Research Article

Numerical Analysis of the Klein-Gordon Equations by Using the New Iteration Transform Method

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This paper presents an analysis based on a mixture of the Laplace transform and the new iteration method to obtain new approximate results of the fractional-order Klein-Gordon equations in the Caputo-Fabrizio sense. So, a general system to investigate the approximate results of the fractional-order Klein-Gordon equations is obtained. This technique's effectiveness is demonstrated by comparing the actual results of the fractional-order equations suggested with the results achieved.

1. Introduction

Fractional partial differential equations (FPDEs) are critical tools for analyzing and simulating numerous narrative models in physics and mathematical models, such as electrical circuits, fluid dynamics, damping, induction, mathematical biology, ad relaxation, (Klimek, 2005; Baleanu et al., 2009; Kilbas et al., 2010; Jumarie, 2009; Mainardi, 2010; Ortigueira, 2010). Fractional derivatives provide more precise representations of real-world problems than integer-order derivatives; they are regarded as an effective technique for describing such physical problems. The subject of fractional calculus is an important and valuable branch of mathematics that plays a critical and severe role in explaining complex dynamic behavior in a wide range of application areas, helps to understand the essence of the matter as well as simplify the control design without any lack of inherited behavior, and describes even more complex structures [1, 2].

The Klein-Gordon equations (KGEs) play an important role in physics, nonlinear optics, quantum field theory and solid state physics, plasma physics, kinematics, mathematical biology, and the recurrence of the initial state. The modeling of many phenomena, including the behavior of elementary particles and dislocation of crystals propagation, is the important applications of KGEs. To study solitons [3], examining nonlinear wave equations [4] and condensed matter physics equations gained the attention of scholars. In the previous few years, mathematicians have made many considerable efforts to find the solutions to these equations. There are many methods introduced to find the solution of these equations such as the radial basis functions [5], B-spline collocation method [5], auxiliary approach [6], and exponential-type potential, and there are some more methods mentioned in [7–11] for the solution of these equations. To solve the KGEs of a nonlinear type got tremendous attention of scholar, and a verity of methods were developed as mentioned in [12–14]. Some other methods are the stationary solution [15], the Homotopy perturbation technique [16], the tanh technique [17], the variation iteration technique [18], the traveling wave solutions, and so on.

In the recent paper, we are applying new iterative transform method to KGEs of both linear and nonlinear orders of the following form:

$$\frac{\partial^{\varrho}\mu}{\partial\tau^{\varrho}} - \frac{\partial^2\mu}{\partial\zeta^2} + \Psi = g(\zeta), 1 < \rho \le 2, \tag{1}$$

with the boundary conditions

$$\mu(\zeta, 0) = 0 \; \mu_\tau(\zeta, 0) = k(\zeta). \tag{2}$$

Daftardar-Gejji and Jafari developed a new iterative approach for solving nonlinear equations in 2006 [19, 20]. Jafari et al. first applied the Laplace transformation in the iterative technique. They proposed a new straightforward technique called the iterative Laplace transform method (ILTM) [21] to look for the numerical solution of the FPDE system. The iterative Laplace transform method was used to solve linear and nonlinear partial differential equations such as the time-fractional Fokker-Planck equation [22], Zakharov-Kuznetsov equation [23], and Fornberg-Whitham equation [24]. The Elzaki transform was used to modify the iterative technique, known as the new iterative transform method.

The new iterative transform method is implemented to investigate the fractional-order of the Klein-Gordon equations. The solution of the fractional-order problems and integral-order models is calculated applying the current techniques. The proposed approach is also helpful for dealing with other fractional-orders of linear and nonlinear PDEs.

2. Fractional Calculus

This section provides some fundamental concepts of fractional calculus.

Definition 1. The Liouville-Caputo operator (C) is given as [25]

$$D^\varrho_{\mathfrak{I}} u(\zeta, \mathfrak{I}) = \frac{1}{\Gamma(n-\beta)} \int_0^{\mathfrak{I}} (\mathfrak{I}-\theta)^{n-1} u^n(\zeta, \theta) d\theta, n-1 < \rho < n, \tag{3}$$

where $u^n(\zeta, \theta)$ is the derivative of integer nth order of $u(\zeta, \mathfrak{I})$, $n = 1, 2, \cdots \in N$ and $n - 1 < \varrho \le n$. If $0 < \varrho \le 1$; then, we defined the Laplace transformation for the Caputo fractional derivative as follows:

$$\mathscr{L}\left[D^\varrho_{\mathfrak{I}} u(\zeta, \mathfrak{I})\right](s) = s^\varrho \mathscr{L}[u(\zeta, \mathfrak{I})](s) - s^{\varrho-1}[u(\zeta, 0)]. \tag{4}$$

Definition 2. The Caputo-Fabrizio operator (CF) is define as given [25]:

$$D^\varrho_{\mathfrak{I}} u(\zeta, \mathfrak{I}) = \frac{(2-\varrho)M(\varrho)}{2(n-\varrho)} \int_0^{\mathfrak{I}} \exp\left(-\varrho \frac{(\mathfrak{I}-\theta)}{n-\varrho}\right) u^{(n)}(\zeta, \theta) d\theta, n < \varrho \le n+1. \tag{5}$$

$M(\varrho)$ is a normalization form, and $M(0) = M(1) = 1$. The exponential law is used as the nonsingular kernel in this fractional operator.

If $0 < \rho \le 1$, then we define the Caputo-Fabrizio of the Laplace transformation for the fractional derivative is given as

$$\mathscr{L}\left[D^\varrho_{\mathfrak{I}} u(\zeta, \mathfrak{I})\right](s) = \left(\frac{s\mathscr{L}[u(\zeta, \mathfrak{I})](s) - u(\zeta, 0)}{s + \varrho(1-s)}\right). \tag{6}$$

3. The Iterative Transform Method Basic Procedure

Consider a particular type of a FPDE.

$$D^\varrho_\tau v(\zeta, \tau) + Mv(\zeta, \tau) + Nv(\zeta, \tau) = h(\zeta, \tau), n \in N, n-1 < \varrho \le n, \tag{7}$$

where the functions of linear and nonlinear are M and N, respectively.

