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Research Article

The Multiplicity of Nontrivial Solutions for a New $p(x)$-Kirchhoff-Type Elliptic Problem

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In the paper, we study the existence of weak solutions for a class of new nonlocal problems involving a $p(x)$-Laplacian operator. By using Ekeland's variational principle and mountain pass theorem, we prove that the new $p(x)$-Kirchhoff problem has at least two nontrivial weak solutions.

1. Introduction and Main Result

In this paper, we consider the following nonlocal $p(x)$-Kirchhoff problem:

$$\begin{cases} -\left(a-b\int_{\Omega}\frac{1}{p(x)}|\nabla u|^{p(x)}dx\right)\Delta_{p(x)}u=\lambda\left(c(x)|u|^{q(x)-2}u+d(x)|u|^{r(x)-2}u\right), & \text{in } \Omega, \\ u=0, & \text{on } \partial\Omega, \end{cases} \quad (1)$$

where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain with boundary $\partial\Omega$, $a \geq b > 0$ are constants, $p \in C(\bar{\Omega})$ with $1 < p^- := \inf_{x\in\bar{\Omega}} p(x) \leq p(x) \leq p^+ := \sup_{x\in\bar{\Omega}} p(x) < N$, $\Delta_{p(x)}u := \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$ is called $p(x)$-Laplacian operator. λ is a real parameter, $c(x)$, $d(x) > 0$ for $x \in \bar{\Omega}$, $1 < q(x) < p(x) < r(x) < p^*(x) = Np(x)/N - p(x)$.

The study of variational problems with nonstandard $p(x)$-growth conditions has been a new and interesting topic. These problems arise from the image processing model and stationary thermorheological viscous flows; we can refer to [1, 2]. Problem (1) is related to the stationary problem

$$\rho h \frac{\partial^2 u}{\partial t^2} + \delta \frac{\partial u}{\partial t} + f_1\left(\frac{\partial u}{\partial t}\right) = \left(p_0 + \frac{Eh}{2L}\int_0^L \left|\frac{\partial u}{\partial x}\right|^2 dx\right)\frac{\partial^2 u}{\partial x^2} + f_2(x,u), \quad (2)$$

where ρ, h, δ, p_0, L are constants which represent some physical meanings; E is Young's modulus; $u(x,t)$ is the lateral displacement; and f_1, f_2 are the external forces. When $f_1 = f_2$, it extends the classical D'Alembert wave equation for free vibrations of elastic strings. Since problem (2) is no longer a pointwise identity, it is often called a nonlocal problem. In recent years, nonlocal elliptic problems have attracted wide attention, and some important and interesting results have been established (see [3–5]).

In the past few decades, many people studied the following $p(x)$-Kirchhoff problem:

$$\begin{cases} -M\left(\int_{\Omega}\frac{1}{p(x)}|\nabla u|^{p(x)}dx\right)\Delta_{p(x)}u = f(x,u), & \text{in } \Omega, \\ u=0, & \text{on } \partial\Omega, \end{cases} \quad (3)$$

where Ω be a bounded open subset of $\mathbb{R}^N$ with a C^1-boundary $\partial\Omega$, $M(t) \in C^1([0,+\infty))$ is a bounded below function, $f:\Omega\times\mathbb{R}\longrightarrow\mathbb{R}$ is a Carathéodory function. For example, Dai in [4] using the three-critical-point theorem obtained the existence of solutions for problem (3). Moreover, when $f(x,u)=\lambda(a(x)|u|^{q(x)-2}u+b(x)|u|^{r(x)-2}u)$, Zhou and Ge in [6] studied the existence of the solution for the nonlocal problem (3) by using the fibering map approach for the corresponding Nehari manifold. In recent years, some people are starting to pay attention to the case $M(t)=a-b\,t(a\geq b>0)$ in problem (3). Obviously, $M(t)=a-bt(a\geq b>0)$ is not bounded below. Therefore, this is a new class of nonlocal problems. In fact, for the case $p(x)\equiv p$, some results are given in [7–9]. However, few literatures have considered this new nonlocal $p(x)$-Kirchhoff problem. Recently, the authors in [10] have considered the following $p(x)$-Kirchhoff-type problem:

$$\begin{cases}-\left(a-b\int_\Omega\frac{1}{p(x)}|\nabla u|^{p(x)}dx\right)\Delta_{p(x)}u=\lambda|u|^{p(x)-2}u+g(x,u), & \text{in }\Omega,\\ u=0, & \text{on }\partial\Omega,\end{cases}\tag{4}$$

where $a\geq b>0$ are constants, $\Omega\in\mathbb{R}^N$ is a bounded smooth domain, $p\in C(\bar{\Omega})$ with $1<p(x)<N$, λ is a real parameter, and g is a continuous function. Under appropriate hypotheses, the author used a mountain pass theorem and fountain theorem to obtain the existence and multiplicity of nontrivial solutions for problem (4).

