Research Article

Random Power Series in $Q_{p,0}$ Spaces

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Aulaskari et al. proved if $0 < p < 1$ and $\{\varepsilon_n\}$ is sequence of independent, identically distributed Rademacher random variables on a probability space, then the condition $\sum_{n=0}^{\infty} n^{1-p} |a_n|^2 < \infty$ implies that the random power series $(\mathcal{R}f)(z) = \sum_{n=0}^{\infty} a_n \varepsilon_n z^n \in Q_p$ almost surely. In this paper, we improve this result showing that the condition $\sum_{n=0}^{\infty} n^{1-p} |a_n|^2 < \infty$ actually implies $\mathcal{R}f \in Q_{p,0}$ almost surely.

1. Introduction

Let $\mathbb{D} = \{z : |z| < 1\}$ be the unit disk in the extended complex plane $\widehat{\mathbb{C}}$, $S^1 = \partial\mathbb{D}$ and let $H(\mathbb{D})$ be the space of all analytic functions on $\mathbb{D}$.

If $0 < r < 1$ and $f \in H(\mathbb{D})$, we set

$$M_p(r, f) = \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta \right)^{1/p}, \quad 0 < p < \infty, \quad (1)$$

and $M_{\infty}(r, f) = \sup_{|z|=r} |f(z)|$. A function $f \in H(\mathbb{D})$ is said to belong to the Hardy space H^p , if

$$\|f\|_{H^p} = \sup_{0 < r < 1} M_p(r, f) < \infty. \quad (2)$$

For $a \in \mathbb{D}$, let

$$\sigma_a(z) = \frac{a-z}{1-\bar{a}z}, \quad z \in \mathbb{D}, \quad (3)$$

then σ_a is a Möbius transformation of $\mathbb{D}$ which interchanges a and 0. For $-1 < p < \infty$, it is said that f belongs to the Q_p space if $f \in H(\mathbb{D})$ and

$$\|f\|_{Q_p}^2 = \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |f'(z)|^2 (1 - |\sigma_a(z)|^2)^p dA(z) < \infty. \quad (4)$$

Furthermore, we say that f belongs to the little Q_p space, denoted by $Q_{p,0}$, if $f \in Q_p$ and

$$\lim_{|a| \rightarrow 1^-} \int_{\mathbb{D}} |f'(z)|^2 (1 - |\sigma_a(z)|^2)^p dA(z) = 0. \quad (5)$$

Q_p spaces were introduced by Aulaskari et al. in [1] and have been fully studied in recent years. It is known that Q_0 is the classical Dirichlet space, Q_1 is BMOA, and $Q_{1,0}$ is VMOA (see [2]). It is proved in [3] that Q_p is the Bloch space if $1 < p < \infty$ and in [4] that Q_p contains only constants if $-1 < p < 0$. The proper inclusions

$$H^\infty \subset \text{BMOA} \subset \bigcap_{p < \infty} H^p, \quad (6)$$

and $Q_p \subset \text{BMOA}$ ($0 < p < 1$) are well known. We refer to [1, 5–7] for more information on Q_p spaces.

Recall that an analytic function $f(z) = \sum_{k=0}^{\infty} a_k z^{n_k}$ is called a lacunary series if

$$\lambda = \inf_k \frac{n_{k+1}}{n_k} > 1. \quad (7)$$

For lacunary series, the following equivalent relations ([1]) hold for Q_p , ($0 < p < 1$):

$$\sum_{k=0}^{\infty} n_k^{1-p} |a_k|^2 < \infty \iff f \in Q_p \iff f \in Q_{p,0}, \quad (8)$$

and for BMOA:

$$\sum_{k=0}^{\infty} |a_k|^2 < \infty \iff f \in \text{BMOA} \iff f \in \text{VMOA}. \quad (9)$$

Let $\{\varepsilon_n\}_{n=0}^{\infty}$ be a sequence of independent, identically distributed (i.i.d.) Rademacher random variables on a probability space $(\Omega, \mathcal{F}, P)$; that is, $P(\varepsilon_n = 1) = P(\varepsilon_n = -1) = 1/2$ for all $n \geq 0$. For each $f(z) = \sum_{n=0}^{\infty} a_n z^n \in H(\mathbb{D})$, we define the random power series:

$$(\mathcal{R}f)(z) = \sum_{n=0}^{\infty} a_n \varepsilon_n z^n. \quad (10)$$

Usually, random power series are similar to lacunary series. For example, they are very well behaved if the coefficients are square-summable and very badly behaved if not.