With the initial condition

$$v^k(\zeta, 0) = g_k(\zeta), k = 0, 1, 2 \cdots n-1, \tag{8}$$

implementing the Laplace transformation of Equation (7), we have

$$L[D^\varrho_\tau v(\zeta, \tau)] + L[Mv(\zeta, \tau) + Nv(\zeta, \tau)] = L[h(\zeta, \tau)]. \tag{9}$$

Applying the Laplace differentiation is given to

$$L[v(\zeta, \tau)] = \frac{1}{s} v(\zeta, 0) + \frac{s + \varrho(1-s)}{s^2} L[h(\zeta, \tau)] - \frac{s + \varrho(1-s)}{s^2} L[Mv(\zeta, \tau) + Nv(\zeta, \tau)], \tag{10}$$

using the inverse Laplace transformation of Equation (10) into

$$v(\zeta, \tau) = L^{-1}\left[\left(\frac{1}{s} v(\zeta, 0) + \frac{s + \varrho(1-s)}{s^2} L[h(\zeta, \tau)]\right)\right] - L^{-1}\left[\frac{s + \varrho(1-s)}{s^2} L[Mv(\zeta, \tau) + Nv(\zeta, \tau)]\right]. \tag{11}$$

As through the iterative technique, we have

$$v(\zeta, \tau) = \sum_{m=0}^{\infty} v_m(\zeta, \tau). \tag{12}$$

Further, the operator M is linear; therefore

$$M\left(\sum_{m=0}^{\infty} v_m(\zeta, \tau)\right) = \sum_{m=0}^{\infty} M[v_m(\zeta, \tau)], \tag{13}$$

and the operator N is nonlinear; we have the following

$$N\left(\sum_{m=0}^{\infty} v_m(\zeta, \tau)\right) = v_0(\zeta, \tau) + M\left(\sum_{k=0}^{m} v_k(\zeta, \tau)\right) - N\left(\sum_{k=0}^{m} v_k(\zeta, \tau)\right). \tag{14}$$

Putting Equations (12)–(14) in Equation (11), we obtain

$$\sum_{m=0}^{\infty} v_m(\zeta, \tau) = L^{-1}\left[\left(\frac{1}{s}v(\zeta, 0) + \frac{s + \varrho(1-s)}{s^2}L[h(\zeta, \tau)]\right)\right]$$
$$- L^{-1}\left[\frac{s + \varrho(1-s)}{s^2}E\left[M\left(\sum_{k=0}^{m} v_k(\zeta, \tau)\right) - N\left(\sum_{k=0}^{m} v_k(\zeta, \tau)\right)\right]\right].$$

(15)

The new iterative transform method is defined as

$$v_0(\zeta, \tau) = L^{-1}\left[\left(\frac{1}{s}v(\zeta, 0) + \frac{s + \varrho(1-s)}{s^2}L(g(\zeta, \tau))\right)\right],$$

$$v_1(\zeta, \tau) = -L^{-1}\left[\frac{s + \varrho(1-s)}{s^2}L[M[v_0(\zeta, \tau)] + N[v_0(\zeta, \tau)]]\right],$$

$$v_{m+1}(\zeta, \tau) = -L^{-1}\left[\frac{s + \varrho(1-s)}{s^2}L\left[-M\left(\sum_{k=0}^{m} v_k(\zeta, \tau)\right) - N\left(\sum_{k=0}^{m} v_k(\zeta, \tau)\right)\right]\right], m \geq 1.$$

(16)

Finally, Equations (7) and (8) provide the m-terms solution in a series form given as

$$v(\zeta, \tau) \cong v_0(\zeta, \tau) + v_1(\zeta, \tau) + v_2(\zeta, \tau) + \cdots + v_m(\zeta, \tau), m = 1, 2, \cdots.$$

(17)

4. Applications of the Proposed Method

4.1. Example. Consider the fractional-order Klein-Gordon equation [18]

$$\frac{^{CF}\partial^{\varrho+1}\mu(\zeta, \tau)}{\partial\tau^{\varrho+1}} - \frac{\partial^2\mu(\zeta, \tau)}{\partial\zeta^2} + \mu(\zeta, \tau) = 0 \quad 0 < \zeta \quad \tau < 0 \quad 0 < \varrho \leq 1,$$

(18)

with the initial conditions

$$\mu(\zeta, 0) = 0, \mu_\tau(\zeta, 0) = \zeta.$$

(19)

Applying the Laplace transform to Equation (18), we have

$$s^\varrho L[\mu(\zeta, \tau)] = \mu_{(0)}(\zeta, 0)s^{-1} + \mu_{(\tau)}(\zeta, 0)s^{-2} + L\left[\frac{\partial^2\mu(\zeta, \tau)}{\partial\zeta^2} - \mu(\zeta, \tau)\right],$$

(20)

$$L[\mu(\zeta, \tau)] = s^{-1}\mu(\zeta, 0) + s^{-2}\mu_{(\tau)}(\zeta, 0) + \frac{s + \varrho(1-s)}{s^2}L\left[\frac{\partial^2\mu(\zeta, \tau)}{\partial\zeta^2} - \mu(\zeta, \tau)\right].$$

(21)

Applying the inverse Laplace transform of Equation (21), we have

$$\mu(\zeta, \tau) = L^{-1}\left[s^{-1}\mu(\zeta, 0) + s^{-2}\mu_{(\tau)}(\zeta, 0)\right]$$
$$+ L^{-1}\left[\frac{s + \varrho(1-s)}{s^2}L\left(\frac{\partial^2\mu(\zeta, \tau)}{\partial\zeta^2} - \mu(\zeta, \tau)\right)\right].$$

(22)

Now, by using the suggested analytical method, we get

$$\mu_0(\zeta, \tau) = \zeta\tau,$$

$$\mu_1(\zeta, \tau) = L^{-1}\left[\frac{s + \rho(1-s)}{s^2}L\left(\frac{\partial^2\mu_0(\zeta, \tau)}{\partial\zeta^2} - \mu_0(\zeta, \tau)\right)\right]$$
$$= -\frac{\zeta\tau^2}{6}(\tau\rho + 3 - 3\rho),$$

$$\mu_2(\zeta, \tau) = L^{-1}\left[\frac{s + \varrho(1-s)}{s^2}L\left(\frac{\partial^2\mu_1(\zeta, \tau)}{\partial\zeta^2} - \mu_1(\zeta, \tau)\right)\right]$$
$$= \frac{\zeta\tau^3}{120}(10\tau\varrho - 10\tau\varrho^2 - 40\varrho + 20 + 20\varrho^2 + \tau^2\varrho^2),$$

$$\mu_3(\zeta, \tau) = L^{-1}\left[\frac{s + \varrho(1-s)}{s^2}L\left(\frac{\partial^2\mu_2(\zeta, \tau)}{\partial\zeta^2} - \mu_2(\zeta, \tau)\right)\right],$$

$$\mu_3(\zeta, \tau) = -\frac{\zeta\tau^4}{5040}(630\varrho^2 - 630\varrho + 210 - 210\varrho^3 + \tau^3\varrho^3$$
$$+ 21\tau^2\varrho^2 - 21\tau^2\varrho^3 - 252\tau\varrho^2 + 126\tau\varrho + 126\tau\varrho^3),$$

$$\mu_4(\zeta, \tau) = L^{-1}\left[\frac{s + \varrho(1-s)}{s^2}L\left(\frac{\partial^2\mu_3(\zeta, \tau)}{\partial\zeta^2} - \mu_3(\zeta, \tau)\right)\right],$$

$$\mu_4(\zeta, \tau) = \frac{\zeta\tau^5}{362880}(\tau\varrho + 6 - 6\varrho) * (\tau^3\varrho^3 - 30\tau^2\varrho^3 + 30\tau^2\varrho^2$$
$$+ 252\tau\varrho^3 - 504\tau\varrho^2 + 252\tau\varrho, -504\varrho^3$$
$$+ 1512\varrho^2 - 1512\varrho + 504),$$

$$\vdots$$

$$\mu_n + 1(\zeta, \tau) = L^{-1}\left[\frac{(s + \varrho(1-s))^n}{s^{2n+2}}L\left(\frac{\partial^2\mu_n(\zeta, \tau)}{\partial\zeta^2} - \mu_n(\zeta, \tau)\right)\right].$$

