Hindawi
Journal of Function Spaces
Volume 2021, Article ID 1865172, 9 pages
https://doi.org/10.1155/2021/1865172

Research Article

Periodic and Fixed Points for Caristi-Type G-Contractions in Extended b-Gauge Spaces

Nosheen Zikria,[1] Maria Samreen,[1] Tayyab Kamran,[1] and Seher Sultan Yeşilkaya[2]

[1]*Department of Mathematics, Quaid-i-Azam University, Islamabad, Pakistan*
[2]*Division of Applied Mathematics, Thu Dau Mot University, Binh Duong Province, Vietnam*

Correspondence should be addressed to Seher Sultan Yeşilkaya; yesilkaya@tdmu.edu.vn

Received 22 April 2021; Accepted 7 June 2021; Published 11 June 2021

Academic Editor: Santosh Kumar

In this paper, we introduce extended b-gauge spaces and the extended family of generalized extended pseudo-b-distances. Moreover, we define the sequential completeness and construct the Caristi-type G-contractions in the framework of extended b -gauge spaces. Furthermore, we develop periodic and fixed point results in this new setting endowed with a graph. The obtained results of this paper not only generalize but also unify and improve the existing results in the corresponding literature.

1. Introduction and Preliminaries

The famous Caristi fixed point theorem [1] states that a self-mapping T on a complete metric space (U, p) possesses a fixed point w in U if

$$p(u, Tu) \leq f(u) - f(Tu), \tag{1}$$

for all $u \in U$, where $f : U \longrightarrow [0,\infty)$ is a lower semicontinuous function.

Indeed, Caristi [1] observed these results when he searched for alternative proof of the outstanding fixed point theorem of Banach. It is known also Caristi-Kirk fixed point theorem [2]. In fact, Caristi's theorem is equivalent to metric completeness [3]. For some other contributions to this topic, we refer to [4–10].

In view of extending the concept of Banach contraction, Banach G-contraction was introduced by Jachymaski [11] in complete metric space accompanied with the graph G where the set of vertex matches with the metric space (see also [12–22]).

The notion of metric space has been refined and extended in several distinct directions, by many authors [23–25]. Among all, the notion of gauge space was initiated by Dugundji [26] as a generalization of a metric space. In 1973, Reilly [27] studied quasi-gauge spaces and proved that it generalizes topological spaces, quasi uniform spaces, and quasi metric spaces. This notion was extended as b-gauge spaces by Ali et al. [28] in 2015. For further facts on gauge spaces, we recommend the reader to [29–36].

In 2013, Wlodarczyk and Plebaniak [37] have given the notion of left (right) $\mathcal{J}$-families of generalized pseudo distances in quasi-gauge spaces that generalizes the abovementioned distances and provides powerful and useful tools for finding solutions to various problems of nonlinear analysis.

This paper is aimed at introducing extended b-gauge spaces $(U, Q_{\varphi;\Omega})$ and the extended $\mathcal{J}_{\varphi;\Omega}$-family of generalized extended pseudo-b-distances generated by $(U, Q_{\varphi;\Omega})$. Moreover, by using extended $\mathcal{J}_{\varphi;\Omega}$-family, we define the extended $\mathcal{J}_{\varphi;\Omega}$-sequential completeness and construct the Caristi-type G-contractions $T : U \longrightarrow Cl^{\mathcal{J}_{\varphi;\Omega}}(U)$. Furthermore, we investigate periodic and fixed point results for these mappings in the new setting endowed with a graph, which generalizes and improves the existing results in the literature of fixed point theory.

