Research Article

Numerical Analysis of the Fractional-Order Nonlinear System of Volterra Integro-Differential Equations

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This paper presents the nonlinear systems of Volterra-type fractional integro-differential equation solutions through a Chebyshev pseudospectral method. The proposed method is based on the Caputo fractional derivative. The results that we get show the accuracy and reliability of the present method. Different nonlinear systems have been solved; the solutions that we get are compared with other methods and the exact solution. Also, from the presented figures, it is easy to conclude that the CPM error converges quickly as compared to other methods. Comparing the exact solution and other techniques reveals that the Chebyshev pseudospectral method has a higher degree of accuracy and converges quickly towards the exact solution. Moreover, it is easy to implement the suggested method for solving fractional-order linear and nonlinear physical problems related to science and engineering.

1. Introduction

Fractional calculus has a long history as classical calculus. The concept of fractional calculus arose when Leibnitz used a proper representation $d^n f/dx^n$ for the n th derivative in his publications. L'hôpital raises a question on the particular notation on what happens if " n " is a noninteger. It was the beginning of fractional calculus [1]. Recently, mathematicians focused on fractional calculus due to its numerous applications in every field of science: viscoelastic materials [2], economics [3], continuum and statistical mechanics [4], dynamics of interfaces between soft nanoparticles and rough substrates [5], solid mechanics [6], and much more [7–14].

Mathematical formulations solve many problems of nature with the help of converting the physical phenomena to the equation form. Differential equations (DEs) are among those that play the main role in modeling various phenomena. However, some problems are complex and cannot be handled with the help of a differential equation. In this regard, the researchers utilized fractional differential equations (FDEs) that model the phenomenon more accurately than differential equations having order integers. Nowadays, FDEs got the importance of real-world modeling problems: such as electrode-electrolyte polarization [15], electrochemistry of corrosion [16], circuit systems [17], optics and signal processing [18], heat conduction [19], diffusion wave [20], control theory of dynamical systems [21],

fluid flow [22], probability and statistics [23, 24], and so on (see [25–28]).

The role of fractional integral and integrodifferential equations is found in every field of engineering and science. When a physical phenomenon is modeled under the differential equation, it finally gives a differential equation, an integral equation, or an integrodifferential equation. Some applications of these types of equations are nanohydrodynamics [29], glass-forming process [30], wind ripple in the desert [31], and drop-wise condensation [32]. The analytical solution of integral and integrodifferential equations does not exist in most cases. Even if it exists in certain cases, it is hard to find. Different numerical methods have been developed for finding an approximate solution of integral and integro-differential equations. The most common among these methods are the Chebyshev polynomials [33], Haar wavelet [34], triangular function method [35], collocation method [36], Legendre wavelet operational method [37], Taylor series expansion method [38], homotopy perturbation method [39], reproducing kernel Hilbert space method [40], Adomian decomposition method [41], Euler wavelet method [42], variational iteration method [43], spectral collocation method [44], least square method [45], homotopy analysis method [46], and differential transform method [47].

We apply Chebyshev pseudospectral method (CPM) to solve nonlinear Volterra integro-differential equation systems in the present work. CPM is a powerful technique for solving linear and nonlinear problems. The obtained results show the higher convergence rate of the present technique. The solution that we get shows that CPM has good agreement with the exact solution. Error analysis reveals the efficiency of the proposed technique that CPM has greater accuracy than other methods.

2. Definitions and Preliminary Concept

This unit shows the preliminary concept and some essential definitions taken from fractional calculus and used in our present research work.

2.1. Definition. The definition for fractional derivative by Caputo of order α is showed by the following mathematical expression [48]:

$$D^\alpha j(s) = \frac{1}{\Gamma(n-\alpha)} \int_0^s (s-t)^{n-\alpha-1} j^{(n)}(t) dt, \quad (1)$$

for $n-1 < \alpha \leq n$, $n \in \mathbb{N}$, $s > 0$, $j \in C_{-1}^m$.

2.2. Definition. The fractional derivatives by Jin-Hunan He are described as [48]

$$\frac{D^\alpha j(s)}{Ds^\alpha} = \Gamma(1+\alpha) \lim_{\Delta s=s_1-s_2 \rightarrow L} \frac{f(s_1)-f(s_2)}{(s_1-s_2)^\alpha}, \quad (2)$$

where Δs does not approach zero.