(23)

The series form result is

$$\mu(\zeta, \tau) = \mu_0(\zeta, \tau) + \mu_1(\zeta, \tau) + \mu_2(\zeta, \tau) + \mu_3(\zeta, \tau) + \cdots \mu_n(\zeta, \tau),$$

$$\mu(\zeta, \tau) = \zeta\tau - \frac{\zeta\tau^2}{6}(\tau\varrho + 3 - 3\varrho) + \frac{\zeta\tau^3}{120}(10\tau\varrho - 10\tau\varrho^2 - 40\varrho$$
$$+ 20 + 20\varrho^2 + \tau^2\varrho^2) - \frac{\zeta\tau^4}{5040}(630\varrho^2 - 630\varrho + 210$$
$$- 210\varrho^3 + \tau^3\varrho^3 + 21\tau^2\varrho^2 - 21\tau^2\varrho^3 - 252\tau\varrho^2$$
$$+ 126\tau\varrho + 126\tau\varrho^3) + \frac{\zeta\tau^5}{362880}(\tau\varrho + 6 - 6\varrho)$$
$$\cdot (\tau^3\varrho^3 - 30\tau^2\varrho^3 + 30\tau^2\varrho^2 + 252\tau\varrho^3 - 504\tau\varrho^2$$
$$+ 252\tau\varrho - 504\varrho^3 + 1512\varrho^2 - 1512\varrho + 504) - \cdots.$$

(24)

The problem has the exact solution at $\varrho = 1$:

$$\mu(\zeta, \tau) = \zeta \sin(\tau).$$

(25)

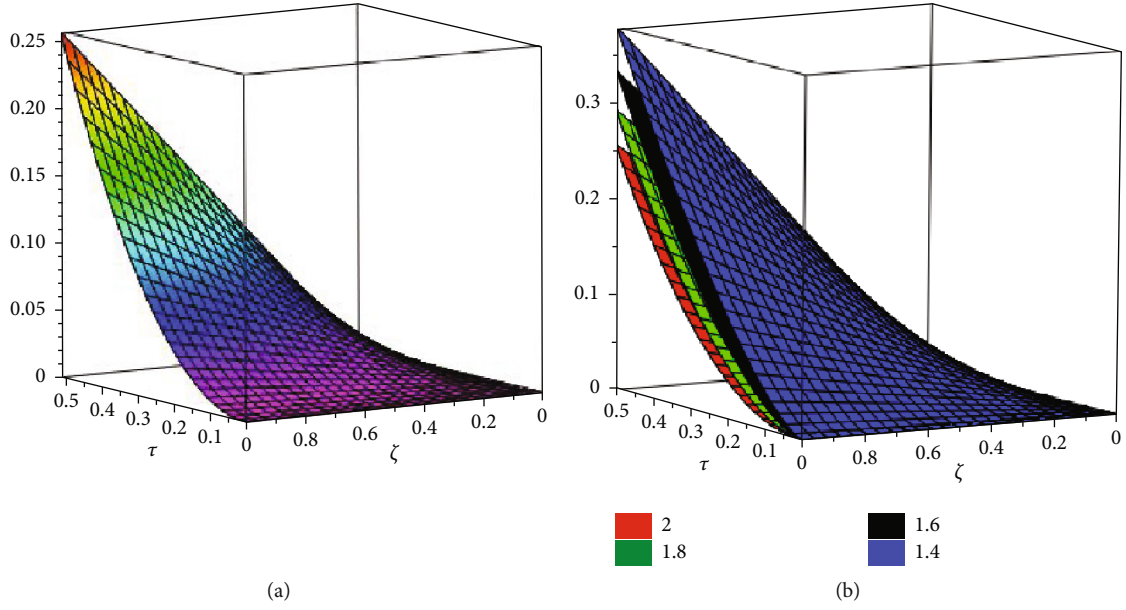In Figure 1, the exact and the approximate solutions of

Figure 1: (a) Exact and an approximate graph of problem 1 and (b) different fractional-order graphs of problem 1.

example 1 at $\varrho = 1$ are shown, and the second graph shows the 3D graph of different fractional-order ρ, respectively. From the given graphs, it can be shown that both the approximate and exact solutions are in close relation with each other. Also, in Figure 2, the 2D figure of the approximate solutions of problem 1 is analysis at different fractional-order ρ for ζ and τ. It is demonstrated that the outcomes of time-fractional problems converge to an integer-order effect as the time-fractional evaluation to integer-order.

4.2. Example. Consider the fractional-order Klein-Gordon equation [18]:

$$\frac{^{CF}\partial^{\varrho+1}\mu(\zeta,\tau)}{\partial\tau^{\varrho+1}} - \frac{\partial^2\mu(\zeta,\tau)}{\partial\zeta^2} + \mu(\zeta,\tau) = 2\sin(\zeta) \quad 0 < \zeta \quad \tau < 0 \quad 0 < \varrho \leq 1,$$

(26)

with the initial conditions

$$\mu(\zeta,0) = \sin(\zeta), \mu_\tau(\zeta,0) = 1. \tag{27}$$

We apply the Laplace transformation to Equation (26), and we get

$$\frac{s^2}{s+\varrho(1-s)}L[\mu(\zeta,\tau)] = \mu_{(0)}(\zeta,0)s^{-1} + \mu_{(\tau)}(\zeta,0)s^{-2}$$
$$+ L\left[\frac{\partial^2\mu(\zeta,\tau)}{\partial\zeta^2} - \mu(\zeta,\tau) + 2\sin(\zeta)\right],$$

(28)

$$L[\mu(\zeta,\tau)] = s^{-1}\mu(\zeta,0) + s^{-2}\mu_{(\tau)}(\zeta,0) + \frac{s+\varrho(1-s)}{s^2}L$$
$$\cdot \left[\frac{\partial^2\mu(\zeta,\tau)}{\partial\zeta^2} - \mu(\zeta,\tau) + 2\sin(\zeta)\right].$$

(29)

Now, using the inverse Laplace transformation of Equation (29), we have

$$\mu(\zeta,\tau) = L^{-1}\left[s^{-1}\mu(\zeta,0) + s^{-2}\mu_{(\tau)}(\zeta,0)\right]$$
$$+ L^{-1}\left[\frac{s+\varrho(1-s)}{s^2}L\left(\frac{\partial^2\mu(\zeta,\tau)}{\partial\zeta^2} - \mu(\zeta,\tau) + 2\sin(\zeta)\right)\right].$$