Aulaskari et al. [8] have shown that for $0 < p < 1$, $\sum_{n=0}^{\infty} n^{1-p} |a_n|^2 < \infty$ implies $\mathcal{R}f \in Q_p$ a.s., a result which is similar to the one for lacunary series. However, the condition $\sum_{n=0}^{\infty} |a_n|^2 < \infty$ implies $f \in \bigcap_{p < \infty} H^p$ a.s. [9], while there exists a sequence $\{a_n\}$ with $\sum_{n=0}^{\infty} |a_n|^2 < \infty$ and yet $\mathcal{R}f \notin \text{BMOA}$ for any choice of $\{\varepsilon_n\}$ ([10]). Sledd [11] and Duren [12] have shown that the condition $\sum_{n=0}^{\infty} |a_n|^2 \log n < \infty$ implies that $\mathcal{R}f \in \text{BMOA}$ a.s. Furthermore, this condition actually gives the stronger conclusion that $\mathcal{R}f \in \text{VMOA}$ a.s.

In this paper, we further discuss the random power series in $Q_{p,0}$ space for $0 < p < 1$. We get the following:

Theorem 1. Suppose that $0 < p < 1$ and let $f(z) = \sum_{n=0}^{\infty} a_n z^n \in H(\mathbb{D})$. If $\sum_{n=0}^{\infty} n^{1-p} |a_n|^2 < \infty$ for $0 < p < 1$, then

$$(\mathcal{R}f)(z) = \sum_{n=0}^{\infty} a_n \varepsilon_n z^n \in Q_{p,0} \text{ a.s.} \quad (11)$$

In what follows, C always denotes some positive constant which may differ at different situations.

2. Preliminaries and Basic Properties

The sequence of Rademacher functions $\{\varphi_n\}_{n=0}^{\infty}$ is defined as follows

$$\varphi_1(t) = \begin{cases} 1, & 0 < t < \frac{1}{2}, \\ -1, & \frac{1}{2} < t < 1, \\ 0, & t = 0, \frac{1}{2}, 1. \end{cases} \quad (12)$$

and for $n \geq 2$,

$$\varphi_n(t) = \text{sgn} \{ \sin(2^n \pi t) \}, 0 \leq t \leq 1. \quad (13)$$

In general, $\varphi_n(t)$ vanishes at all multiples of 2^{-n} and takes the values ± 1 elsewhere. See [13, 14] for properties of these functions.

The following Khintchine's inequality (Appendix A of [13, 15]) for Rademacher functions will be used later.

Lemma 2 (Khintchine's inequality). Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of complex numbers with $\sum_{n=0}^{\infty} |a_n|^2 < \infty$, then the series $\sum_{n=0}^{\infty} a_n \varphi_n(t)$ converges a.s. Furthermore, let $0 < p < \infty$, then for all $N \geq 0$,

$$A_p \left(\sum_{n=0}^N |a_n|^2 \right)^{1/2} \leq \left(\mathbb{E} \left| \sum_{n=0}^N a_n \varphi_n(t) \right|^p \right)^{1/p} \leq B_p \left(\sum_{n=0}^N |a_n|^2 \right)^{1/2}, \quad (14)$$

for some positive constants A_p and B_p depend only on p .

For $\alpha \geq 1$ and $\beta \geq 1$, the following inequality holds for the beta function $B(\alpha, \beta)$.