Inspired by the above facts and aforementioned papers, the main purpose of this paper is to study the existence of two nontrivial solutions for problem (1). Before stating our main results, we make some assumptions on the functions c, d, p, and r.

(H_1) $c(x)\in L^{\alpha(x)}(\Omega)$, $d(x)\in L^{\beta(x)}(\Omega)$, with $1<\alpha(x)\in C(\bar{\Omega})$, $1<\beta(x)\in C(\bar{\Omega})$

(H_2) $q(x)<(\alpha(x)-1)p^*(x)/\alpha(x)$ and $r(x)<(\beta(x)-1)p^*(x)/\beta(x)$

(H_3) $1<p^-<(p^+)^2<2(p^-)^2$

Now, our main result is as follows:

Theorem 1. *Assume that the function $p(x), q(x), r(x)\in C(\bar{\Omega})$, $c(x)$, $d(x)>0$ satisfies $1<q^+<p^-\leq p(x)\leq p^+<2p^-<r^-<r(x)<p^*(x)$. If (H_1)–(H_3) hold, then there exists $\lambda_*>0$ such that problem (1) has at least two nontrivial weak solutions for any $\lambda\in(0,\lambda_*)$.*

2. Preliminaries

In order to discuss problem (1), we need some theories on spaces $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$ which we will call generalized Lebesgue-Sobolev spaces. For more details on the basic properties of these spaces, we refer the readers to Fan and Zhao [11]. Set $C_+(\bar{\Omega})=\{p(x);p(x)\in C(\bar{\Omega}),p(x)>1,\forall x\in\bar{\Omega}\}$; $\zeta(\Omega)$ denoted the set of all measurable real functions defined on Ω.

For any $p\in C_+(\bar{\Omega})$, the variable exponent Lebesgue space $L^{p(x)}(\Omega)$ is defined as

$$L^{p(x)}(\Omega)=\left\{u\in\zeta(\Omega):\int_\Omega|u|^{p(x)}dx<+\infty\right\},\tag{5}$$

with the norm

$$\|u\|_{L^{p(x)}(\Omega)}=|u|_{p(x)}=\inf\left\{\lambda>0:\int_\Omega\left|\frac{u(x)}{\lambda}\right|^{p(x)}dx\leq 1\right\}.\tag{6}$$

The variable exponent Sobolev space $W^{1,p(x)}(\Omega)$ is defined as

$$W^{1,p(x)}(\Omega)=\left\{u\in L^{p(x)}(\Omega):|\nabla u|\in L^{p(x)}(\Omega)\right\},\tag{7}$$

with the norm

$$\|u\|_{W^{1,p(x)}(\Omega)}=|u|_{p(x)}+|\nabla u|_{p(x)}.\tag{8}$$

Define $W_0^{1,p(x)}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$. From [11], we know that the spaces $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$ are all separable and reflexive Banach spaces.

Moreover, there is a constant $C>0$, such that

$$|u|_{p(x)}\leq C|\nabla u|_{p(x)},\tag{9}$$

for all $u\in W_0^{1,p(x)}(\Omega)$. Therefore, $\|u\|=|\nabla u|_{p(x)}$ and $\|u\|_{W^{1,p(x)}(\Omega)}$ are equivalent norms on $W_0^{1,p(x)}(\Omega)$. In addition, we can recall the following properties of the variable exponent spaces.

Lemma 2 (see [11]). *For any $u\in L^{p(x)}(\Omega)$, $v\in L^{p'(x)}(\Omega)$ with $1/p(x)+1/p'(x)=1$, the inequality holds as follows:*

$$\left|\int_\Omega uv\,dx\right|\leq\left(\frac{1}{p^-}+\frac{1}{p'^-}\right)|u|_{p(x)}|v|_{p'(x)}.\tag{10}$$

Moreover, if p_1, p_2, $p_3\in C_+(\bar{\Omega})$ and $1/p_1(x)+1/p_2(x)+1/p_3(x)=1$, then for any $u\in L^{p_1(x)}(\Omega)$, $v\in L^{p_2(x)}(\Omega)$, and $w\in L^{p_3(x)}(\Omega)$, the following inequality holds:

$$\left|\int_\Omega uvw\,dx\right|\leq\left(\frac{1}{p_1^-}+\frac{1}{p_2^-}+\frac{1}{p_3^-}\right)|u|_{p_1(x)}|v|_{p_2(x)}|w|_{p_3(x)}.\tag{11}$$