2.3. Definition. Xiao-Jun explains derivatives having fractional order as [48]

$$D_s^\alpha j(s_0) = j^\alpha(s_0) = \left. \frac{d^\alpha j(s)}{ds^\alpha} \right|_{s=s_0} = \lim_{s \rightarrow s_0} \frac{\Delta^\alpha(j(s) - j(s_0))}{(s-s_0)^\alpha}, \quad (3)$$

where

$$\Delta^\alpha(j(s) - j(s_0)) \cong \Gamma(1+\alpha) \Delta(j(s) - j(s_0)). \quad (4)$$

2.4. Definition. The integral operator by Riemann-Liouville for order α is [48]

$$I^\alpha j(s) = \frac{1}{\Gamma(\alpha)} \int_0^s (s-t)^{\alpha-1} j(t) dt. \quad (5)$$

The Caputo derivative operator and Riemann-Liouville integral operator have the following properties

$$\begin{aligned} D^\alpha I^\alpha j(s) &= j(s), \\ I^\alpha D^\alpha j(s) &= j(s) - \sum_{k=0}^{n-1} \frac{j^{(k)}(0^+)}{k!} s^k, \quad s \geq 0, \quad n-1 < \alpha < n. \end{aligned} \quad (6)$$

3. Chebyshev Pseudospectral Method (CPM)

The Chebyshev polynomials are defined in the $[-1, 1]$ interval and can be described by the following recurrence formula:

$$R_{n+1}(t) = 2uR_n(s) - R_{n-1}(s), \quad n = 1, 2, \dots, \quad (7)$$

where

$$\begin{aligned} R_0(s) &= 1, \\ R_1(s) &= s. \end{aligned} \quad (8)$$

To apply the Chebyshev polynomials in the $[0, 1]$ interval, we define the Chebyshev shifted polynomials $\hat{R}_n(s)$ which are defined in the manner of Chebyshev polynomials $R_n(s)$ by relation

$$\hat{R}_n(s) = R_n(2s-1). \quad (9)$$

And the recurrence formula is as follows:

$$\hat{R}_{n+1}(s) = 2(2s-1)\hat{R}_n(s) - \hat{R}_{n-1}(s), \quad n = 1, 2, \dots, \quad (10)$$

where

$$\begin{aligned} \hat{R}_0(s) &= 1, \\ \hat{R}_1(s) &= 2s-1. \end{aligned} \quad (11)$$

TABLE 1: Exact versus CPM solution of problem 1 at $m = 10$.

s	Exact $j(s)$	Exact $k(s)$	CPM solution $j(s)$	CPM solution $k(s)$
0	0.0000000000	1.0000000000	0.0000000000	1.0000000000
0.1	0.1001667500	1.0050041680	0.1001667500	1.0050041680
0.2	0.2013360025	1.0200667556	0.2013360025	1.0200667556
0.3	0.3045202934	1.0453385141	0.3045202934	1.0453385141
0.4	0.4107523258	1.0810723718	0.4107523258	1.0810723718
0.5	0.5210953054	1.1276259652	0.5210953054	1.1276259652
0.6	0.6366535821	1.1854652182	0.6366535821	1.1854652182
0.7	0.7585837018	1.2551690056	0.7585837018	1.2551690056
0.8	0.8881059821	1.3374349463	0.8881059821	1.3374349462
0.9	1.0265167257	1.4330863854	1.0265167253	1.4330863853
1.0	1.1752011936	1.5430806348	1.1752011918	1.5430806343

TABLE 2: Error comparison of CPM versus other methods of Section 4.1 at $m = 10$.