In what follows, we recollect some essential concepts and basic results which shall be used in the sequel. For a nonempty set U, we use the notation 2^U to denote the set of all nonempty subsets of the space U. If $T : U \longrightarrow 2^U$ is a multivalued map, then the sets of all fixed points are denoted by Fix (T), that is, Fix $(T) = \{u \in U : u \in T(u)\}$. In addition, the set of all periodic points of T is denoted by Per (T), that is, Per $(T) = \{u \in U : u \in T^{[k]}(u) \text{forsome } k \text{ in } N\}$, where $T^{[k]} = T \circ T \circ T \circ \cdots .. \circ T$ (k-times). A dynamic process of the system (U, T) starting at $w^0 \in U$ is a sequence $\{w^m : m \in \{0\} \cup N\}$ defined by $\forall_{m \in N}\{w^m \in T(w^{m-1})\}$.

One of the most interesting extension of a metric is the notion of b-metric [38, 39].

Definition 1. Let U be a nonempty set and $s \geq 1$. A map $q : U \times U \longrightarrow [0,\infty)$ is b-metric, if it satisfies the following properties:

(a) $q(e, f) = 0 \Leftrightarrow e = f$

(b) $q(e, f) = q(f, e)$

(c) $q(e, g) \leq s\{q(e, f) + q(f, g)\}$

for all $e, f, g \in U$. Here, the pair (U, q, s) is called b-metric space.

Indeed, b-metric is one of the most interesting and original generalizations of the notion metric. As it is seen obviously, in the case of $s = 1$, the notions b-metric and standard metric coincide. On the other hand, despite the standard metric, b-metric is not continuous despite metric. Further, an open (closed) ball is not an open (closed) set. For more details on b-metric and interesting examples, we refer to, e.g., [40–49].

In 2017, Kamran et al. [50] refined the notion of b-metric under the name "extended b-metric."

Definition 2. Suppose U be a nonvoid set and let $\varphi : U \times U \longrightarrow [1,\infty)$. A map $q : U \times U \longrightarrow [0,\infty)$ is said to be an extended b -metric, if it satisfies the following properties:

(a) $q(e, f) = 0 \Leftrightarrow e = f$

(b) $q(e, f) = q(f, e)$

(c) $q(e, g) \leq \varphi(e, g)\{q(e, f) + q(f, g)\}$, for all $e, f, g \in U$

For given extended b-metric q on U, a pair (U, q) is called extended b-metric space.

Definition 3. Let U be a nonvoid set. The map $q : U \times U \longrightarrow [0,\infty)$ is called to be pseudo metric, if it satisfies the following properties:

(a) $q(e, e) = 0$

(b) $q(e, f) = q(f, e)$

(c) $q(e, g) \leq q(e, f) + q(f, g)$, for all $e, f, g \in U$

The pair (U, q) is said to be pseudo metric space.

In 2015, Ali et al. [28] have defined gauge spaces in the setting of b_s-pseudo metrics called b-gauge spaces. In order to introduce extended b-gauge spaces, we start here with the introduction of the notion of extended pseudo-b-metric.

Definition 4. Let U be a nonempty set and $\varphi : U \times U \longrightarrow [1,\infty)$. A map $q : U \times U \longrightarrow [0,\infty)$ is an extended pseudo-b -metric, if it satisfies the following properties:

(a) $q(e, e) = 0$

(b) $q(e, f) = q(f, e)$

(c) $q(e, g) \leq \varphi(e, g)\{q(e, f) + q(f, g)\}$, for all $e, f, g \in U$

The pair (U, q) is called extended pseudo-b-metric space.

Example 1. Let $U = [0, 1]$. Define $q : U \times U \longrightarrow [0,\infty)$ and $\varphi : U \times U \longrightarrow [1,\infty)$ for all $e, f \in U$ as follows:

$$\begin{aligned} q(e, f) &= (e - f)^2, \\ \varphi(e, f) &= e + f + 2, \end{aligned} \tag{2}$$

for all $e, f, g \in U$. Then, q is an extended pseudo-b-metric on U. Indeed, $q(e, e) = 0$ and $q(e, f) = q(f, e)$. Further, $q(e, g) \leq \varphi(e, g)\{q(e, f) + q(f, g)\}$ holds.