Let us now recall the modular function $\rho_{p(x)}(u):L^{p(x)}(\Omega)\longrightarrow\mathbb{R}$ which plays an important role in the variable order Lebesgue spaces and which is defined by

$$\rho_{p(x)}(u)=\int_\Omega|u|^{p(x)}dx.\tag{12}$$

Lemma 3 (see [10]). *For any $u_n, u\in L^{p(x)}(\Omega)$, the following properties hold:*

(i) $|u|_{p(x)} < 1(= 1; > 1) \Leftrightarrow \rho_{p(x)}(u) < 1(= 1; > 1)$

(ii) $|u|_{p(x)} < 1 \Rightarrow |u|_{p(x)}^{p^+} \le \rho_{p(x)}(u) \le |u|_{p(x)}^{p^-}$, $|u|_{p(x)} > 1 \Rightarrow |u|_{p(x)}^{p^-} \le \rho_{p(x)}(u) \le |u|_{p(x)}^{p^+}$

(iii) $\lim_{n\longrightarrow\infty} |u_n|_{p(x)} = 0 \Leftrightarrow \lim_{n\longrightarrow\infty} \rho_{p(x)}(u_n) = 0$, $\lim_{n\longrightarrow\infty} |u_n - u| = 0 \Leftrightarrow \lim_{n\longrightarrow\infty} \rho_{p(x)}(u_n - u) = 0$

Lemma 4 (see [10]). *For p, $q \in C_+(\bar{\Omega})$ such that $1 \le q(x) \le p^*(x)$ for all $x \in \bar{\Omega}$, there is a continuous embedding*

$$W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega). \tag{13}$$

If we replace $\le$ with $<$, the embedding is compact.

Lemma 5 (see [12]). *Assume that $h \in L_+^\infty(\Omega)$, $p \in C_+(\bar{\Omega})$. If $|u|^{h(x)} \in L^{p(x)}(\Omega)$, then we have*

$$\min\left\{|u|_{h(x)p(x)}^{h^-}, |u|_{h(x)p(x)}^{h^+}\right\} \le \left||u|^{h(x)}\right|_{p(x)} \le \max\left\{|u|_{h(x)p(x)}^{h^-}, |u|_{h(x)p(x)}^{h^+}\right\}. \tag{14}$$

Lemma 6 (see [10, 13]). *Let $A(u) = \int_\Omega (1/p(x))|\nabla u|^{p(x)}dx$, for all $u \in W_0^{1,p(x)}(\Omega)$, then $A(u) \in C^1(W_0^{1,p(x)}(\Omega), R)$, $\langle A'(u), v\rangle = \int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla v dx$, and for all u, $v \in W_0^{1,p(x)}(\Omega)$, and the following properties hold:*

(i) *A is convex and sequentially weakly lower semicontinuous*

(ii) *A' : $W_0^{1,p(x)}(\Omega) \longrightarrow (W_0^{1,p(x)}(\Omega))^*$ is a mapping of type (S_+); that is, if $u_n \rightharpoonup u$ in $W_0^{1,p(x)}(\Omega)$ and $\overline{\lim}_{n\longrightarrow\infty} \langle A'(u_n), u_n - u\rangle \le 0$, then $u_n \longrightarrow u$ in $W_0^{1,p(x)}(\Omega)$*

(iii) *A' : $W_0^{1,p(x)}(\Omega) \longrightarrow (W_0^{1,p(x)}(\Omega))^*$ is a strictly monotone operator and homeomorphism*

Definition 7. We say that $u \in W_0^{1,p(x)}(\Omega)$ is a weak solution of problem (1), if for any $\varphi \in W_0^{1,p(x)}(\Omega)$, it satisfies the following:

$$\left(a - b\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\right)\int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla\varphi dx - \lambda\int_\Omega c(x)|u|^{q(x)-2}u\varphi dx - \lambda\int_\Omega d(x)|u|^{r(x)-2}u\varphi dx = 0. \tag{15}$$

The energy functional $J_\lambda(u)$: $W_0^{1,p(x)}(0) \longrightarrow \mathbb{R}$ associated to problem (1) is defined as

$$J_\lambda(u) = a\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx - \frac{b}{2}\left(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\right)^2 - \lambda\int_\Omega \frac{c(x)}{q(x)}|u|^{q(x)}dx - \lambda\int_\Omega \frac{d(x)}{r(x)}|u|^{r(x)}dx. \tag{16}$$

Obviously, $J_\lambda(u)$ is a $C^1(W_0^{1,p(x)}(\Omega), \mathbb{R})$ functional and

$$\left\langle J_\lambda'(u), \varphi\right\rangle = \left(a - b\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\right)\int_\Omega |\nabla u|^{p(x)-2}\nabla u\nabla\varphi dx - \lambda\int_\Omega c(x)|u|^{q(x)-2}u\varphi dx - \lambda\int_\Omega d(x)|u|^{r(x)-2}u\varphi dx, \tag{17}$$

for all $u, \varphi \in W_0^{1,p(x)}(\Omega)$. It is well known that the weak solution of (1) corresponds to the critical point of the functional J_λ on $W_0^{1,p(x)}(\Omega)$.