(30)

Now, by using the suggested analytical method, we get

$$\mu_0(\zeta,\tau) = \sin(\zeta) + \tau,$$

$$\mu_1(\zeta,\tau) = L^{-1}\left[\frac{s+\varrho(1-s)}{s^2}L\left(\frac{\partial^2\mu_0(\zeta,\tau)}{\partial\zeta^2} - \mu_0(\zeta,\tau) + 2\sin(\zeta)\right)\right]$$
$$= -\frac{\tau^2}{6}(\tau\varrho + 3 - 3\varrho),$$

$$\mu_2(\zeta,\tau) = L^{-1}\left[\frac{s+\varrho(1-s)}{s^2}L\left(\frac{\partial^2\mu_1(\zeta,\tau)}{\partial\zeta^2} - \mu_1(\zeta,\tau) + 2\sin(\zeta)\right)\right],$$

$$\mu_2(\zeta,\tau) = \frac{\tau^2}{120}(20\tau - 60\tau\varrho + 20\tau\varrho^2 + \tau^3\varrho^2 - 60$$
$$+ 60\varrho + 10\tau^2\varrho - 10\tau^2\varrho^2),$$

$$\mu_3(\zeta,\tau) = L^{-1}\left[\frac{s+\varrho(1-s)}{s^2}L\left(\frac{\partial^2\mu_2(\zeta,\tau)}{\partial\zeta^2} - \mu_2(\zeta,\tau) + 2\sin(\zeta)\right)\right],$$

$$\mu_3(\zeta,\tau) = -\frac{\tau^2}{5040}(4200\tau\varrho - 1680\tau - 1680\tau\varrho^2 - 336\tau^3\varrho^2$$
$$+ 126\varrho\tau^3 + 126\tau^3\varrho^3 + \varrho^3\tau^5 + 2520 - 2520\varrho$$
$$+ 1470\tau^2\varrho^2 - 1470\tau^2\varrho + 210\tau^2 - 210\tau^2\varrho^3$$
$$+ 21\varrho^2\tau^4 - 21\tau^4\varrho^3),$$

FIGURE 2: The different fractional-orders with respect to ζ and τ of problem 1.

$$\mu_4(\zeta, \tau) = L^{-1}\left[\frac{s + \varrho(1 - s)}{s^2} L\left(\frac{\partial^2 \mu_3(\zeta, \tau)}{\partial \zeta^2} - \mu_3(\zeta, \tau) + 2\sin(\zeta)\right)\right],$$

$$\begin{aligned}
\mu_4(\zeta, \tau) = \frac{\tau^2}{362880} \Big(&181440\tau\varrho^2 + 181440\tau - 423360\tau\varrho \\
&+ 81648\tau^3\varrho^2 - 39312\tau^3\varrho^3 - 39312\varrho\tau^3 + 3024\tau^3 \\
&+ 3024\tau^3\varrho^4 + 36\varrho^3\tau^6 - 36\varrho^4\tau^6 - 1080\varrho^3\tau^5 \\
&+ 432\varrho^2\tau^5 + 432\tau^5\varrho^4 + \tau^7\varrho^4 - 181440 + 11440\varrho \\
&+ 226800\tau^2\varrho + 45360\tau^2\varrho^3 - 226800\tau^2\varrho^2 \\
&- 45360\tau^2 - 10584\varrho^2\tau^4 + 10584\tau^4\varrho^3 \\
&- 2016\tau^4\varrho^4 + 2016\varrho\tau^4\Big),
\end{aligned}$$

$$\vdots$$

$$\mu_n(\zeta, \tau) = L^{-1}\left[\frac{(s + \varrho(1 - s))^n}{s^{2n+2}} L\left(\frac{\partial^2 \mu_n(\zeta, \tau)}{\partial \zeta^2} - \mu_n(\zeta, \tau) + 2\sin(\zeta)\right)\right]. \tag{31}$$

The series form result is

$$\mu(\zeta, \tau) = \mu_0(\zeta, \tau) + \mu_1(\zeta, \tau) + \mu_2(\zeta, \tau) + \mu_3(\zeta, \tau) + \cdots \mu_n(\zeta, \tau),$$

$$\begin{aligned}
\mu(\zeta, \tau) == \sin(\zeta) + \tau &- \frac{\tau^2}{6}(\tau\varrho + 3 - 3\varrho) \\
&+ \frac{\tau^2}{120}\big(20\tau - 60\tau\varrho + 20\tau\varrho^2 + \tau^3\varrho^2 - 60 \\
&+ 60\varrho + 10\tau^2\varrho - 10\tau^2\varrho^2\big) + - \frac{\tau^2}{5040} \\
&\cdot \big(4200\tau\varrho - 1680\tau - 1680\tau\varrho^2 - 336\tau^3\varrho^2 \\
&+ 126\varrho\tau^3 + 126\tau^3\varrho^3 + \varrho^3\tau^5 + 2520 - 2520\varrho \\
&+ 1470\tau^2\varrho^2 - 1470\tau^2\varrho + 210\tau^2 - 210\tau^2\varrho^3 \\
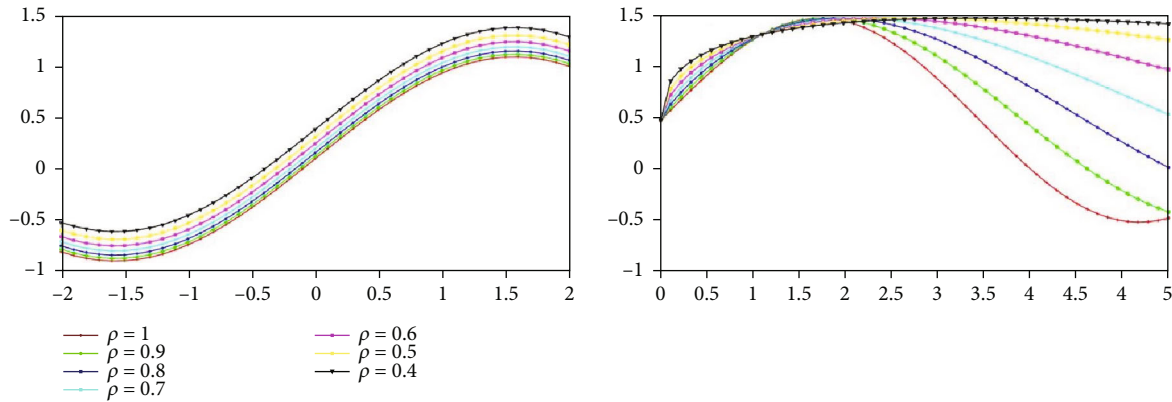&+ 21\varrho^2\tau^4 - 21\tau^4\varrho^3\big) + \frac{\tau^2}{362880} \\
&\cdot \big(181440\tau\varrho^2 + 181440\tau - 423360\tau\varrho + 81648\tau^3\varrho^2 \\
&- 39312\tau^3\varrho^3 - 39312\varrho\tau^3 + 3024\tau^3 + 3024\tau^3\varrho^4 \\
&+ 36\varrho^3\tau^6 - 36\varrho^4\tau^6 - 1080\varrho^3\tau^5 + 432\varrho^2\tau^5 + 432\tau^5\varrho^4 \\
&+ \tau^7\varrho^4 - 181440 + 11440\rho + 226800\tau^2\varrho + 45360\tau^2\varrho^3 \\
&- 226800\tau^2\varrho^2 - 45360\tau^2 - 10584\varrho^2\tau^4 + 10584\tau^4\varrho^3 \\
&- 2016\tau^4\varrho^4 + 2016\varrho\tau^4\big) + \cdots.
\end{aligned} \tag{32}$$

