Hindawi
Journal of Function Spaces
Volume 2021, Article ID 1667865, 7 pages
https://doi.org/10.1155/2021/1667865

Research Article

Integrability on the Abstract Wiener Space of Double Sequences and Fernique Theorem

Jeong-Gyoo Kim

Hongik University, Sejong, Republic of Korea

Correspondence should be addressed to Jeong-Gyoo Kim; jgkim@hongik.ac.kr

Received 15 July 2021; Accepted 17 August 2021; Published 12 October 2021

Academic Editor: Kwok-Pun Ho

The integrability of a function defined on the abstract Wiener space of double Fourier coefficients is explored. The abstract Wiener space is also a Hilbert space. We define an orthonormal system of the Hilbert space to establish a measure and integration on the abstract Wiener space. We examine the integrability of a function $e^{\alpha\|\cdot\|^2}$ defined on the abstract Wiener space for Fernique theorem. With respect to the abstract Wiener measure, the integral of the function turns out to be convergent for $\alpha < 1/2$. The result provides a wider choice of the constant α than that of Fernique.

1. Introduction

We explore an abstract Wiener space that consists of double Fourier coefficients focused on the integrability of functions defined on the space. The space is also a Hilbert space. We define an orthonormal system of the Hilbert space and utilise the system to define a probability measure on the abstract Wiener space of double sequences and develop to integration. Using the probability measure, we examine the integral of $e^{\alpha\|\cdot\|^2}$, i.e., Fernique theorem in the abstract Wiener space. It is proved that the integral of the function with respect to the abstract Wiener measure converges for $\alpha < 1/2$. The specified range of α that we verified in this paper provides a wider choice of the constant α than that of Fernique. We expect that the results can be applied to check integrability of related functions.

The concepts of an abstract Wiener space are known to be first appeared on [1], and we mainly refer to [2] for necessary definitions and theorems of an abstract Wiener space. An analogue Wiener space of sequences is studied in [3]. An abstract Wiener space of sequences is discussed in [4]. Both of them are for the single-indexed sequences. An abstract Wiener space of double Fourier coefficients is defined in the author's work [5] where detailed development of the space can be found. Fernique theorem is introduced in [6]; we use an English version for the theorem. There is a work [7] that generalises the Fernique theorem to functions having values in the extended real number system.

For background knowledge, we introduce an abstract Wiener space as well as the Hilbert spaces of double Fourier coefficients in the next section. An orthonormal system for the Hilbert space is defined in Section 3. In Section 4, we define an abstract Wiener measure using the orthonormal system and provide the main theorems related to Fernique theorem. In the last section, conclusions and discussions are given.

2. Preliminaries

In this paper, underlying concepts are overlapped with the author's recent work [5], such as the setting of an abstract Wiener space based on double Fourier coefficients. We briefly introduce definitions, notations, and basic properties of the space; some of which are in common with the author's previous work.

(1) Let S and T be real numbers. A function $f : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ is said to be (S, T)-periodic in variables s and t provided that $f(s + S, t + T) = f(s, t)$ holds for all s and t

(2) Trigonometric form [8]: let a function $f : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ be continuous on $[-S, S] \times [-T, T]$ and

periodic with period $(2S, 2T)$. Then, the double Fourier series of f is as follows:

$$f(x,y) = \sum_{m,n=0}^{\infty} \left\{ A_{m,n}(f) \cos \frac{m\pi x}{S} \cos \frac{n\pi y}{T} + B_{m,n}(f) \sin \frac{m\pi x}{S} \cos \frac{n\pi y}{T} + C_{m,n}(f) \cos \frac{m\pi x}{S} \sin \frac{n\pi y}{T} + D_{m,n}(f) \sin \frac{m\pi x}{S} \sin \frac{n\pi y}{T} \right\}, \quad (1)$$

where

$$\begin{aligned} A_{m,n}(f) &= \frac{1}{ST} \int_{-S}^{S} \int_{-T}^{T} f(x,y) \cos \frac{m\pi x}{S} \cos \frac{n\pi y}{T} \, dxdy, \\ B_{m,n}(f) &= \frac{1}{ST} \int_{-S}^{S} \int_{-T}^{T} f(x,y) \sin \frac{m\pi x}{S} \cos \frac{n\pi y}{T} \, dxdy, \\ C_{m,n}(f) &= \frac{1}{ST} \int_{-S}^{S} \int_{-T}^{T} f(x,y) \cos \frac{m\pi x}{S} \sin \frac{n\pi y}{T} \, dxdy, \\ D_{m,n}(f) &= \frac{1}{ST} \int_{-S}^{S} \int_{-T}^{T} f(x,y) \sin \frac{m\pi x}{S} \sin \frac{n\pi y}{T} \, dxdy. \end{aligned} \quad (2)$$

The derivation of double Fourier series can be found in [9].