Definition 8. Let $(W_0^{1,p(x)}(\Omega), \|\cdot\|)$ be a Banach space and $J_\lambda \in C^1(W_0^{1,p(x)}(\Omega))$.

We say that J_λ satisfies the $(PS)_c$ for $c \in \mathbb{R}$, if any sequence $\{u_n\} \subset W_0^{1,p(x)}(\Omega)$ satisfying $J_\lambda(u_n) \longrightarrow c$ and $J_\lambda'(u_n) \longrightarrow 0$ in $(W_0^{1,p(x)}(\Omega))^*$ as $n \longrightarrow \infty$ has a convergent subsequence.

3. Proof of Main Result

In this section, the existence of nontrivial solutions for problem (1) is obtained by using a mountain pass lemma combined with Ekeland's variational principle.

Lemma 9. *Assume that the conditions of Theorem 1 hold. Then, the function J_λ satisfies the $(PS)_c$ condition with $c < a^2/2b$ for $\lambda > 0$ small enough.*

Proof. Firstly, we prove that $\{u_n\}$ is bounded in $W_0^{1,p(x)}(\Omega)$. Let $\{u_n\} \subset W_0^{1,p(x)}(\Omega)$ be a $(PS)_c$ sequence such that $c < a^2/2b$. Arguing by contradiction, we assume that $\|u_n\| \longrightarrow +\infty$ as $n \longrightarrow +\infty$.

According to (H_2), we have $1 < q(x)\alpha'(x) < p^*(x)$ for all $x \in \bar{\Omega}$, where $1/\alpha(x) + 1/\alpha'(x) = 1$, by Lemmas 3–5, we have

$$\begin{aligned} p^+c + 1 \ge p^+J_\lambda(u_n) - \left\langle J_\lambda'(u_n), u_n\right\rangle &\ge p^+\left(aA(u_n) - \frac{b}{2}A(u_n)^2 - \lambda\int_\Omega \frac{c(x)}{q(x)}|u_n|^{q(x)}dx - \lambda\int_\Omega \frac{d(x)}{r(x)}|u_n|^{r(x)}dx\right) \\ &\quad - \left[(a - bA(u_n))\int_\Omega |\nabla u_n|^{p(x)}dx - \lambda\int_\Omega c(x)|u_n|^{q(x)}dx - \lambda\int_\Omega d(x)|u_n|^{r(x)}dx\right] \ge \frac{p^+a}{p^+}\int_\Omega |\nabla u_n|^{p(x)}dx \end{aligned}$$

$$\begin{aligned}
&-\frac{bp^+}{2(p^-)^2}\left(\int_\Omega |\nabla u_n|^{p(x)}dx\right)^2 - \frac{\lambda p^+}{q^-}\int_\Omega c(x)|u_n|^{q(x)}dx \\
&\quad - \frac{\lambda p^+}{r^-}\int_\Omega d(x)|u_n|^{r(x)}dx - a\int_\Omega |\nabla u_n|^{p(x)}dx \\
&\quad + \frac{b}{p^+}\left(\int_\Omega |\nabla u_n|^{p(x)}dx\right)^2 + \lambda\int_\Omega c(x)|u_n|^{q(x)}dx \\
&\quad + \lambda\int_\Omega d(x)|u_n|^{r(x)}dx \geq b\left(\frac{1}{p^+} - \frac{p^+}{2(p^-)^2}\right) \\
&\quad \cdot \left(\int_\Omega |\nabla u_n|^{p(x)}dx\right)^2 + \lambda\left(1 - \frac{p^+}{q^-}\right)\int_\Omega c(x)|u_n|^{q(x)}dx \\
&\quad + \lambda\left(1 - \frac{p^+}{r^-}\right)\int_\Omega d(x)|u_n|^{r(x)}dx \\
&\geq b\left(\frac{1}{p^+} - \frac{p^+}{2(p^-)^2}\right)\|u_n\|^{2p^-} - \lambda\left(\frac{p^+}{q^-} - 1\right)|c|_{a(x)} \\
&\quad \cdot \left||u_n|^{q(x)}\right|_{\alpha'(x)} \geq b\left(\frac{1}{p^+} - \frac{p^+}{2(p^-)^2}\right)\|u_n\|^{2p^-} \\
&\quad - 2C\lambda\left(\frac{p^+}{q^-} - 1\right)|c|_{a(x)}\|u_n\|^{q^+}.
\end{aligned} \tag{18}$$

Dividing the above inequality by $\|u_n\|^{q+}$, passing to the limit as $n \longrightarrow +\infty$, note that $2(p^-)^2 > (p^+)^2$ and $2p^- > q^+$, we obtain a contradiction with $\lambda > 0$ being sufficiently small. It follows that $\{u_n\}$ is bounded in $W_0^{1,p(x)}(\Omega)$.