s	Error(j_{CPM})	Error(k_{CPM})	Error(j_{OTM})	Error(k_{OTM})
0	0.0000000000E + 00	0.0000000000E + 00	0.00E + 00	0.00E + 00
0.1	8.2692537238E - 17	1.4541999963E - 17	1.39E - 17	0.00E + 00
0.2	9.1822482293E - 16	1.6662883612E - 16	5.27E - 16	0.00E + 00
0.3	1.2443994903E - 15	2.2699378388E - 16	4.45E - 14	1.11E - 15
0.4	1.2607906760E - 15	2.3113111594E - 16	1.05E - 12	3.51E - 14
0.5	7.2897283903E - 16	1.2454060032E - 16	1.23E - 11	5.10E - 13
0.6	2.5175064758E - 13	5.5241090380E - 14	9.11E - 11	4.55E - 12
0.7	5.8189005235E - 12	1.3209327312E - 12	4.97E - 10	2.90E - 11
0.8	5.8725732247E - 11	1.3786485036E - 11	2.16E - 9	1.44E - 10
0.9	3.7796170567E - 10	9.1669608328E - 11	7.90E - 9	5.92E - 10
1.0	1.8157658168E - 09	4.5450903393E - 10	2.52E - 8	2.10E - 9

FIGURE 1: The solution graph of example 1. (a) Exact solution and (b) CPM solution.

FIGURE 2: The solution graph of example 1. (a) Exact solution and (b) CPM solution.

FIGURE 3: CPM and OTM error graph of Section 4.1.

FIGURE 5: The absolute error graph of Section 4.1 at a different fractional order.

FIGURE 4: CPM and OTM error graph of Section 4.1.

FIGURE 6: The absolute error graph of Section 4.1 at a different fractional order.

A function $j(s) \in L_2[0, 1]$, in terms of Chebyshev shifted polynomials described as

$$j(s) = \sum_{n=1}^{\infty} c_n \hat{R}_n(s). \quad (12)$$

TABLE 3: Exact and CPM solutions of Section 4.2 at $m = 3$.

s	Exact $j(s)$	Exact $k(s)$	CPM solution $j(s)$	CPM solution $k(s)$
0.05	0.047500000000000	-0.002375000000000	0.047500000050000	-0.002375000000000
0.15	0.127500000000000	-0.019125000000000	0.127500000100000	-0.019124999970000
0.25	0.187500000000000	-0.046875000000000	0.187500000100000	-0.046874999940000
0.35	0.227500000000000	-0.079625000000000	0.227500000100000	-0.079624999950000
0.45	0.247500000000000	-0.111375000000000	0.247500000000000	-0.111374999900000
0.55	0.247500000000000	-0.136125000000000	0.247500000000000	-0.136125000000000
0.65	0.227500000000000	-0.147875000000000	0.227500000000000	-0.147875000000000
0.75	0.187500000000000	-0.140625000000000	0.187500000000000	-0.140625000000000
0.85	0.127500000000000	-0.108375000000000	0.127500000000000	-0.108375000000000
0.95	0.047500000000000	-0.045125000000000	0.047499999999999	-0.045125000100000

TABLE 4: Error comparison of CPM versus other methods of Section 4.2.

s	Error(j_{CPM})	Error(k_{CPM})	Error(j_{SCM})	Error(k_{SCM})
0.05	5.0000000000E - 11	0.0000000000E + 00	8.17886E - 8	1.71222E - 7
0.15	1.0000000000E - 10	3.0000000000E - 11	4.14502E - 8	5.20012E - 8
0.25	1.0000000000E - 10	6.0000000000E - 11	3.00945E - 9	1.53079E - 7
0.35	1.0000000000E - 10	5.0000000000E - 11	5.66834E - 8	1.82626E - 7
0.45	0.0000000000E + 00	1.0000000000E - 10	3.81977E - 8	6.42170E - 7
0.55	0.0000000000E + 00	0.0000000000E + 00	3.16220E - 8	6.19236E - 7
0.65	0.0000000000E + 00	0.0000000000E + 00	6.05974E - 8	1.37882E - 7
0.75	0.0000000000E + 00	0.0000000000E + 00	9.63834E - 9	1.53242E - 7
0.85	0.0000000000E + 00	0.0000000000E + 00	4.55344E - 8	1.18939E - 8
0.95	5.0000000000E - 11	1.0000000000E - 10	8.32363E - 8	9.10621E - 8

The Chebyshev shifted polynomials first $(m + 1)$ terms are considered as

$$j_m(s) = \sum_{n=0}^m c_n \hat{R}_n(s),$$

$$\frac{d^\alpha}{ds^\alpha} \left(\sum_{n=0}^m c_n \hat{R}_n(s) \right) + \sum_{n=0}^m c_n \hat{R}_n(s) + \int_{u_0}^s \left(\sum_{n=0}^m c_n \hat{R}_n(s) \right) ds = g(s, j). \quad (13)$$