(3) Let $R = [0, S] \times [0, T]$ and $F(R)$ be the space of two-variable functions on R. For x in $F(R)$, we define $\tilde{x}(s,t)$ as a periodic function having period $(2S, 2T)$ and to be symmetric with respect to the first argument and symmetric with respect to the second argument within the rectangle $[-S, S] \times [-T, T]$; $\tilde{x}(s,t) = x(s,t)$ in R and $\tilde{x}$ is $(2S, 2T)$-periodic in the whole plane $\mathbb{R} \times \mathbb{R}$

(4) The coefficients B, C, D of $\tilde{x}$ are all zero due to the symmetries in s and t

$$\begin{aligned} \tilde{x}(s,t) &= \sum_{m,n=0}^{\infty} \left\{ A_{m,n}(\tilde{x}) \cos \frac{m\pi s}{S} \cos \frac{n\pi t}{T} \right\}, \\ \sum_{m,n=0}^{\infty} A_{m,n}(\tilde{x})^2 &= \frac{1}{ST} \int_{-S}^{S} \int_{-T}^{T} \tilde{x}(s,t)^2 \, dsdt. \end{aligned} \quad (3)$$

The underlying spaces of this paper are the following three types of double sequences.

Definition 1 (see [5]). A double sequence is denoted by $\{a_{m,n}\}$.

(a) For $1 \le p < \infty$, let $\ell^p = \{\{a_{m,n}\} \mid (\sum_{m,n=0}^{\infty} |a_{m,n}|^p)^{1/p} < \infty\}$

(b) Let $\mathscr{H}$ be the space of all double sequences $\{a_{m,n}\}$ in ℓ^1 with an inner product

$$\langle \{a_{m,n}\}, \{b_{m,n}\} \rangle = \sum_{k=0}^{\infty} \left(\sum_{l=0}^{k} \left(\frac{k(k+1)}{2} + l + 1 \right)^2 a_{l,k-l} b_{l,k-l} \right). \quad (4)$$

For $\{a_{m,n}\}$ in $\mathscr{H}$, we let $\|\{a_{m,n}\}\|_{\mathscr{H}} = \sqrt{\langle \{a_{m,n}\}, \{a_{m,n}\} \rangle}$.

(c) Let $\mathscr{U}$ be the space of all double sequences $\{a_{m,n}\}$ in ℓ^2 such that the limit

$$x(s,t) := \lim_{p \longrightarrow \infty} \frac{1}{(p(p+1)/2) + q + 1} \sum_{m=0}^{p} \cdot \left(\sum_{u=0}^{m} \sum_{k=0}^{q} a_{k,u-k} \cos \frac{k\pi s}{S} \cos \frac{(u-k)\pi t}{T} \right) \quad (5)$$

converges uniformly on $\mathbb{R} \times \mathbb{R}$.

Remark 2 (see [5]).

(a) It is obvious that $\ell^p \subset \ell^q$ holds for $1 \le p \le q$

(b) In this work, $\sum_{m,n=0}^{\infty} a_{m,n}$ represents a double sum. A double sum may not coincide with iterated sums. As known well, if a double series converges absolutely, then the double sum and iterated sums exist and are all equivalent. We deal with sequences in ℓ^1 for the paper, and rearranging the order of summation does not affect the convergence of double series in the definition

(c) The double summation $\sum_{k=0}^{\infty} \sum_{l=0}^{k} a_{k,k-l}$ is carried out by diagonal directions, $a_{0,0} + (a_{0,1} + a_{1,0}) + (a_{0,2} + a_{1,1} a_{2,0}) + \cdots$

(d) The space $\mathscr{U}$ is defined in association with a double Fourier series; the limit of the arithmetic means of partial sums of the series is called a Cesáro mean. The denominator $(p(p+1)/2) + q + 1$ in its limit is the number of terms involved in a partial sum for each diagonal explained in (c). The inner product defined for $\mathscr{H}$ in (b) is also motivated by this

Proposition 3 (see [5]). *If $\{a_{m,n}\}$ is in ℓ^1, then, the series $\sum_{m,n=0}^{\infty} a_{m,n} \cos(m\pi s/S) \cos(n\pi t/T)$ converges uniformly on $[-S, S] \times [-T, T]$.*

Proposition 4 (see [5]).

$$\mathscr{H} \subsetneq \ell^1 \subsetneq \mathscr{U} \subsetneq \ell^2. \quad (6)$$

The following notations are for the definitions from [2].