The problem has the exact solution at $\varrho = 1$:

$$\mu(\zeta, \tau) = \sin(\zeta) + \sin(\tau). \tag{33}$$

In Figure 3, the exact and the approximate solutions of example 2 at $\varrho = 1$ are shown. From the given figures, it can be seen that both the approximate and exact solutions are in close contact with each other. Also, in Figure 4, the 2D graph of the approximate results of problem 2 is investigated at different fractional-order ρ for ζ and τ. It is demonstrated that the outcomes of time-fractional problems converge to an integer-order effect as the time-fractional evaluation to integer-order.

4.3. Example. Consider the fractional-order nonlinear Klein-Gordon equation [18]:

$$\frac{{}^{CF}\partial^{\varrho+1}\mu(\zeta, \tau)}{\partial \tau^{\varrho+1}} - \frac{\partial^2 \mu(\zeta, \tau)}{\partial \zeta^2} + \mu^2(\zeta, \tau) = \zeta^2\tau^2 \quad 0 < \zeta \quad \tau < 0 \quad 0 < \varrho \leq 1, \tag{34}$$

with the initial conditions

$$\mu(\zeta, 0) = 0, \mu_\tau(\zeta, 0) = \zeta. \tag{35}$$

Using the Laplace transform to Equation (34), we get

$$\begin{aligned}
\frac{s^2}{s + \varrho(1 - s)} L[\mu(\zeta, \tau)] = &\mu_{(0)}(\zeta, 0)s^{-1} + \mu_{(\tau)}(\zeta, 0)s^{-2} \\
&+ L\left[\frac{\partial^2 \mu(\zeta, \tau)}{\partial \zeta^2} - \mu^2(\zeta, \tau) + \zeta^2\tau^2\right],
\end{aligned} \tag{36}$$

$$\begin{aligned}
L[\mu(\zeta, \tau)] = &s^{-1}\mu(\zeta, 0) + s^{-2}\mu_{(\tau)}(\zeta, 0) + \frac{s + \varrho(1 - s)}{s^2} L \\
&\cdot \left[\frac{\partial^2 \mu(\zeta, \tau)}{\partial \zeta^2} - \mu^2(\zeta, \tau) + \zeta^2\tau^2\right].
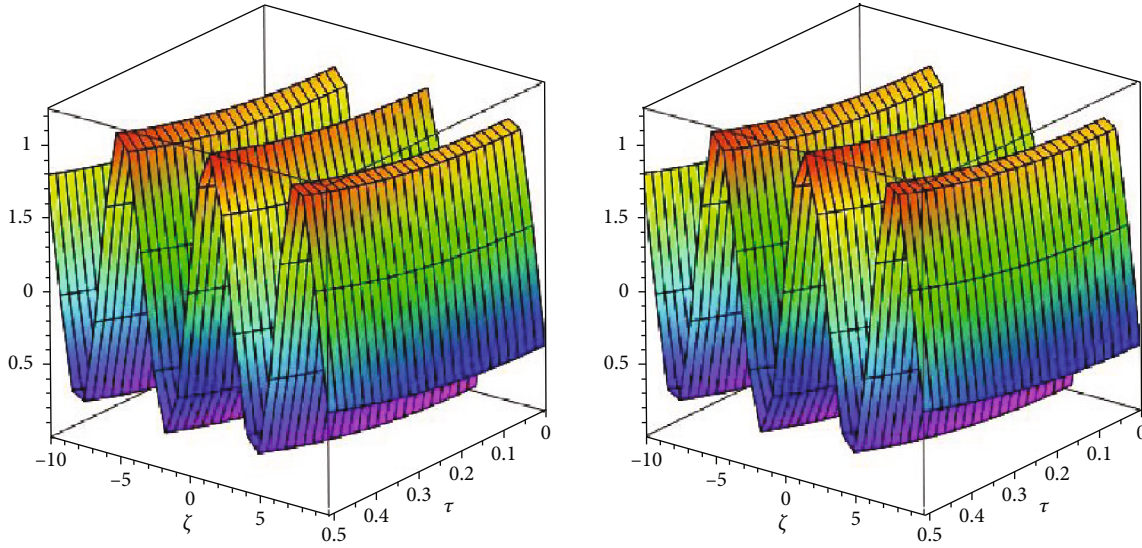\end{aligned} \tag{37}$$
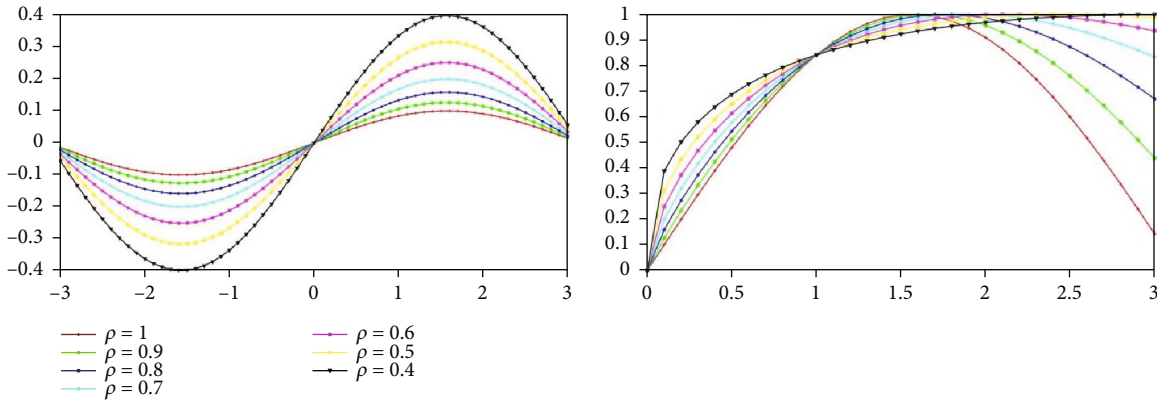
Figure 3: The exact and an approximate graph of problem 2.

