambda_2^4((X_3 u)^2/l^2)$.*

*Proof.* Using similar discussions in [4, 5], we assume that all points satisfy $\omega(\dot{\gamma}_i(t)) \neq 0$ on the curve $\gamma_i$. Recalling the result in Proposition 12 indicates

$$\kappa_{\gamma_i,\Sigma}^{L,s} = \kappa_{\gamma_i,\Sigma}^{\infty,s} + O\left(L^{-1/2}\right). \tag{116}$$

According to the Gauss-Bonnet theorem, we get

$$\int_\Sigma \mathscr{K}^{\Sigma,L}\frac{1}{\sqrt{L}}d\sigma_{\Sigma,L} + \sum_{i=1}^n \int_{\gamma_i} \kappa_{\gamma_i,\Sigma}^{L,s}\frac{1}{\sqrt{L}}ds_L = 2\pi\frac{\chi(\Sigma)}{\sqrt{L}}. \tag{117}$$

So by (115), (116), (104), (89), (96), and (97), we get

$$\left(\int_\Sigma A d\sigma_\Sigma + \sum_{i=1}^n \int_{\gamma_i} \kappa_{\gamma_i,\Sigma}^{\infty,s} ds_L\right) + O\left(L^{-1/2}\right) = 2\pi\frac{\chi(\Sigma)}{\sqrt{L}}, \tag{118}$$

where $A = -|e_1, \nabla_H(X_3 u/|\nabla_H u|)|_L - \lambda_2^4((X_3 u)^2/l^2)$. Let $L$ go to the infinity, and use the dominated convergence theorem, we get the desired result. □ □

## Data Availability

No data were used to support this study.

## Conflicts of Interest

The authors declare that they have no conflicts of interests.

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