(i) *H be a real separable Hilbert space with norm* $\|\cdot\| = \sqrt{\langle\cdot,\cdot\rangle}$

(ii) *P is an orthogonal projection on H*

(iii) $\mathcal{F}$ *is the partially ordered set of orthogonal projections P of H* ($P > Q$ *means* $P(H) \supset Q(H)$ *for* $P, Q \in \mathcal{F}$)

Definition 5 (see [2] p59). A seminorm $\|\cdot\|$ in Hilbert space H is called measurable if for $\varepsilon > 0$, there exists $P_0 \in \mathcal{F}$ such that $\mu\{\|Px\| > \varepsilon\} < \varepsilon$ for all $P \perp P_0$, $P \in \mathcal{F}$.

Definition 6 (see [2] p63). Let B be the completion of H with respect to a measurable norm $\|\cdot\|$. i will denote the inclusion map of H into B. The triple (i, H, B) is called an abstract Wiener space.

Proposition 7 (see [5]). *$(i, \mathcal{H}, \mathcal{U})$ is an abstract Wiener space.*

3. An Orthonormal System of the Hilbert Space $\mathcal{H}$

From Equation (6), the space of double sequences $\mathcal{U}$ contains $\mathcal{H}$, which is a Hilbert space. We find an orthonormal system in $\mathcal{H}$ to define an appropriate probability and integration in the space $\mathcal{U}$ that will be used in the next section.

Definition 8. For nonnegative integers m and n, we define the following:

$$a^{m,n}(u,v) := \begin{cases} \dfrac{1}{((m+n)(m+n-1)/2)+n+1}, & \text{if } (u,v) = (m,n), \\ 0, & \text{otherwise.} \end{cases} \tag{7}$$

Here, the argument (u, v) is for a component of $a^{m,n}$. That is, $a^{m,n}$ is a member of $\mathcal{H}$ whose components are all 0 except for the (m, n)-th component.

Theorem 9. *The set $\{a^{m,n}\}$ in Definition 8 constitutes an orthonormal system in $\mathcal{H}$.*

Proof. Let $c = \{c_{u,v}\}$ be in $\mathcal{H}$. By Definition 1, the inner product of c and $a^{m,n}$ is as follows:

$$\langle c, a^{m,n}\rangle = \langle \{c_{u,v}\}, \{a^{m,n}_{u,v}\}\rangle = \begin{cases} \dfrac{c_{m,n}}{((m+n)(m+n-1)/2)+n+1}, & \text{if } (u,v) = (m,n), \\ 0, & \text{otherwise.} \end{cases} \tag{8}$$

We have $\langle a^{i,j}, a^{m,n}\rangle = 0$ for $(i, j) \neq (m, n)$, and

$$\begin{aligned} \langle a^{m,n}, a^{m,n}\rangle &= \sum_{k=0}^{\infty}\left(\sum_{l=0}^{k}\left(\frac{k(k+1)}{2}+l+1\right)^2 a^{m,n}_{l,k-l}a^{m,n}_{l,k-l}\right) \\ &= \left(\frac{(m+n)(m+n+1)}{2}+n+1\right)^2 \\ &\cdot \left(\frac{1}{((m+n)(m+n-1)/2)+n+1}\right)^2 = 1. \end{aligned} \tag{9}$$

Therefore, $\{a^{m,n}\}$ is a complete orthonormal system in $\mathcal{H}$. □

Remark 10. By Theorem 9, any $c = \{c_{u,v}\}$ in $\mathcal{U}$ can be expressed as the following limit:

$$c = \lim_{m,n\longrightarrow\infty}\sum_{i=0}^{m}\sum_{j=0}^{n}\langle c, a^{i,j}\rangle a^{i,j} = \lim_{m\longrightarrow\infty}\sum_{k=0}^{m}\sum_{l=0}^{k}\left\langle c, a^{l,k-l}\right\rangle a^{l,k-l}. \tag{10}$$

Then, Theorem 9 is also valid for $\mathcal{U}$.

We introduce an operator for an expression of an inner product in $\mathcal{U}$. Let $T^{m,n} : \mathcal{U} \longrightarrow \mathcal{U}$ be an operator defined by the following:

$$T^{m,n}(\{d_{u,v}\}) := \begin{cases} \dfrac{d_{m,n}}{((m+n)(m+n-1)/2)+n+1}, & \text{if } (u,v) = (m,n), \\ 0, & \text{otherwise.} \end{cases} \tag{11}$$

If d is in $\mathcal{H}$, then $T^{m,n}(d) = \langle d, a^{m,n}\rangle$ obviously. We want to extend the concept of this $\mathcal{H}$ inner product to members of the larger space $\mathcal{U}$ or ℓ^2.