$\rho = 1$
$\rho = 0.9$
$\rho = 0.8$
$\rho = 0.7$
$\rho = 0.6$
$\rho = 0.5$
$\rho = 0.4$

Figure 4: The different fractional-orders with respect to ζ and τ of problem 2.

Applying the inverse Laplace transform of Equation (37), we have

$$\mu(\zeta, \tau) = L^{-1} \left[s^{-1}\mu(\zeta, 0) + s^{-2}\mu_{(\tau)}(\zeta, 0) \right] + L^{-1}$$
$$\cdot \left[\frac{s + \varrho(1 - s)}{s^2} L \left(\frac{\partial^2 \mu^2(\zeta, \tau)}{\partial \zeta^2} - \mu^2(\zeta, \tau) + \zeta^2\tau^2 \right) \right].$$

(38)

Now, by using the suggested an approximate method, we get

$$\mu_0(\zeta, \tau) = \zeta\tau,$$
$$\mu_1(\zeta, \tau) = L^{-1} \left[\frac{s + \varrho(1 - s)}{s^2} L \left(\frac{\partial^2 \mu_0(\zeta, \tau)}{\partial \zeta^2} - \mu_0^2(\zeta, \tau) + \zeta^2\tau^2 \right) \right] = 0,$$
$$\vdots$$
$$\mu_n(\zeta, \tau) = L^{-1} \left[\frac{s + \varrho(1 - s)}{s^2} L \left(\frac{\partial^2 \mu_n(\zeta, \tau)}{\partial \zeta^2} - \mu_n^2(\zeta, \tau) + \zeta^2\tau^2 \right) \right].$$

(39)

The series form result is

$$\mu(\zeta, \tau) = \mu_0(\zeta, \tau) + \mu_1(\zeta, \tau) + \mu_2(\zeta, \tau) + \mu_3(\zeta, \tau) + \cdots \mu_n(\zeta, \tau),$$
$$\mu(\zeta, \tau) = \zeta\tau + 0 + \cdots.$$

(40)

The problem has the exact solution at $\varrho = 2$:

$$\mu(\zeta, \tau) = \zeta\tau.$$

(41)

Figure 5 compares the exact solution and approximate solution of example 3 for the nonlinear fractional-order Klein-Gordon equation at $\varrho = 1$. The figure shows the close relationship between the exact and an approximate solution.

4.4. Example. Consider the fractional-order nonlinear Klein-Gordon equation [18]:

$$\frac{^{CF}\partial^{\varrho+1}\mu(\zeta, \tau)}{\partial\tau^{\varrho+1}} - \frac{\partial^2\mu(\zeta, \tau)}{\partial\zeta^2} + \mu^2(\zeta, \tau) = 2\zeta^2 - 2\tau^2 + \zeta^4\tau^4 \quad 0 < \zeta \quad \tau < 0 \quad 0 < \varrho \le 1,$$
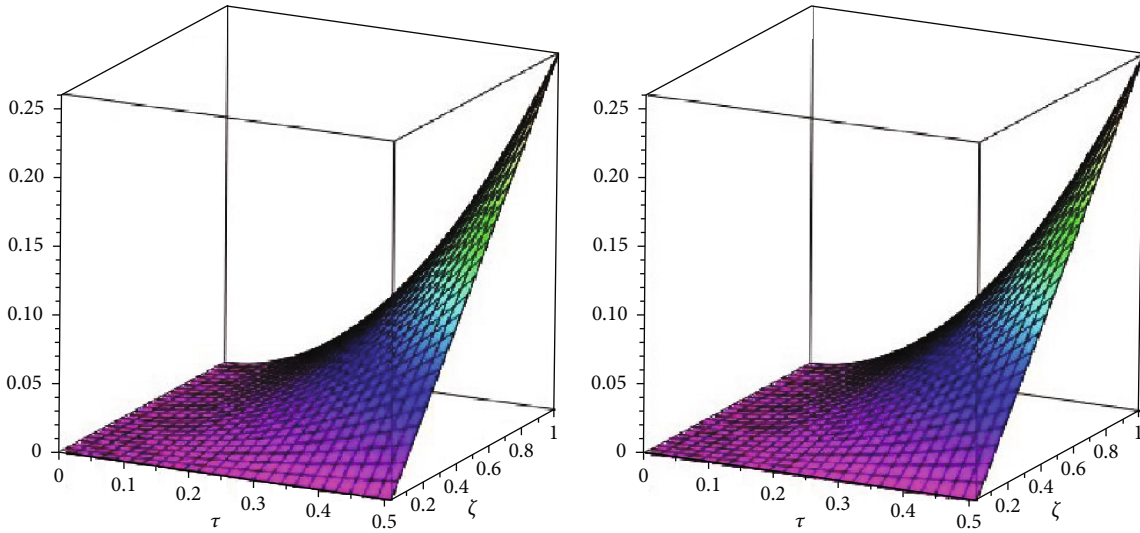
(42)

FIGURE 5: The exact and an approximate graph of problem 3.

with the initial conditions

$$\mu(\zeta, 0) = \mu_\tau(\zeta, 0) = 0. \tag{43}$$

Using the Laplace transform to Equation (42), we get

$$\frac{s^2}{s + \varrho(1-s)} L[\mu(\zeta, \tau)] = \mu_{(0)}(\zeta, 0)s^{-1} + \mu_{(\tau)}(\zeta, 0)s^{-2} + L\left[\frac{\partial^2 \mu(\zeta, \tau)}{\partial \zeta^2} - \mu^2(\zeta, \tau) + 2\zeta^2 - 2\tau^2 + \zeta^4\tau^4\right], \tag{44}$$

$$L[\mu(\zeta, \tau)] = s^{-1}\mu(\zeta, 0) + s^{-2}\mu_{(\tau)}(\zeta, 0) + \frac{s + \varrho(1-s)}{s^2} L \cdot \left[\frac{\partial^2 \mu(\zeta, \tau)}{\partial \zeta^2} - \mu^2(\zeta, \tau) + 2\zeta^2 - 2\tau^2 + \zeta^4\tau^4\right]. \tag{45}$$

Applying the inverse Laplace transform of Equation (45), we have

$$\mu(\zeta, \tau) = L^{-1}\left[s^{-1}\mu(\zeta, 0) + s^{-2}\mu_{(\tau)}(\zeta, 0)\right] + L^{-1} \cdot \left[\frac{s + \varrho(1-s)}{s^2} L\left(\frac{\partial^2 \mu^2(\zeta, \tau)}{\partial \zeta^2} - \mu^2(\zeta, \tau) + 2\zeta^2 - 2\tau^2 + \zeta^4\tau^4\right)\right]. \tag{46}$$