Secondly, we will prove that $\{u_n\}$ has a convergent subsequence in $W_0^{1,p(x)}(\Omega)$.

Since $\{u_n\}$ is bounded, there exists a subsequence, still denoted by $\{u_n\}$ and $u \in W_0^{1,p(x)}(\Omega)$ such that

$$\begin{cases} u_n \longrightarrow u, & \text{weakly in } W_0^{1,p(x)}(\Omega), \\ u_n \longrightarrow u, & \text{strongly in } L^{s(x)}(\Omega), 1 \leq s(x) < p^*(x), \\ u_n(x) \longrightarrow u(x), & \text{a.e. in } \Omega. \end{cases} \tag{19}$$

In view of (19) and Lemma 2, we have

$$\begin{aligned}
&\left|\int_\Omega c(x)|u_n|^{q(x)-2}u_n(u_n - u)dx\right| \leq \int_\Omega \left|c(x)|u_n|^{q(x)-1}\right| \\
&\quad \cdot |u_n - u|dx \leq 3|c(x)|_{\alpha(x)}\left||u_n|^{q(x)-1}\right|_{p(x)/(q(x)-1)} \\
&\quad \cdot |u_n - u|_{\alpha_1(x)} \longrightarrow 0,
\end{aligned} \tag{20}$$

as $n \longrightarrow \infty$, where $\alpha_1(x) \in C_+(\bar{\Omega})$ and $1/\alpha_1(x) + 1/\alpha(x) + (q(x) - 1)/p(x) = 1$. Similarly, we also have $|\int_\Omega d(x) |u_n|^{r(x)-2}u_n(u_n - u)dx| \longrightarrow 0$ as $n \longrightarrow \infty$.

Thus,

$$\begin{cases} \lim_{n\longrightarrow\infty}\int_\Omega c(x)|u_n|^{q(x)-2}u_n(u_n - u)dx = 0, \\ \lim_{n\longrightarrow\infty}\int_\Omega d(x)|u_n|^{r(x)-2}u_n(u_n - u)dx = 0. \end{cases} \tag{21}$$

According to $J_\lambda'(u_n) \longrightarrow 0$, we have $\langle J_\lambda'(u_n), u_n - u\rangle \longrightarrow 0$. Therefore,

$$\begin{aligned}
\left\langle J_\lambda'(u_n), u_n - u\right\rangle &= \left(a - b\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}dx\right)\int_\Omega \\
&\quad \cdot |\nabla u_n|^{p(x)-2}\nabla u_n(\nabla u_n - \nabla u)dx \\
&\quad - \lambda\int_\Omega c(x)|u_n|^{q(x)-2}u_n(u_n - u)dx \\
&\quad - \lambda\int_\Omega d(x)|u_n|^{r(x)-2}u_n(u_n - u)dx \longrightarrow 0,
\end{aligned} \tag{22}$$

as $n \longrightarrow \infty$. So, we can deduce from (21) that

$$\left(a - b\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}dx\right)\int_\Omega |\nabla u_n|^{p(x)-2}\nabla u_n(\nabla u_n - \nabla u)dx \longrightarrow 0. \tag{23}$$

Since $\{u_n\}$ is bounded in $W_0^{1,p(x)}(\Omega)$, passing to a subsequence, if necessary, we assume that

$$\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}dx \longrightarrow \sigma \geq 0 \quad \text{as } n \longrightarrow +\infty. \tag{24}$$

If $\sigma = a/b$, then $a - b\int_\Omega (1/p(x))|\nabla u_n|^{p(x)}dx \longrightarrow 0$. For any $v \in W_0^{1,p(x)}(\Omega)$, by (19) and the Hölder inequality, it implies that

$$\begin{aligned}
\lim_{n\longrightarrow\infty}\int_\Omega &\left(c(x)\left(|u_n|^{q(x)-2}u_n - |u|^{q(x)-2}u\right)\right. \\
&\left. + d(x)\left(|u_n|^{r(x)-2}u_n - |u|^{r(x)-2}u\right)\right)vdx = 0.
\end{aligned} \tag{25}$$

Since

$$\begin{aligned}
\left\langle J_\lambda'(u_n), v\right\rangle &= \left(a - b\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}dx\right)\int_\Omega \\
&\quad \cdot |\nabla u_n|^{p(x)-2}\nabla u_n\nabla v dx - \lambda\int_\Omega \\
&\quad \cdot \left(c(x)|u_n|^{q(x)-2}u_n + d(x)|u_n|^{r(x)-2}u_n\right)vdx,
\end{aligned} \tag{26}$$

and $\langle J_\lambda'(u_n), v\rangle \longrightarrow 0$ as $n \longrightarrow \infty$, we have $\int_\Omega (c(x)|u_n|^{q(x)-2} u_n + d(x)|u_n|^{r(x)-2}u_n)vdx \longrightarrow 0$ as $n \longrightarrow \infty$ for fix $\lambda > 0$. Therefore, $\int_\Omega (c(x)|u|^{q(x)-2}u + d(x)|u|^{r(x)-2}u)vdx = 0$.