Theorem 11. *Let $c = \{c_{u,v}\}$ be in $\mathcal{H}$, $d = \{d_{u,v}\}$ in $\mathcal{U}$, and $\{a^{m,n}\}$ is the orthonormal system in Definition 8. The limit $\lim_{m\longrightarrow\infty}\sum_{k=0}^{m}\sum_{l=0}^{k}\langle c, T^{l,k-l}(d)\rangle a^{l,k-l}$ exists in the ℓ^2-sense.*

Proof. Let $d = \{d_{u,v}\}$ be in $\mathcal{U}$. We want to show that $\sum_{m,n=0}^{\infty}\langle\{c_{u,v}\}, T^{m,n}(\{d_{u,v}\})\rangle a^{l,k-l}$ exists for any d in $\mathcal{U}$ (or in ℓ^2). We express the double sum $\sum_{m,n=0}^{\infty}$ as $\sum_{k=0}^{\infty}\sum_{l=0}^{k}$. It suffices to show that $\sum_{k=0}^{m}\sum_{l=0}^{k}\langle c, T^{l,k-l}(d)\rangle$ is Cauchy.

$$\begin{aligned} &\left|\left\langle c, \sum_{k=m_1+1}^{m_2}\sum_{l=0}^{k}T^{l,k-l}(d)\right\rangle\right| \\ &= \left|\left\langle \{c_{u,v}\}, \sum_{k=m_1+1}^{m_2}\sum_{l=0}^{k}T^{l,k-l}(\{d_{u,v}\})\right\rangle\right| \end{aligned}$$

$$= \sum_{k=m_1+1}^{m_2} \sum_{l=0}^{k} \left(\frac{k(k+1)}{2} + l + 1\right)^2 c_{l,k-l} \frac{d_{l,k-l}}{(k(k+1)/2) + l + 1}$$
$$= \sum_{k=m_1+1}^{m_2} \sum_{l=0}^{k} \left\{ \left(\left(\frac{k(k+1)}{2} + l + 1\right) c_{l,k-l} \right) d_{l,k-l} \right\}$$
$$\le \|\{c_{u,v}\}\|_{\mathscr{H}} \sqrt{\sum_{k=m_1+1}^{m_2} \sum_{l=0}^{k} d_{l,k-l}^2}. \quad (12)$$

The last inequality comes from Schwarz inequality and the norm defined in Definition 1. □

Definition 12. Let $c = \{c_{u,v}\}$ be in $\mathscr{U}$ and $d = \{d_{u,v}\}$ be in $\mathscr{U}$. We define the following:

$$\langle c, d \rangle = \lim_{m \longrightarrow \infty} \sum_{k=0}^{m} \sum_{l=0}^{k} \left\langle c, T^{l,k-l}(d) \right\rangle = \sum_{k=0}^{\infty} \sum_{l=0}^{k} \left\langle c, T^{l,k-l}(d) \right\rangle, \quad (13)$$

where the limit is in ℓ^2-sense, which has been verified in Theorem 11.

4. Integrability of $e^{\alpha\|\cdot\|^2}$ in $\mathscr{U}$

We defined an orthonormal system $\{a^{m,n}\}$ in Definition 8 for Fernique theorem in the abstract Wiener space $\mathscr{U}$. Each member of the orthonormal system $\{a^{m,n}\}$ is a double sequence; the components of $a^{m,n}$ are all 0 except for the (m, n)-th component. We define an abstract Wiener measure with the dual of the orthonormal system. Then, we can examine the integrability of $e^{\alpha\|\cdot\|^2}$ in $\mathscr{U}$ as in Fernique theorem.

Proposition 13 (Kuo and Fernique [2, 6]). *There exists $\alpha > 0$ such that $\int_B e^{\alpha\|x\|^2} d\mu(x) < \infty$, where B is a Banach space and μ is an abstract Wiener measure.*

Let $b^* : \mathscr{U} \longrightarrow \mathbb{R}$ be a member of $\mathscr{U}^*$, the dual space of $\mathscr{U}$, which has been defined using the orthonormal system $a^{m,n}$ in Definition 8; $b_{m,n}(d) = \langle d, a^{m,n} \rangle$ as in Equation (8). That is, for $d = \{d_{u,v}\}$ in $\mathscr{U}$

$$b_{m,n}(d) \coloneqq \frac{d_{m,n}}{((m+n)(m+n-1)/2) + n + 1}. \quad (14)$$

4.1. Single Indexing for a Double Sequence. The underlying space of integral for Fernique theorem will be $\mathscr{U}$. Since the orthonormal system and its dual on the space are double sequences, we need to deal with double indexing. This makes calculation and development of the functional on $\mathscr{U}$ very complicated. Hence, we convert double indexing of sequences in $\mathscr{U}$ into single indexing to alleviate the complexity in manipulation.