For finding the system of equations, we have

$$\frac{d^\alpha}{ds^\alpha} \left(\sum_{n=0}^m c_n \hat{R}_n(s_i) \right) + \sum_{n=0}^m c_n \hat{R}_n(s_i) + \int_{s_0}^s \left(\sum_{n=0}^m c_n \hat{R}_n(s_i) \right) du = g(s_i, j). \quad (14)$$

whereas

$$s_i = \frac{i - 0.5}{2^{k-1}M}. \quad (15)$$

I solved the resultant system using maple software, which provide CPM solution for the given problem.

4. Numerical Representation

4.1. *Problem.* Consider the nonlinear FIDE system having B.Cs $j(0) = 0$, $k(0) = 1$

$$D^\alpha j(s) + \frac{1}{2} \left(\frac{dk}{ds} \right)^2 - \int_0^s [(s-t)k(t) + k(t)j(t)] dt = 1,$$

$$D^\alpha k(s) + sj(s) - \int_0^s [(s-t)j(t) + k^2(t)] dt = 2s, \quad (16)$$

having $j(s) = \sinh(s)$, $k(s) = \cosh(s)$ as the exact solution at $\alpha = 1$.

The exact solution and numerical results obtained by means of the proposed method are shown in Table 1. The absolute error comparison of our method and those obtained from OTM is given in Table 2. The behavior of the exact solution and approximate solution (our method) of this example when $\alpha = 1$ is presented in Figures 1 and 2 whereas the error comparison of CPM and OTM can be observed in Figures 3 and 4. The graphical representation for different fractional order of α is seen in Figures 5 and 6 which confirm that the solution converge to the exact solution as the value of α converges from the fractional order to the integer order.

FIGURE 7: The solution graph of example 2. (a) Exact solution and (b) CPM solution.

FIGURE 8: The solution graph of example 2. (a) Exact solution and (b) CPM solution.

FIGURE 9: CPM and SCM error graph of Section 4.2.

FIGURE 10: CPM and SCM error graph of Section 4.2.

4.2. *Problem.* Consider the FIDE system with B.Cs $j(0) = j(1) = 0$, $k(0) = k(1) = 0$

$$D^\alpha j(s) + k^2(s) + \frac{s}{2} \frac{dk}{ds} - \int_0^s ((s-t)k(t) + j(t)k(t))dt = g_2(s),$$

$$D^\alpha k(s) + j^2(s) - \int_0^s ((s-t)j(t) - k^2(t) + j^2(t))dt = g_1(s),$$

(17)

TABLE 5: Exact versus CPM solution of Section 4.3 at $m = 10$.

s	Exact $j(s)$	Exact $k(s)$	CPM solution $j(s)$	CPM solution $k(s)$
0	1.0000000000	-1.0000000000	1.0000000000	-1.0000000000
0.1	1.2051709180	-1.0051709180	1.2051709180	-1.0051709180
0.2	1.4214027580	-1.0214027580	1.4214027580	-1.0214027580
0.3	1.6498588080	-1.0498588075	1.6498588080	-1.0498588075
0.4	1.8918246980	-1.0918246976	1.8918246980	-1.0918246976
0.5	2.1487212710	-1.1487212707	2.1487212710	-1.1487212707
0.6	2.4221188000	-1.2221188003	2.4221188000	-1.2221188003
0.7	2.7137527070	-1.3137527074	2.7137527070	-1.3137527074
0.8	3.0255409280	-1.4255409284	3.0255409280	-1.4255409284
0.9	3.3596031110	-1.5596031111	3.3596031111	-1.5596031109
1.0	3.7182818280	-1.7182818284	3.7182818284	-1.7182818275

TABLE 6: Error comparison of CPM versus other methods of Section 4.3 at $m = 10$.