By the fundamental lemma of the variational method, we obtain that $c(x)|u(x)|^{q(x)-2}u(x) + d(x)|u(x)|^{r(x)-2}u(x) = 0$ for a.e.$x \in \Omega$.

It follows from $c(x) > 0$ and $d(x) > 0$ that $u = 0$. So

$$\lambda \int_\Omega \frac{c(x)}{q(x)} |u_n|^{q(x)} dx + \lambda \int_\Omega \frac{d(x)}{r(x)} |u_n|^{r(x)} dx \longrightarrow \lambda \cdot \int_\Omega \frac{c(x)}{q(x)} |u|^{q(x)} dx + \lambda \int_\Omega \frac{d(x)}{r(x)} |u|^{r(x)} dx = 0. \tag{27}$$

Hence, for $\sigma = a/b$, we see that

$$J_\lambda(u_n) = a \int_\Omega \frac{1}{p(x)} |\nabla u_n|^{p(x)} dx - \frac{b}{2} \left(\int_\Omega \frac{1}{p(x)} |\nabla u_n|^{p(x)} dx \right)^2 - \lambda \int_\Omega \frac{c(x)}{q(x)} |u_n|^{q(x)} dx - \lambda \int_\Omega \frac{d(x)}{r(x)} |u_n|^{r(x)} dx \longrightarrow \frac{a^2}{2b}, \tag{28}$$

which contradicts the fact that $J_\lambda(u_n) \longrightarrow c < a^2/2b$. Then, $a - b\int_\Omega (1/p(x))|\nabla u_n|^{p(x)} dx \longrightarrow 0$ is not true. Hence,

$$a - b \int_\Omega \frac{1}{p(x)} |\nabla u_n|^{p(x)} dx \longrightarrow a - b\sigma \neq 0. \tag{29}$$

By (23), we obtain that

$$\int_\Omega |\nabla u_n|^{p(x)-2} \nabla u_n (\nabla u_n - \nabla u) dx \longrightarrow 0 \text{ as } n \longrightarrow +\infty. \tag{30}$$

Invoking the S_+ condition (see Lemma 6), we can deduce that $u_n \longrightarrow u$ strongly in $W_0^{1,p(x)}(\Omega)$ as $n \longrightarrow \infty$. So, J_λ satisfies the $(PS)_c$ condition. The proof is complete. □

Lemma 10. *Assume that the conditions of Theorem 1 hold; then, there exist $\rho > 0$, $\beta > 0$, $\lambda_* > 0$ such that $J_\lambda(u) \geq \beta > 0$ for any $\lambda \in (0, \lambda_*)$ and $u \in W_0^{1,p(x)}(\Omega)$ with $\|u\| = \rho$.*

Proof. Let $\rho \in (0,1)$ and $u \in W_0^{1,p(x)}(\Omega)$ with $\|u\| = \rho$. By Hölder inequality and Lemmas 3–5, we can deduce that

$$\begin{aligned} J_\lambda(u) &= a \int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)} dx - \frac{b}{2} \left(\int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)} dx \right)^2 - \lambda \int_\Omega \frac{c(x)}{q(x)} |u|^{q(x)} dx - \lambda \int_\Omega \frac{d(x)}{r(x)} |u|^{r(x)} dx \\ &\geq \frac{a}{p^+} \int_\Omega |\nabla u|^{p(x)} dx - \frac{b}{2(p^-)^2} \left(\int_\Omega |\nabla u|^{p(x)} dx \right)^2 - \frac{\lambda}{q^-} \int_\Omega c(x)|u|^{q(x)} dx - \frac{\lambda}{r^-} \int_\Omega d(x)|u|^{r(x)} dx \\ &\geq \frac{a}{p^+} \rho_{p(x)}(\nabla u) - \frac{b}{2(p^-)^2} \left(\rho_{p(x)}(\nabla u) \right)^2 - \frac{\lambda}{q^-} |c|_{\alpha(x)} \left| |u|^{q(x)} \right|_{\alpha'(x)} - \frac{\lambda}{r^-} |d|_{\beta(x)} \left| |u|^{r(x)} \right|_{\beta'(x)} \\ &\geq \frac{a}{p^+} \rho_{p(x)}(\nabla u) - \frac{b}{2(p^-)^2} \left(\rho_{p(x)}(\nabla u) \right)^2 - \frac{\lambda}{q^-} |c|_{\alpha(x)} \left[|u|^{q^+}_{\alpha'(x)q(x)} + |u|^{q^-}_{\alpha'(x)q(x)} \right] - \frac{\lambda}{r^-} |d|_{\beta(x)} \cdot \left[|u|^{r^+}_{\beta'(x)r(x)} + |u|^{r^-}_{\beta'(x)r(x)} \right] \\ &\geq \frac{a}{p^+} \|u\|^{p^+} - \frac{b}{2(p^-)^2} \|u\|^{2p^-} - 2C \frac{\lambda}{q^-} |c|_{\alpha(x)} \|u\|^{q^-} - 2C \frac{\lambda}{r^-} \cdot |d|_{\beta(x)} \|u\|^{r^-} \\ &\geq \left(\frac{a}{p^+} - \frac{b}{2(p^-)^2} \|u\|^{2p^- - p^+} - \frac{\lambda 2C}{q^-} \cdot |c|_{\alpha(x)} \|u\|^{q^- - p^+} - \frac{\lambda 2C}{r^-} |d|_{\beta(x)} \|u\|^{r^- - p^+} \right) \|u\|^{p^+}. \end{aligned} \tag{31}$$