Definition 14. We define $\psi : \mathbb{N} \cup \{0\} \longrightarrow (\mathbb{N} \cup \{0\}) \times (\mathbb{N} \cup \{0\})$, $\psi(p) \coloneqq (q_p, r_p)$ for $p \in \mathbb{N} \cup \{0\}$. Here,

$$q_p = \max \left\{ k \in \mathbb{N} \,\middle|\, \frac{k(k+1)}{2} \le p \right\},$$
$$r_p = p - \frac{q_p\left(q_p + 1\right)}{2}. \quad (15)$$

Remark 15. For Definition 14, we have the following:

(1) The function ψ is a bijection

(2) We adopt the following notation for a double sequence $\{d_{m,n}\}$

$$\sum_{p=0}^{\infty} d_p = \sum_{\psi(p) \in (\mathbb{N}\cup\{0\})\times(\mathbb{N}\cup\{0\})} d_{\psi(p)} = \text{diag} - \sum_{m,n=0}^{\infty} d_{m,n}, \quad (16)$$

where $\text{diag} - \sum_{m,n=0}^{\infty}$ denotes summation by diagonals as in Definition 14 for a double sequence.

Using Definition 14, any double sequence can be regarded as a single indexed sequence. For the integrability of a function in relation to Fernique theorem, we use the expression of single indexing in Equation (16) in this section.

4.2. Orthogonal System of Dual Members with Single Indexing. The orthonormal system $\{a^{m,n}\}$ in $\mathscr{U}$ in Definition 8 can be regarded as a single indexed sequence using Equation (16); we denote the system by $\{f_n\}$, the new name of $\{a^{m,n}\}$. Applying Definition 14 to Equation (14), the n-th component of f_n is $1/(n+1)$ and all other components are zero:

$$f_n(k) = \begin{cases} \dfrac{1}{n+1}, & \text{if } k = n, \\ 0, & \text{otherwise,} \end{cases} \quad (17)$$

where k is the k-th component of f_n.

According to this conversion with Equation (16), we rewrite the properties of the orthonormal system $\{a^{m,n}\}$ in terms of a single indexed sequence $\{f_n\}$. As is $\{a_{m,n}\}$, $\langle f_n, f_m \rangle = 0$ for $n \ne m$, and $\|f_n\|^2 = \langle f_n, f_n \rangle = \sum_{j=0}^{\infty} (n+1)^2 (f_n(j))^2 = (n+1)^2/(n+1)^2 = 1$, $\{f_n\}$ consists of an orthonormal system in $\mathscr{U}$. Here, $\|\cdot\|$ is the ℓ^2-norm.

Consider the dual space $\mathscr{U}^*$ of $\mathscr{U}$; for b^* in $\mathscr{U}^*$ $(b = \{b_n\})$, its norm is defined by $\|b^*\| = \sup_{\|\{c_n\}\|=1} \langle \{c_n\}, \{b_n\} \rangle / \|\{c_n\}\| = \sup_{\|\{c_n\}\|=1} |\sum_{n=0}^{\infty} c_n b_n|$.

Let f_n^*, the dual of f_n, be a member of $\mathscr{U}^*$. Then, $f_n^* : \mathscr{U} \longrightarrow \mathbb{R}$ maps $\{d_k\}$ to its n-th component divided by $n+1$; i.e., $f_n^*(\{d_k\}) = d_n/(n+1)$. Let us compute the norm of f_n^*:

$$\|f_n^*\| = \sup_{\|\{c_m\}\|=1} \frac{|f_n^*(\{c_m\})|}{\|\{c_m\}\|_{\mathscr{U}}} = \sup_{\|\{c_m\}\|=1} \left|\frac{c_n}{n+1}\right| = \frac{1}{n+1}. \quad (18)$$

Then, the set of f_n^* consists of an orthogonal system in $\mathscr{U}^*$.

4.3. Fernique Theorem on $\mathscr{U}$. In order to be used in Fernique theorem, we need a functional for a measure (density) on the abstract Wiener space $\mathscr{U}$. We regard each f_n^* as a random variable defined on $\mathscr{U}$ and have the following theorems.