Example 2. Let $U = \{e, f, g\}$ and $\varphi : U \times U \longrightarrow [1,\infty)$ with $\varphi(e, f) = |e| + |f| + 2$. Define $q : U \times U \longrightarrow [0,\infty)$ as follows:

$$\begin{aligned} q(e, e) &= 0, \\ q(e, f) &= q(f, e) = 1, \\ q(f, g) &= q(g, f) = \frac{1}{2}, \\ \text{and } q(e, g) &= q(g, e) = 2, \end{aligned} \tag{3}$$

for all $e, f, g \in U$. Further, $q(e, g) \leq \varphi(e, g)\{q(e, f) + q(f, g)\}$ holds. In conclusion, q is an extended pseudo-b-metric on U. Notice that $2 = q(e, g) > 3/2 = q(e, f) + q(f, g)$; thus, q is not a pseudo metric on U. This example shows that extended pseudo-b-metric is more general than pseudo metric.

Definition 5. Each family $Q_{\varphi;\Omega} = \{q_\beta : \beta \in \Omega\}$ of extended pseudo-b-metrics $q_\beta : U \times U \longrightarrow [0,\infty)$, $\beta \in \Omega$, is called an extended b-gauge on U.

Definition 6. Let the family $Q_{\varphi;\Omega} = \{q_\beta : \beta \in \Omega\}$ be an extended b-gauge on U. The topology $\mathcal{T}(Q_{\varphi;\Omega})$ whose subbase is defined by the family $\mathcal{B}(Q_{\varphi;\Omega}) = \{B(e, \varepsilon_\beta): e \in U, \varepsilon_\beta > 0, \beta \in \Omega\}$ of all balls $B(e, \varepsilon_\beta) = \{f \in U : q_\beta(e, f) < \varepsilon_\beta\}$ is called the topology induced by $Q_{\varphi;\Omega}$ on U.

Definition 7. Suppose $(U, \mathcal{T})$ be a topological space and $Q_{\varphi;\Omega}$ be an extended b-gauge on U such that $\mathcal{T} = \mathcal{T}(Q_{\varphi;\Omega})$. Then, the topological space is called to be an extended b-gauge space, which is denoted by $(U, Q_{\varphi;\Omega})$.

Remark 8. (a) Each gauge space is b_s-gauge space (for $s = 1$), and each b-gauge space is an extended b-gauge space (for $\varphi_\beta(u, v) = s$, for each $\beta \in \Omega$, where $s \geq 1$). Therefore, we can term extended b-gauge spaces as the largest general spaces.

(b) We observe that if $\varphi_\beta(u,v)=s$, for each $\beta\in\Omega$, where $s\geq 1$, the above definitions turn down to the corresponding definitions in b-gauge spaces, and if $\varphi_\beta(u,v)=1$ for each $\beta\in\Omega$, the above definitions turn down to the corresponding definitions in gauge spaces.

We now introduce the notion of extended $\mathcal{J}_{\varphi;\Omega}$-families of generalized extended pseudo-b-distances on U (extended $\mathcal{J}_{\varphi;\Omega}$-families is the generalization of extended quasi-b-gauges).

Definition 9. Let $(U,Q_{\varphi;\Omega})$ be an extended b-gauge space. The family $\mathcal{J}_{\varphi;\Omega}=\{J_\beta:\beta\in\Omega\}$ where $J_\beta:U\times U\longrightarrow[0,\infty)$, $\beta\in\Omega$ is said to be the extended $\mathcal{J}_{\varphi;\Omega}$-family of generalized extended pseudo- b-distances on U (for short, extended $\mathcal{J}_{\varphi;\Omega}$-family on U) if there exists $\varphi=\{\varphi_\beta\}_{\beta\in\Omega}$, where $\varphi_\beta:U\times U\longrightarrow[1,\infty)$ such that for each $\beta\in\Omega$ and for all $u,v,w\in U$, the following hold:

(J1) $J_\beta(u,w)\leq\varphi_\beta(u,w)\{J_\beta(u,v)+J_\beta(v,w)\}$

(J2) For each sequences $(u_m:m\in N)$ and $(v_m:m\in N)$ in U fulfilling

$$\lim_{m\longrightarrow\infty}\sup_{n>m}J_\beta(u_m,u_n)=0, \tag{4}$$

$$\lim_{m\longrightarrow\infty}J_\beta(v_m,u_m)=0, \tag{5}$$

the following holds:

$$\lim_{m\longrightarrow\infty}q_\beta(v_m,u_m)=0. \tag{6}$$

We denote $J_{(U,Q_{\varphi;\Omega})}=\{\mathcal{J}_{\varphi;\Omega}:\mathcal{J}_{\varphi;\Omega}=\{J_\beta:\beta\in\Omega\}\}$.

We mention here some trivial properties of extended $\mathcal{J}_{\varphi;\Omega}$-families in the following remark.

Remark 10. Let $(U,Q_{\varphi;\Omega})$ be an extended b-gauge space. Then, for each $\beta\in\Omega$ and for all $u,v,w\in U$, the following hold:

(a) $Q_{\varphi;\Omega}\in J_{(U,Q_{\varphi;\Omega})}$

(b) Let $\mathcal{J}_{\varphi;\Omega}\in J_{(U,Q_{\varphi;\Omega})}$. If $J_\beta(v,v)=0$ and $J_\beta(u,v)=J_\beta(v,u)$, then J_β is an extended pseudo-b-metric

(c) There exist examples of $\mathcal{J}_{\varphi;\Omega}\in J_{(U,Q_{\varphi;\Omega})}$ which show that the maps J_β are not an extended pseudo-b-metrics (see following Example 3)

Example 3. Suppose $U=[0,1]\subset R$. Let $Q_{\varphi;\Omega}=\{q\}$ be the family of pseudo-b-metric where $q:U\times U\longrightarrow[0,\infty)$ be defined as in Example 1.

Let the set $F=[1/8,1]\subset U$. Let $d\in(0,\infty)^\Omega$ satisfies $\{\delta(F)<d\}$, where $\{\delta(F)=\sup\{q(e,f): e,f\in F\}\}$. Let $J:U\times U\longrightarrow[0,\infty)$ and $\varphi:U\times U\longrightarrow[1,\infty)$ for all $e,f\in U$ define as follows:

$$J(e,f)=\begin{cases} q(e,f) & \text{if } F\cap\{e,f\}=\{e,f\},\\ d=4 & \text{if } F\cap\{e,f\}\neq\{e,f\},\end{cases} \tag{7}$$

and $\varphi(e,f)=e+f+2$. Then, $\mathcal{J}_{\varphi;\Omega}=\{J\}\in J_{(U,Q)}$.

We observe that $J(e,g)\leq\varphi(e,g)\{J(e,f)+J(f,g)\}$ for all $e,f,g\in U$; thus, condition $(\mathcal{J}_1)$ holds. Indeed, condition $(\mathcal{J}_1)$ will not hold in case if there exists some $e,f,g\in U$ such that $J(e,g)=d$, $J(e,f)=q(e,f)$, $J(f,g)=q(f,g)$, and $\varphi(e,g)\{q(e,f)+q(f,g)\}\leq d$. However, then this implies the existence of $h\in\{e,g\}$ with $h\notin F$, and on other hand, $e,f,g\in F$, which is impossible.