Since $\rho \in (0,1)$, we can choose ρ sufficiently small such that $a/p^+ - (b/2(p^-)^2)\rho^{2p^- - p^+} > 0$. Set

$$\lambda_* = \frac{a/p^+ - \left(b/2(p^-)^2\right)\rho^{2p^- - p^+}}{\left(2C|c|_{\alpha(x)}/q^-\right)\rho^{q^- - p^+} + \left(C|d|_{\beta(x)}/r^-\right)\rho^{r^- - p^+}}, \quad \beta = \lambda_* \rho^{p^+}. \tag{32}$$

We can deduce that for any $\lambda \in (0, \lambda_*)$, there exists $\beta > 0$ such that for any $u \in W_0^{1,p(x)}(\Omega)$ with $\|u\| = \rho$, we have $J_\lambda(u) \geq \beta > 0$.□

Lemma 11. *Assume that the conditions of Theorem 1 hold; then, there exists $e \in W_0^{1,p(x)}(\Omega)$ with $\|e\| > \rho > 0$ such that $J_\lambda(e) < 0$.*

Proof. Choosing $\psi \in C_0^\infty(\Omega)$ satisfies $\int_\Omega d(x)|\psi|^{r(x)} dx > 0$ and $t > 1$. We have

$$\begin{aligned} J_\lambda(t\psi) &= a \int_\Omega \frac{t^{p(x)}}{p(x)} |\nabla \psi|^{p(x)} dx - \frac{b}{2} \left(\int_\Omega \frac{t^{p(x)}}{p(x)} |\nabla \psi|^{p(x)} dx \right)^2 - \lambda \int_\Omega \frac{t^{q(x)} c(x)}{q(x)} |\psi|^{q(x)} dx - \lambda \int_\Omega \frac{t^{r(x)} d(x)}{r(x)} |\psi|^{r(x)} dx \\ &\leq \frac{a t^{p+}}{p^-} \int_\Omega |\nabla \psi|^{p(x)} dx - \frac{b t^{2p^-}}{2(p^+)^2} \left(\int_\Omega |\nabla \psi|^{p(x)} dx \right)^2 - \frac{\lambda t^{q^-}}{q^+} \int_\Omega c(x)|\psi|^{q(x)} dx - \frac{\lambda t^{r^-}}{r^+} \int_\Omega d(x)|\psi|^{r(x)} dx. \end{aligned} \tag{33}$$

Since $r^- > 2p^- > p^+$, we obtain $J_\lambda(t\psi) \longrightarrow -\infty$ as $t \longrightarrow +\infty$. Then, for $t > 1$ large enough, we can take $e = t\psi$ so that $\|e\| > \rho$ and $J_\lambda(e) < 0$.□

Lemma 12. *Assume that the conditions of Theorem 1 hold; there exists $\xi \in W_0^{1,p(x)}(\Omega)$ such that $\xi \geq 0$, $\xi \neq 0$, and $J_\lambda(t\xi) < 0$ for all $t > 0$ small enough.*