Theorem 16. *Each f_n^* of the orthogonal system is normally distributed; $f_n^* \sim N(0, \|f_n^*\|^2)$.*

Proof. For a real number a, we calculate the probability $w(\{c \in \mathscr{U} | f_n^*(c) < a\})$ as follows. Let $c = \{c_m\}$ be in $\mathscr{U}$. Then, for nonzero $f_n^* \in \mathscr{U}^*$,

$$\begin{aligned} &w\{c \in \mathscr{U} : \langle c, f_n\rangle < a\} \\ &= w\left\{c \in \mathscr{U} : \left\langle c, \frac{f_n}{\|f_n^*\|}\right\rangle < \frac{a}{\|f_n^*\|}\right\} \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{a/\|f_n^*\|} \exp\left\{-\frac{u^2}{2}\right\} du \\ &= \frac{1}{\sqrt{2\pi\|f_n^*\|^2}} \int_{-\infty}^{a} \exp\left\{-\frac{u^2}{2\|f_n^*\|^2}\right\} du. \end{aligned} \quad (19)$$

Hence, f_n^* is normally distributed with mean 0 and variance $\|f_n^*\|^2$. □

Then, the space $\mathscr{U}$ is with the measure w defined in the proof of Theorem 16.

Theorem 17. *$K_m = f_0^* + f_1^* + \cdots + f_m^*$ is normally distributed; $K_m \sim N(0, \sum_{k=0}^{m} \|f_k^*\|^2)$.*

Proof. We first show that the random variables f_n^*'s are independent to each other using characteristic functions, i.e., Fourier transforms: if X is a random variable and $X \sim N(\mu, \sigma^2)$, then the Fourier transform of X is $[\mathscr{F}(X)](t) = e^{(i\mu t - \sigma^2 t^2)/2}$.

Suppose $m \neq n$. Then, $\|f_m^* + f_n^*\|^2 = \sup_{\|\{c_k\}\|=1} |(c_m/(m+1)) + (c_n/(n+1))| = (1/(m+1)^2) + (1/(n+1)^2)$. We have $f_m^* + f_n^* \sim N(0, (1/(m+1)^2) + (1/(n+1)^2))$, and $[\mathscr{F}(f_m^* + f_n^*)](t) = e^{-(1/(m+1)^2 + 1/(n+1)^2)t^2/2}$. As $f_m^* \sim N(0, 1/(m+1)^2)$ and $f_n^* \sim N(0, 1/(n+1)^2)$, we have $[\mathscr{F}(f_m^*)](t_1) \cdot [\mathscr{F}(f_n^*)](t_2) = e^{-(t_1^2/(m+1)^2 + t_2^2/(n+1)^2)/2}$. We obtained $[\mathscr{F}(f_m^*)](t) \cdot [\mathscr{F}(f_n^*)](t) = [\mathscr{F}(f_m^* + f_n^*)](t)$. Hence, f_m^* and f_n^* are stochastically independent by Theorem 16.13 in [10].

As $\{f_n^*\}$ is an independent system, $\|K_m\|^2 = \sum_{k=0}^{m} \|f_k^*\|^2$. Therefore, $K_m = f_0^* + f_1^* + \cdots + f_m^*$ follows a normal distribution and we have $K_m \sim N(0, \sum_{k=0}^{m} \|f_k^*\|^2)$. □

Now, we need the convergence of an infinite product for Fernique theorem in the abstract Wiener space $\mathscr{U}$, which will be discussed and provided here.

Lemma 18. *The infinite product $\prod_{k=0}^{\infty}((k+1)/(\sqrt{(k+1)^2 - 2\alpha}))$ converges for all $\alpha < 1/2$.*

Proof. The convergence of an infinite product $\prod_{k=0}^{\infty} z_n$ is defined as $\lim_{m\to\infty} \prod_{k=0}^{n} z_n$. We use the theorem that $\prod_{k=0}^{\infty} z_n$ converges if and only if $\sum_{k=0}^{\infty} \ln(z_n)$ converges. We take a logarithm for the infinite product:

$$\begin{aligned} \ln\left(\prod_{k=0}^{\infty} \frac{k+1}{\sqrt{(k+1)^2 - 2\alpha}}\right) &= \sum_{k=0}^{\infty} \ln\left(\frac{k+1}{\sqrt{(k+1)^2 - 2\alpha}}\right) \\ &= \sum_{k=0}^{\infty} -\frac{1}{2} \ln\left(1 - \frac{2\alpha}{(k+1)^2}\right). \end{aligned} \quad (20)$$

Therefore, it suffices to show that $\sum_{k=0}^{\infty} \ln(1 - (2\alpha/(k+1)^2))$ converges for all $\alpha < 1/2$. As the antilogarithm $(1 - (2\alpha/(k+1)^2))$ is less than 1, each logarithmic term of the series is negative. For each k of the finite series $\sum_{k=0}^{m} \ln(1 - (2\alpha/(k+1)^2))$, we use a Taylor series $\ln(1+x) = \sum_{i=1}^{\infty} ((-1)^{i+1}/i)x^i$, where $2\alpha/(k+1)^2$ is plugged into x of the Taylor series. Then, the finite series $\sum_{k=0}^{m}$ is regarded as a double series as follows:

$$\begin{aligned} \sum_{k=0}^{m} \ln\left(1 - \frac{2\alpha}{(k+1)^2}\right) &= \sum_{k=0}^{m} \sum_{i=1}^{\infty} \frac{(-1)^{i+1}}{i} \left(-\frac{2\alpha}{(k+1)^2}\right)^i \\ &= -\sum_{i=1}^{\infty} \sum_{k=0}^{m} \frac{(2\alpha)^i}{i} \frac{1}{(k+1)^2}. \end{aligned} \quad (21)$$