Now, by using the suggested analytical method, we get

$$\mu_0(\zeta, \tau) = 0,$$

$$\mu_1(\zeta, \tau) = L^{-1}\left[\frac{s + \varrho(1-s)}{s^2} L\left(\frac{\partial^2 \mu_0(\zeta, \tau)}{\partial \zeta^2} - \mu_0^2(\zeta, \tau) + 2\zeta^2 - 2\tau^2 + \zeta^4\tau^4\right)\right],$$

$$\mu_1(\zeta, \tau) = \frac{\tau}{30}\left(-5\varrho\tau^3 + 20\tau^2\varrho\zeta^4\tau^5\varrho + 60\zeta^2 - 60\varrho\zeta^2 + 30\zeta^2\tau\varrho + 6\tau^4\zeta^4 - 6\tau^4\zeta^4\varrho\right),$$

$$\mu_2(\zeta, \tau) = L^{-1}\left[\frac{s + \varrho(1-s)}{s^2} L\left(\frac{\partial^2 \mu_1(\zeta, \tau)}{\partial \zeta^2} - \mu_1^2(\zeta, \tau) + 2\zeta^2 - 2\tau^2 + \zeta^4\tau^4\right)\right],$$

$$\mu_2(\zeta, \tau) = -\frac{\tau}{162116200}\left(-32432400\tau\varrho^2 - 2702700\tau^3\varrho^2\right.$$
$$- 32432400\tau^2\varrho + 21621600\tau^2\varrho^2 - 228228\tau^9\varrho^3\zeta^4$$
$$+ 456456\tau^9\varrho^2\zeta^4 - 228228\tau^9\varrho\zeta^4 + 12012\tau^9\zeta^6\varrho^3$$
$$- 32760\varrho^2\tau^{10}\zeta^4 + 32760\varrho^3\tau^{10}\zeta^4 - 1365\tau^{11}\varrho^3\zeta^4$$
$$+ 22932\tau^{11}\zeta^8\varrho^3 + 22932\tau^{11}\zeta^8\varrho + 5005\tau^9\varrho^3$$
$$+ 100100\varrho^2\tau^8 - 100100\varrho^3\tau^8 - 1158300\tau^7\varrho^2$$
$$+ 579150\tau^7\varrho - 480480\tau^8\zeta^4 + 58968\tau^{10}\zeta^8$$
$$+ 1853280\tau^6\zeta^6 + 3088800\tau^6\varrho^2 - 3088800\tau^6\varrho$$
$$+ 21621600\tau^2\zeta^4 + 579150\varrho^3\tau^7 - 8648640\tau^4\zeta^2$$
$$- 6486480\zeta^2\tau^5 + 1029600\tau^6 + 240240\varrho^2\tau^8\zeta^6$$
$$- 240240\varrho^3\tau^8\zeta^6 + 1312740\tau^7\zeta^6\varrho^3 - 2625480\tau^7\zeta^6\varrho^2$$
$$- 96525\tau^7\varrho^3\zeta^2 + 1312740\tau^7\zeta^6\varrho - 115830\tau^7\varrho^2\zeta^2$$
$$- 6486480\varrho^3\tau^4\zeta^4 + 6486480\varrho^2\tau^4\zeta^4 - 43243200\tau^3\varrho^2\zeta^4$$
$$+ 21621600\tau^3\varrho^3\zeta^4 + 21621600\tau^3\varrho\zeta^4 + 99\zeta^8\varrho^3\tau^{13}$$
$$- 1853280\tau^6\varrho\zeta^2 + 308880\tau^6\varrho^2\zeta^2 + 1544400\tau^6\varrho^3\zeta^2$$
$$+ 540540\tau^5\varrho^3\zeta^4 - 6846840\tau^5\varrho^3\zeta^2 + 6126120\tau^5\varrho\zeta^2$$
$$+ 7207200\tau^5\varrho^2\zeta^2 + 2772\varrho^2\tau^{12}\zeta^8 - 2772\varrho^3\tau^{12}\zeta^8$$
$$+ 480480\tau^8\varrho^3\zeta^4 - 1441440\tau^8\varrho^2\zeta^4 + 1441440\tau^8\varrho\zeta^4$$
$$- 58968\tau^{10}\zeta^8\varrho^3 - 176904\tau^{10}\zeta^8\varrho - 1853280\tau^6\zeta^6\varrho^3$$
$$+ 5559840\tau^6\zeta^6\varrho^2 - 5559840\tau^6\zeta^6\varrho + 64864800\tau^2\varrho^2\zeta^4$$
$$- 21621600\tau^2\varrho^3\zeta^4 - 64864800\tau^2\varrho\zeta^4 + 8648640\tau^4\varrho^3\zeta^2$$
$$- 25945920\tau^4\varrho^2\zeta^2 + 176904\tau^{10}\zeta^8\varrho^2 + 25945920\zeta^2\varrho\tau^4$$
$$- 45864\tau^{11}\zeta^8\varrho^2 + 2702700\varrho\tau^3 - 32432400\zeta^2$$
$$- 540540\zeta^4\tau^5\varrho - 16216200\zeta^2\tau\varrho + 3243240\tau^4\zeta^4\rho$$
$$- 1029600\rho^3\tau^6 - 3243240\tau^4\zeta^4 - 32432400\tau$$
$$\left.+ 32432400\varrho\zeta^2 + 64864800\tau\varrho + 10810800\tau^2\right), \vdots$$

$$\mu_n(\zeta, \tau) = L^{-1}\left[\frac{s + \varrho(1-s)}{s^2} L\left(\frac{\partial^2 \mu_n(\zeta, \tau)}{\partial \zeta^2} - \mu_n^2(\zeta, \tau) + 2\zeta^2 - 2\tau^2 + \zeta^4\tau^4\right)\right]. \tag{47}$$
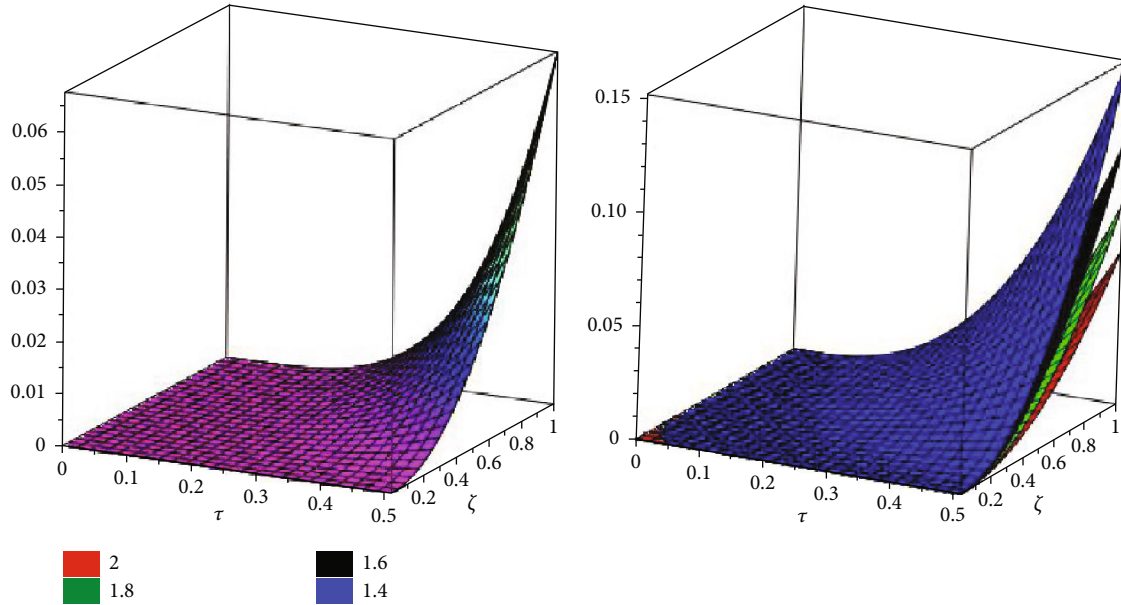
FIGURE 6: The exact and an approximate graph and different fractional-order graphs of problem 4.