Lemma 3 (beta function inequality). Let $\alpha \geq 1$ and $\beta \geq 1$. Then there exist two positive constants C_1 and C_2 depending only on β such that

$$C_1 \left(\frac{1}{\alpha} \right)^{\beta} \leq B(\alpha, \beta) = \int_0^1 t^{\alpha-1} (1-t)^{\beta-1} dt \leq C_2 \left(\frac{1}{\alpha} \right)^{\beta}. \quad (15)$$

Proof. By Stirling's formula, it is not difficult to get

$$\int_0^1 t^{\alpha-1} (1-t)^{\beta-1} dt = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \leq C_2 \left(\frac{1}{\alpha} \right)^{\beta}. \quad (16)$$

On the other hand, for $a \geq 1$,

$$\begin{aligned} \int_0^1 t^{\alpha-1} (1-t)^{\beta-1} dt &\geq \int_{1-1/a}^1 t^{\alpha-1} (1-t)^{\beta-1} dt \\ &\geq \left(1 - \frac{1}{a}\right)^{\alpha-1} \int_{1-1/a}^1 (1-t)^{\beta-1} dt \geq C_1 \left(\frac{1}{\alpha}\right)^\beta. \end{aligned} \quad (17)$$

The proof is complete. $\square$

An important tool to study function spaces is Carleson type measure. Let I be any arc on S^1 and $|I|$ be the normalized length of I . The Carleson sector $S(I)$, based on I , is defined by

$$S(I) = \left\{ z = re^{i\theta} : 1-|I| \leq r < 1, e^{i\theta} \in I \right\}. \quad (18)$$

For $0 < p < \infty$, a positive Borel measure μ on $\mathbb{D}$ is called a bounded p -Carleson measure if there exists a positive constant C such that

$$\mu(S(I)) \leq C|I|^p. \quad (19)$$

We say that μ is a vanishing p -Carleson measure if

$$\mu(S(I)) = o(|I|^p), \quad |I| \longrightarrow 0. \quad (20)$$

When $p = 1$, we get the standard definition of the original Carleson measure.

Let $f \in H(\mathbb{D})$. It is known that $f \in Q_p$ if and only if $d\mu(z) = (1-|z|)^p |f'(z)|^2 dx dy$ is a bounded p -Carleson measure and $f \in Q_{p,0}$ if and only if $d\mu(z) = (1-|z|)^p |f'(z)|^2 dx dy$ is a vanishing p -Carleson measure, respectively [5]. Notice that a bounded p -Carleson measure is a finite measure; hence, for $0 < p \leq 1$, we have

$$f \in Q_p \Rightarrow \int_0^1 (1-r)^p M_q^2(r, f') dr < \infty, \quad 0 < q \leq 2. \quad (21)$$

Conversely, for $f \in H(\mathbb{D})$, the condition $\int_0^1 (1-r)^p M_\infty^2(r, f') dr < \infty$ implies $f \in Q_p$ for $0 < p < 1$, and $\int_0^1 (1-r) M_\infty^2(r, f') dr < \infty$ implies $f \in \text{BMOA}$ (see [16]).

The following proposition gives the stronger conclusion that $f \in Q_{p,0}$ and $f \in \text{VMOA}$ when $p = 1$, respectively.

Proposition 4. Let $f(z) \in H(\mathbb{D})$ and $0 < p \leq 1$. If

$$\int_0^1 (1-r)^p M_\infty^2(r, f') dr < \infty, \quad (22)$$

then $f \in Q_{p,0}$.

Proof. For any $I \subset S^1$, we have

$$\int_I |f'(re^{i\theta})|^2 d\theta \leq M_\infty^2(r, f') |I|. \quad (23)$$

Let $d\mu(z) = (1-|z|)^p |f'(z)|^2 dx dy$,

$$\begin{aligned} \mu(S(I)) &\leq \int_{1-|I|}^1 \int_I (1-r)^p |f'(re^{i\theta})|^2 d\theta dr \\ &\leq |I| \int_{1-|I|}^1 (1-r)^p M_\infty^2(r, f') dr. \end{aligned} \quad (24)$$