Proof. Letting $\xi \in W_0^{1,p(x)}(\Omega)$ with $\|\xi\| = 1$, for all $t > 0$ small enough, we get the estimate

$$\begin{aligned} J_\lambda(t\xi) &= a\int_\Omega \frac{t^{p(x)}}{p(x)}|\nabla\xi|^{p(x)}dx - \frac{b}{2}\left(\int_\Omega \frac{t^{p(x)}}{p(x)}|\nabla\xi|^{p(x)}dx\right)^2 \\ &\quad - \lambda\int_\Omega \frac{t^{q(x)}c(x)}{q(x)}|\xi|^{q(x)}dx - \lambda\int_\Omega \frac{t^{r(x)}d(x)}{r(x)}|\xi|^{r(x)}dx \\ &\le \frac{at^{p^-}}{p^-}\int_\Omega |\nabla\xi|^{p(x)}dx - \frac{bt^{2p^+}}{2(p^+)^2}\left(\int_\Omega |\nabla\xi|^{p(x)}dx\right)^2 \\ &\quad - \frac{\lambda t^{q^+}}{q^+}\int_\Omega c(x)|\xi|^{q(x)}dx - \frac{\lambda t^{r^+}}{r^+}\int_\Omega d(x)|\xi|^{r(x)}dx \\ &\le \frac{at^{p^-}}{p^-} - \frac{bt^{2p^+}}{2(p^+)^2} - \frac{\lambda t^{q^+}}{q^+}\int_\Omega c(x)|\xi|^{q(x)}dx. \end{aligned} \tag{34}$$

Then, for any $t < \delta^{1/(p^- - q^+)}$, with $0 < \delta < \min\{1, bp^-/2a(p^+)^2 + \lambda p^-/aq^+\int_\Omega c(x)|\xi|^{q(x)}dx\}$, we conclude that $J_\lambda(t\xi) < 0$. □

Proof of Theorem 1. By Lemma 10, we have $\inf_{\partial B_\rho(0)} J_\lambda(u) > 0$, where $\partial B_\rho(0) = \{u \in W_0^{1,p(x)}(\Omega); \|u\| = \rho\}$. And from Lemma 12, there exists $\xi \in W_0^{1,p(x)}(\Omega)$ such that $J_\lambda(t\xi) < 0$ for $t > 0$ small enough. As in the proof of Lemma 10, for any $u \in B(0,\rho)$ and $\lambda \in (0, \lambda_*)$, it follows that

$$\begin{aligned} J_\lambda(u) &\ge \frac{a}{p^+}\|u\|^{p^+} - \frac{b}{2(p^-)^2}\|u\|^{2p^-} - 2C\frac{\lambda_*}{q^-}|c|_{\alpha(x)}\|u\|^{q^-} \\ &\quad - 2C\frac{\lambda_*}{r^-}|d|_{\beta(x)}\|u\|^{r^-}. \end{aligned} \tag{35}$$

Note that for $J_\lambda(0) = 0$, we have

$$-\infty < c_1 := \inf_{B_\rho(0)} J_\lambda(u) < 0. \tag{36}$$

By applying Ekeland's principle in $\overline{B(0,\rho)}$ (see [14–16]), we obtain that there exists a $(PS)_{c_1}$ sequence $\{u_n\} \subset \overline{B(0,\rho)}$ of J_λ. It follows from $c_1 < 0$ and Lemma 9 that J_λ satisfies the $(PS)_{c_1}$ condition. Therefore, one has a subsequence still denoted by $\{u_n\}$ and $u_0 \in W_0^{1,p(x)}(\Omega)$ such that $u_n \longrightarrow u_0$ in $W_0^{1,p(x)}(\Omega)$ and $J_\lambda(u_0) = c_1$, $J'_\lambda(u_0) = 0$, which implies that u_0 is a solution of problem (1).

From Lemmas 10 and 11, we see that the functional J_λ has the mountain pass geometry. Define

$$\begin{aligned} \Gamma &= \left\{\gamma \in C^1\left([0,1], W_0^{1,p(x)}(\Omega)\right) \mid \gamma(0) = 0, \gamma(1) = e\right\}, \\ c_2 &= \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} J_\lambda(\gamma(t)). \end{aligned} \tag{37}$$

Noting that $\max_{t>0}\{at - (b/2)t^2\} = a^2/2b$, for all $u \in W_0^{1,p(x)}(\Omega) \setminus \{0\}$, we have

$$J_\lambda(u) \le a\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx - \frac{b}{2}\left(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\right)^2 \le \frac{a^2}{2b}. \tag{38}$$

So, one has $c_2 < a^2/2b$. Letting $\{u_n\}$ be a $(PS)_{c_2}$ sequence of J_λ, by Lemma 9, we obtain J_λ which satisfies the $(PS)_{c_2}$ condition.

By the mountain pass theorem (see [10, 17]), we obtain that problem (1) has the second solution $\tilde{u}$ with $J_\lambda(\tilde{u}) = c_2$. Noting that $J_\lambda(u_0) = c_1 < 0 = J_\lambda(0) < c_2 = J_\lambda(\tilde{u})$, we know that u_0 and $\tilde{u}$ are two nontrivial solutions of problem (1).

Data Availability

The findings in this research do not make use of data.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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