The series form result is

$$\mu(\zeta, \tau) = \mu_0(\zeta, \tau) + \mu_1(\zeta, \tau) + \mu_2(\zeta, \tau) + \mu_3(\zeta, \tau) + \cdots \mu_n(\zeta, \tau),$$

$$
\begin{aligned}
\mu(\zeta, \tau) = \frac{\tau}{30} & \left(-5\varrho\tau^3 + 20\tau^2\varrho\zeta^4\tau^5\varrho + 60\zeta^2 - 60\varrho\zeta^2 \right. \\
& + 30\zeta^2\tau\varrho + 6\tau^4\zeta^4 - 6\tau^4\zeta^4\varrho\Big) - \frac{\tau}{162116200} \\
& \cdot \left(-32432400\tau\varrho^2 - 2702700\tau^3\varrho^2 - 32432400\tau^2\varrho \right. \\
& + 21621600\tau^2\varrho^2 - 228228\tau^9\varrho^3\zeta^4 + 456456\tau^9\varrho^2\zeta^4 \\
& - 228228\tau^9\varrho\zeta^4 + 12012\tau^9\zeta^6\varrho^3 - 32760\varrho^2\tau^{10}\zeta^4 \\
& + 32760\varrho^3\tau^{10}\zeta^4 - 1365\tau^{11}\varrho^3\zeta^4 + 22932\tau^{11}\zeta^8\varrho^3 \\
& + 22932\tau^{11}\zeta^8\varrho + 5005\tau^9\varrho^3 + 100100\varrho^2\tau^8 \\
& - 100100\varrho^3\tau^8 - 1158300\tau^7\varrho^2 + 579150\tau^7\varrho \\
& - 480480\tau^8\zeta^4 + 58968\tau^{10}\zeta^8 + 1853280\tau^6\zeta^6 \\
& + 3088800\tau^6\varrho^2 - 3088800\tau^6\varrho + 21621600\tau^2\zeta^4 \\
& + 579150\varrho^3\tau^7 - 8648640\tau^4\zeta^2 - 6486480\zeta^2\tau^5 \\
& + 1029600\tau^6 + 240240\varrho^2\tau^8\zeta^6 - 240240\varrho^3\tau^8\zeta^6 \\
& + 1312740\tau^7\zeta^6\varrho^3 - 2625480\tau^7\zeta^6\varrho^2 - 96525\tau^7\varrho^3\zeta^2 \\
& + 1312740\tau^7\zeta^6\varrho - 115830\tau^9\varrho^2\zeta^2 - 6486480\varrho^3\tau^4\zeta^4 \\
& + 6486480\varrho^2\tau^4\zeta^4 - 43243200\tau^3\varrho^2\zeta^4 + 21621600\tau^3\varrho^3\zeta^4 \\
& + 21621600\tau^3\varrho\zeta^4 + 99\zeta^8\varrho^3\tau^{13} - 1853280\tau^6\varrho\zeta^2 \\
& + 308880\tau^6\varrho^2\zeta^2 + 1544400\tau^6\varrho^3\zeta^2 + 540540\tau^5\varrho^3\zeta^4 \\
& - 6846840\tau^5\varrho^3\zeta^2 + 6126120\tau^5\varrho\zeta^2 + 7207200\tau^5\varrho^2\zeta^2 \\
& + 2772\varrho^2\tau^{12}\zeta^8 - 2772\varrho^3\tau^{12}\zeta^8 + 480480\tau^8\varrho^3\zeta^4 \\
& - 1441440\tau^8\varrho^2\zeta^4 + 1441440\tau^8\varrho\zeta^4 - 58968\tau^{10}\zeta^8\varrho^3 \\
& - 176904\tau^{10}\zeta^8\varrho + 1853280\tau^6\zeta^6\varrho^3 + 5559840\tau^6\zeta^6\varrho^2 \\
& - 5559840\tau^6\zeta^6\varrho + 64864800\tau^2\varrho^2\zeta^4 - 21621600\tau^2\varrho^3\zeta^4 \\
& - 64864800\tau^2\varrho\zeta^4 + 8648640\tau^4\varrho^3\zeta^2 - 25945920\tau^4\varrho^2\zeta^2 \\
& + 176904\tau^{10}\zeta^8\varrho^2 + 25945920\zeta^2\varrho\tau^4 - 45864\tau^{11}\zeta^8\varrho^2 \\
& + 2702700\varrho\tau^3 - 32432400\zeta^2 - 540540\zeta^4\tau^5\varrho \\
& - 16216200\zeta^2\tau\varrho + 3243240\tau^4\zeta^4\varrho - 1029600\varrho^3\tau^6 \\
& - 3243240\tau^4\zeta^4 - 32432400\tau + 32432400\varrho\zeta^2 \\
& \left. + 64864800\tau\varrho + 10810800\tau^2\right) + \cdots.
\end{aligned}
$$

(48)

The problem has the exact solution at $\varrho = 1$:

$$\mu(\zeta, \tau) = \zeta^2\tau^2. \tag{49}$$

In Figure 6, the exact and the approximate solutions of example 4 at $\varrho = 1$ are shown, and the second graph shows the 3D graph of different fractional-order ρ, respectively. From the given figures, it can be seen that both the approximate and exact solutions are in close contact with each other. It is demonstrated that the outcomes of time-fractional problems converge to an integer-order effect as the time-fractional evaluation to integer-order.

5. Conclusion

In this paper, the iterative transformation method is implemented to achieve approximate analytical results of the fractional-order Klein-Gordon equations, which is widely applied in problems for spatial effects in applied sciences. In physical models, the technique yields series form results that converge very quickly. The obtained results in this article are expected to be important for further analysis of the sophisticated nonlinear models. The calculations of this method are very simple and straightforward. As a result, we conclude that this technique can be used to solve a variety of nonlinear fractional-order partial differential equation systems.

Data Availability

The numerical data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

Acknowledgments

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