The integral

$$\int_0^1 (1-r)^p M_\infty^2(r, f') dr < \infty, \quad (25)$$

implies that for any $\varepsilon > 0$, there exists a $\delta, 0 < \delta < 1$, such that

$$\int_{1-\delta}^1 (1-r)^p M_\infty^2(r, f') dr < \varepsilon. \quad (26)$$

Thus

$$\mu(S(I)) < |I| \varepsilon \leq |I|^p \varepsilon, \quad (27)$$

holds for any subarc I on S^1 with $|I| < \delta$. This shows that the measure $d\mu(z) = (1-|z|)^p |f'(z)|^2 dx dy$ is a vanishing p -Carleson measure, hence $f \in Q_{p,0}$. $\square$

Remark 5. Aulaskari et al. showed in p. 426 of their paper [16] that the exponent p in the left-hand side of (17) can not be increased.

Substituting $M_\infty(r, f')$ by $M_q(r, f')$ for some $2 < q < \infty$, we can get the following result.

Proposition 6. Let $0 < p < 1$, $2 < q < \infty$, and $1 - (2/q) \geq p$. For $f(z) \in H(\mathbb{D})$, if

$$\int_0^1 (1-r)^p M_q^2(r, f') dr < \infty, \quad (28)$$

then $f \in Q_{p,0}$.

Proof. For any $I \subset S^1$, by Hölder's inequality, we have

$$\begin{aligned} \int_I |f'(re^{i\theta})|^2 d\theta &\leq \left(\int_I |f'(re^{i\theta})|^q d\theta \right)^{2/q} \left(\int_I d\theta \right)^{q-2/q} \\ &\leq M_q^2(r, f') |I|^{1-2/q}, \end{aligned} \quad (29)$$

for $2 < q < \infty$. Let $d\mu(z) = (1-|z|)^p |f'(z)|^2 dx dy$,

$$\begin{aligned} \mu(S(I)) &\leq \int_{1-|I|}^1 \int_I (1-r)^p |f'(re^{i\theta})|^2 d\theta dr \\ &\leq |I|^{1-2/q} \int_{1-|I|}^1 (1-r)^p M_q^2(r, f') dr. \end{aligned} \quad (30)$$

The integral

$$\int_0^1 (1-r)^p M_q^2(r, f') dr < \infty, \quad (31)$$

implies that for any $\varepsilon > 0$, there exists a $\delta, 0 < \delta < 1$, such that

$$\int_{1-\delta}^1 (1-r)^p M_q^2(r, f') dr < \varepsilon. \quad (32)$$

Thus, for $1 - (2/q) \geq p$, we have

$$\mu(S(I)) < |I|^{1-2/q} \varepsilon \leq |I|^p \varepsilon, \quad (33)$$

holds for any subarc I on S^1 with $|I| < \delta$. This shows that the measure $d\mu(z) = (1-|z|)^p |f'(z)|^2 dx dy$ is a vanishing p -Carleson measure, hence $f \in Q_{p,0}$. $\square$

Remark 7. It should be mentioned that the proof of Proposition 4 and 6 are similar to that of Theorem 3.2 and 3.3 in [16]. Furthermore, for $p = 1$ and $2 < q < \infty$, it is unknown whether $\int_0^1 (1-r)^p M_q^2(r, f') dr < \infty$ implies $f \in \text{VMOA}$.

3. Proof of Theorem 1

Theorem 8. Suppose that $f(z) = \sum_{n=0}^{\infty} a_n z^n \in H(\mathbb{D})$. If $\sum_{n=0}^{\infty} n^{1-p} |a_n|^2 < \infty$ for $0 < p < 1$, then

$$(\mathcal{R}f)(z) = \sum_{n=0}^{\infty} a_n \varepsilon_n z^n \in Q_{p,0} \text{ a.s.} \quad (34)$$