We change the order of the two summations in the last equality. The sign of each term of this double series is $(-1)^{i+1+i} = -1$ and it is put into the front of the series in the last equation. We examine the double series for each i:

$$\begin{aligned} &\text{for } i = 1, \sum_{k=0}^{m} \frac{(2\alpha)^1}{1} \frac{1}{(k+1)^2} = \frac{(2\alpha)^1}{1} \sum_{k=0}^{m} \frac{1}{(k+1)^2} < \frac{(2\alpha)^1}{1} \frac{\pi^2}{6}, \\ &\text{for } i = 2, \sum_{k=0}^{m} \frac{(2\alpha)^2}{2} \frac{1}{(k+1)^2} = \frac{(2\alpha)^2}{2} \sum_{k=0}^{m} \frac{1}{(k+1)^2} < \frac{(2\alpha)^2}{2} \frac{\pi^2}{6}, \end{aligned} \quad (22)$$

and so on. For all i, the sum $\sum_{k=0}^{m}(1/(k+1)^2)$ is a partial sum of a p-series ($p = 2$) which converges to $\pi^2/6$. Then, $\sum_{i=1}^{\infty}((2\alpha)^i/i) \sum_{k=0}^{m}(1/(k+1)^2) \leq (\pi^2/6) \sum_{i=1}^{\infty}((2\alpha)^i/i)$. Now the series $\sum_{i=1}^{\infty}((2\alpha)^i/i)$ is of positive terms and smaller than a geometric series which is convergent for $2\alpha < 1$. By a comparison test, the series $\sum_{i=1}^{\infty}((2\alpha)^i/i)$ also converges for $2\alpha <$

1. When $\alpha \geq 1/2$, the series diverges. Therefore, $\prod_{k=0}^{\infty}((k+1)/(\sqrt{(k+1)^2-2\alpha}))$ converges for all $\alpha < 1/2$. It is obvious that the infinite product converges for $\alpha < 0$. □

The function to be integrated in Fernique theorem is $e^{\alpha\|\cdot\|^2}$ for a variable c in $\mathscr{U}$ that is an infinite dimensional space. When a function defined on an abstract Wiener space assigns finite values, its integral on the abstract Wiener space makes sense and can be expressed by a Lebesgue integral. We use a projection P_m for a function of $c=\{c_n\}$ in $\mathscr{U}$, where $P_m : \mathscr{U} \longrightarrow \mathscr{U}$ is an $(m+1)$-dimensional projection; $P_m(c)=(c_0, c_1, c_2, \cdots, c_m, 0, 0, \cdots)$. Then, the integral of $e^{\alpha\|c\|^2}$ for Fernique theorem should be explained via the integral of $e^{\alpha\|P_m(c)\|^2}$.

Theorem 19. $\int_{\mathscr{U}} e^{\alpha\|c\|^2} dw(c) < \infty$ *for all* $\alpha < 1/2$.

Proof. For a sequence $c=\{c_k\}$ in $\mathscr{U}$, $P_m(c) = (c_0, c_1, c_2, \cdots, c_m, 0, 0, \cdots)$. As we have shown in Theorems 16 and 17, random variables f_j^* are independent and the variance of each f_j^* is $\|f_j^*\|^2 = 1/(j+1)^2$. Therefore,

$$\begin{aligned}
&\int_{\mathscr{U}} e^{\alpha\|P_m(c)\|^2} dw(c) \\
&= \frac{1}{\sqrt{(2\pi)^{m+1}\prod_{j=0}^{m}\left\|f_j^*\right\|}} \int_{\mathbb{R}^{m+1}} \exp\left\{\alpha \sum_{j=0}^{m} u_j^2\right\} \\
&\quad \cdot \exp\left\{-\sum_{j=0}^{m} \frac{u_j^2}{2\left\|f_j^*\right\|^2}\right\} du_0 du_1 \cdots du_m \\
&= \prod_{j=0}^{m} \frac{1}{\sqrt{2\pi}\left\|f_j^*\right\|} \int_{\mathbb{R}^{m+1}} \exp\left\{\left(\alpha - \frac{1}{2\left\|f_j^*\right\|^2}\right) u_j^2\right\} du_j \\
&= \prod_{j=0}^{m} \frac{j+1}{\sqrt{2\pi}} \int_{\mathbb{R}} \exp\left\{\left(\alpha - \frac{(j+1)^2}{2}\right) u_j^2\right\} du_j.
\end{aligned} \tag{23}$$