s	Error(j_{CPM})	Error(k_{CPM})	Error(j_{OTM})	Error(k_{OTM})
0	0.0000000000E + 00	0.0000000000E + 00	0.00E + 00	0.00E + 00
0.1	8.1999412724E - 17	8.2624090006E - 17	2.22E - 16	0.00E + 00
0.2	3.8043301351E - 15	3.8490016915E - 15	4.44E - 16	6.66E - 16
0.3	1.9076326419E - 14	1.9470658817E - 14	4.55E - 14	4.55E - 14
0.4	3.8309289099E - 14	3.9767165526E - 14	1.09E - 12	1.09E - 12
0.5	5.6578756405E - 14	6.0124928716E - 14	1.28E - 11	1.28E - 11
0.6	3.2264909603E - 13	3.2822178282E - 13	9.57E - 11	9.57E - 11
0.7	6.8521655232E - 12	6.8380840814E - 12	5.26E - 10	5.26E - 10
0.8	7.2715580869E - 11	7.2515818424E - 11	2.30E - 9	2.30E - 9
0.9	4.8186768333E - 10	4.8069973992E - 10	8.49E - 9	8.49E - 9
1.0	2.3551879379E - 09	2.3502526547E - 9	2.73E - 8	2.73E - 8

FIGURE 11: The solution graph of example 3. (a) Exact solution and (b) CPM solution.

FIGURE 12: The solution graph of example 3. (a) Exact solution and (b) CPM solution.

FIGURE 13: CPM and OTM error graph of Section 4.3.

FIGURE 15: The absolute error graph of Section 4.3 at a different fractional order.

FIGURE 14: CPM and OTM error graph of Section 4.3.

FIGURE 16: The absolute error graph of Section 4.3 at a different fractional order.

with the exact solution ($j(s) = s - s^2$, $k(s) = s^3 - s^2$) at $\alpha = 2$, where

$$\begin{aligned} g_1(s) &= \frac{7}{6}s^6 - \frac{49}{20}s^5 + \frac{4}{3}s^4 + \frac{3}{2}s^3 - s^2 - 2, \\ g_2(s) &= \frac{s^7}{7} - \frac{s^6}{3} + \frac{19}{12}s^4 - \frac{5}{2}s^3 + s^2 + 6s - 2. \end{aligned} \quad (18)$$

In Table 3, we give the numerical values of the exact solution and CPM solution for $m = 3$. The absolute errors

obtained by the present method are compared with SCM in Table 4. We compare the actual and estimated solution in Figures 7 and 8 which tells us that both the solutions are quite close to each other. Also, Figures 9 and 10 display the error comparison of CPM and SCM which verify that our method is in good agreement with the exact solution.

4.3. Problem. Consider the nonlinear FIDE system having B.Cs $j(0) = 1$, $j'(0) = 2$, $k(0) = -1$, and $k'(0) = 0$

$$\begin{aligned} D^\alpha j(s) + \frac{1}{2} \left(\frac{dk}{ds} \right)^2 - \frac{1}{2} \int_0^s [j^2(t) + k^2(t)] dt &= 1 - \frac{1}{3} s^3, \\ D^\alpha k(s) + sj(s) - \frac{1}{4} \int_0^s [j^2(t) - k^2(t)] dt &= s^2 - 1, \end{aligned} \quad (19)$$

having $j(s) = s + e^s$, $k(s) = s - e^s$ as the exact solution.

To solve this example, we implement the method suggested in Section 4 for $\alpha = 2$ with $m = 10$. The exact solution and estimated solution by CPM are presented in Table 5. The absolute error of our method and those obtained from OTM are given in Table 6. In Figures 11 and 12, it is clear that the numerical solution of the proposed method is in good contact with the exact solution. In order to illustrate the effectiveness of CPM, the error comparison with OTM is shown in Figures 13 and 14. Also, in Figures 15 and 16, we can obtain that as $\alpha \rightarrow 2$ the estimated solution approach to the exact solutions.

5. Conclusion

In this work, we implemented the Chebyshev pseudospectral method for solving nonlinear fractional integral and integro-differential equation systems. The proposed technique reduces this type of systems to the solution of the system of linear and nonlinear algebraic equations. Special attention is given to study the convergence of the proposed method. The results that we get by implementing the suggested technique are in excellent agreement with the exact solution and show more accuracy than the solution obtained using other methods. Also, from the presented figures, it is easy to conclude that the CPM error converges quickly as compared to other methods. The computation work in this article is done using Maple.

Data Availability

The numerical data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

All authors jointly worked on the results, and they read and approved the final manuscript.

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