Proof. Assume that $\sum_{n=0}^{\infty} n^{1-p} |a_n|^2 < \infty$ for $0 < p < 1$. For every $2 < q < \infty$, by Fubini's theorem, Jensen's inequality, and Khintchine's inequality, we have

$$\begin{aligned} &\mathbb{E} \left(\int_0^1 M_q^2(r, (\mathcal{R}f)') (1-r)^p dr \right) \\ &= \int_0^1 \mathbb{E} \left(\int_0^{2\pi} \left| \sum_{n=0}^{\infty} n a_n \varepsilon_n r^{(n-1)} e^{i(n-1)\theta} \right|^2 \frac{d\theta}{2\pi} \right)^{2/q} (1-r)^p dr \\ &\leq \int_0^1 \left(\mathbb{E} \int_0^{2\pi} \left| \sum_{n=0}^{\infty} n a_n \varepsilon_n r^{(n-1)} e^{i(n-1)\theta} \right|^2 \frac{d\theta}{2\pi} \right)^{2/q} (1-r)^p dr \\ &\leq B_p \int_0^1 \sum_{n=0}^{\infty} n^2 |a_n|^2 r^{2n-2} (1-r)^p dr \\ &= B_p \sum_{n=0}^{\infty} n^2 |a_n|^2 \int_0^1 r^{2n-2} (1-r)^p dr. \end{aligned} \quad (35)$$

From the beta function inequality of Lemma 3,

$$\int_0^1 r^{2n-2} (1-r)^p dr \leq C_2 \frac{1}{(2n-1)^{(1+p)}} \leq C_2 \frac{1}{n^{1+p}}, \quad (36)$$

for $n \geq 1$. So we get

$$\mathbb{E} \left(\int_0^1 M_q^2(r, (\mathcal{R}f)') (1-r)^p dr \right) \leq C \sum_{n=0}^{\infty} n^{1-p} |a_n|^2 < \infty, \quad (37)$$

which implies that

$$\int_0^1 (1-r)^p M_q^2(r, (\mathcal{R}f)') dr < \infty, \text{ a.s.} \quad (38)$$

Fix $q \geq (2/(1-p))$. By Proposition 6, we have $\mathcal{R}f \in Q_{p,0}$ a.s.

This theorem is proved. $\square$

Let $q \geq 1$ and $0 < \alpha \leq 1$, we say that $f \in H(\mathbb{D})$ belongs to the mean Lipschitz space $\Lambda(q, \alpha)$ (see [13]), if

$$M_q(r, f') = O\left(\frac{1}{(1-r)^{1-\alpha}}\right), \text{ as } r \rightarrow 1. \quad (39)$$

Furthermore, if

$$M_q(r, f') = o\left(\frac{1}{(1-r)^{1-\alpha}}\right), \text{ as } r \rightarrow 1, \quad (40)$$

we say that $f \in \lambda(q, \alpha)$.

Since $M_q(r, f')$ is an increasing function of r , it is easy to see that the condition

$$\int_0^1 M_q^2(r, (\mathcal{R}f)') (1-r)^p dr < \infty, \quad (41)$$

implies that

$$M_q\left(r, (\mathcal{R}f)'\right) = o\left(\frac{1}{(1-r)^{(p+1)/2}}\right), \text{ as } r \longrightarrow 1. \quad (42)$$

By the condition $1 - (2/q) \geq p$ of Proposition 6, we get

$$M_q\left(r, (\mathcal{R}f)'\right) = o\left(\frac{1}{(1-r)^{1-(1/q)}}\right), \text{ as } r \longrightarrow 1. \quad (43)$$

So we get the following result.

Theorem 9. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n \in H(\mathbb{D})$. If $\sum_{n=0}^{\infty} n^{1-p} |a_n|^2 < \infty$ for $0 < p < 1$, then

$$(\mathcal{R}f)(z) = \sum_{n=0}^{\infty} a_n \varepsilon_n z^n \in \Lambda\left(q, \frac{1}{q}\right) \text{ a.s.} \quad (44)$$

where $q > 2$ and $1 - (2/q) \geq p$. Furthermore, $(\mathcal{R}f)(z) \in \lambda(q, 1/q)$ a.s.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there is no conflict of interests regarding the publication of this article.

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