We calculate the k-th integral in the product of the last equality:

$$\begin{aligned}
&\frac{k+1}{\sqrt{2\pi}} \int_{\mathbb{R}} \exp\left\{\left(\alpha - \frac{(k+1)^2}{2}\right) u_k^2\right\} du_k \\
&= \frac{k+1}{\sqrt{2\pi}} \int_{\mathbb{R}} \exp\left\{-\left(\frac{(k+1)^2}{2} - \alpha\right) u_k^2\right\} du_k \\
&\overset{(*)}{=} \frac{k+1}{\sqrt{2\pi}} \frac{\sqrt{2}}{\sqrt{(k+1)^2 - 2\alpha}} \sqrt{\pi} = \frac{k+1}{\sqrt{(k+1)^2 - 2\alpha}}.
\end{aligned} \tag{24}$$

For the equality $(*)$ above, we use the well-known formula $\int_{\mathbb{R}} \exp\{-a^2u^2\} du = \sqrt{\pi}/a$. Then, the integral in Equation (23) is expressed by the following:

$$\int_{\mathscr{U}} e^{\alpha\|P_m(c)\|^2} dw(c) = \prod_{k=0}^{m} \frac{k+1}{\sqrt{(k+1)^2 - 2\alpha}}. \tag{25}$$

We need to show that the limits of both sides of Equation (25) are equal. Taking the limit on its left side, $\lim_{m\longrightarrow\infty} \int_{\mathscr{U}} e^{\alpha\|P_m(c)\|^2} dw(c) = \int_{\mathscr{U}} e^{\alpha\|c\|^2} dw(c)$ by the monotone convergence theorem since $\lim_{m\longrightarrow\infty}\|P_m(c)\| = \|c\|$. Also, taking the limit on its right side, the limit is $\prod_{k=0}^{\infty}((k+1)/(\sqrt{(k+1)^2-2\alpha}))$. The convergence of this infinite product has been shown for all $\alpha < 1/2$ in Lemma 18, and we have the desired result. □

5. Conclusions and Discussions

We explored the abstract Wiener space $\mathscr{U}$ consisting of sequences of double Fourier coefficients for integrability. As it is also a Hilbert space, we found its orthonormal system and used it to define a probability measure on the abstract Wiener space. Then, we examined integrability of a function $e^{\alpha\|\cdot\|^2}$ appeared in Fernique theorem. We verified that the function is integrable with respect to the abstract Wiener measure with a wider choice of a constant α than that of Fernique. As this paper provides a specific range of values for the constant α, the result can be applied to related functions.

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares that there is no conflict of interest regarding the publication of this paper.

Acknowledgments

This research was supported by 2019 Hongik University internal fund.

References

[1] L. Gross, *Abstract Wiener measure*, vol. 140 of Lecture Notes in Mathematics, , Springer-Verlag, 1970.

[2] H.-H. Kuo, "Gaussian measures in Banach spaces," in *Lecture Notes in Mathematics*, vol. 463, Springer-Verlag, Berlin, Heidelberg, New-York, 1975.

[3] K. S. Ryu, "Analogue of Wiener integral in the space of sequences of real numbers," *Journal of the Chungcheong Mathematical Society*, vol. 25, pp. 65–72, 2012.

[4] A. de Andrade and P. R. C. Ruffino, "Wiener integral in the space of sequences of real numbers," *Archivum Mathematicum*, vol. 36, pp. 95–101, 2000.

[5] J.-G. Kim, "The Hilbert space of double Fourier coefficients for an abstract Wiener space," *Mathematics*, vol. 9, no. 4, p. 389, 2021.

[6] X. Fernique, "Intégrabilité des vecteurs gaussiens," *Comptes Rendus de l'Académie des Sciences*, vol. 270, pp. A1698–A1699, 1970.

[7] P. Friz and H. Oberhauser, "A generalized Fernique theorem and applications," *Proceedings of the American Mathematical Society*, vol. 138, no. 10, pp. 3679–3688, 2010.

[8] A. D. Poularikas, "Fourier series," in *The Handbook of Formulas and Tables for Signal Processing*, A. D. Poularikas and B. Raton, Eds., CRC Press LLC, USA, 1999.

[9] H. H. F. Weinberger, *A First Course in Partial Differential Equations: with Complex Variables and Transform Methods*, Dover Publications INC., New York, 1965, Chapter 6, pp. 141–145.

[10] J. Yeh, *Stochastic Processes and the Wiener Integral*, Marcel Dekker Inc., 1973.