Now suppose that (4) and (5) are satisfied by the sequences $(u_m:m\in N)$ and $(v_m:m\in N)$ in U. Then, (5) implies

$$\forall_{0<\varepsilon<d}\exists_{m_1\in N}\forall_{m\geq m_1}\{J(v_m,u_m)<\varepsilon\}. \tag{8}$$

By (8) and (7), we have

$$\begin{gathered}\forall_{m\geq m_1}\{F\cap\{v_m,u_m\}=\{v_m,u_m\}\},\\ \forall_{0<\varepsilon<d}\exists_{m_1\in N}\forall_{m\geq m_1}\{q(v_m,u_m)=J(v_m,u_m)<\varepsilon\}.\end{gathered} \tag{9}$$

Thus, (6) is satisfied by the sequences $(u_m:m\in N)$ and $(v_m:m\in N)$. Therefore, $\mathcal{J}_{\varphi;\Omega}$ is an extended $\mathcal{J}_{\varphi;\Omega}$-family on U.

Now, using extended $\mathcal{J}_{\varphi;\Omega}$-family on U, we establish the following concepts of extended $\mathcal{J}_{\varphi;\Omega}$-completeness in the extended b-gauge space $(U,Q_{\varphi;\Omega})$.

Definition 11. Let $(U,Q_{\varphi;\Omega})$ be an extended b-gauge space. Let $\mathcal{J}_{\varphi;\Omega}=\{J_\beta:\beta\in\Omega\}$ be the extended $\mathcal{J}_{\varphi;\Omega}$-family on U. A sequence $(u_m:m\in N)$ is extended $\mathcal{J}_{\varphi;\Omega}$-cauchy sequence in U if, for all $\beta\in\Omega$, $\lim_{m\longrightarrow\infty}\sup_{n>m}J_\beta(u_m,u_n)=0$.

Definition 12. Let $(U,Q_{\varphi;\Omega})$ be an extended b-gauge space. Let $\mathcal{J}_{\varphi;\Omega}=\{J_\beta:\beta\in\Omega\}$ be the extended $\mathcal{J}_{\varphi;\Omega}$-family on U. The sequence $(u_m:m\in N)$ is called to be extended $\mathcal{J}_{\varphi;\Omega}$-convergent to $u\in U$ if $\lim^{\mathcal{J}_{\varphi;\Omega}}_{m\longrightarrow\infty}u_m=u$, where

$$\lim^{\mathcal{J}_{\varphi;\Omega}}_{m\longrightarrow\infty}u_m=u\Leftrightarrow\lim_{m\longrightarrow\infty}J_\beta(u,u_m)=0=\lim_{m\longrightarrow\infty}J_\beta(u_m,u). \tag{10}$$

Definition 13. Let $(U,Q_{\varphi;\Omega})$ be an extended b-gauge space. Let $\mathcal{J}_{\varphi;\Omega}=\{J_\beta:\beta\in\Omega\}$ be the extended $\mathcal{J}_{\varphi;\Omega}$-family on U. If $S^{\mathcal{J}_{\varphi;\Omega}}_{(u_m:m\in N)}\neq\varnothing$, where $S^{\mathcal{J}_{\varphi;\Omega}}_{(u_m:m\in N)}=\{u\in U:\lim^{\mathcal{J}_{\varphi;\Omega}}_{m\longrightarrow\infty}u_m=u\}$. Then, the sequence $(u_m:m\in N)$ in U is extended $\mathcal{J}_{\varphi;\Omega}$-convergent in U.

Definition 14. Let $(U,Q_{\varphi;\Omega})$ be an extended b-gauge space. Let $\mathcal{J}_{\varphi;\Omega}=\{J_\beta:\beta\in\Omega\}$ be the extended $\mathcal{J}_{\varphi;\Omega}$-family on U.

The space $(U, Q_{\varphi;\Omega})$ is called $\mathcal{J}_{\varphi;\Omega}$-sequentially complete extended b-gauge space, if every extended $\mathcal{J}_{\varphi;\Omega}$-Cauchy in U is an extended $\mathcal{J}_{\varphi;\Omega}$-convergent in U.

Remark 15. Let $(U, Q_{\varphi;\Omega})$ be an extended b-gauge space.

(a) For each subsequence $(v_m : m \in N)$ of $(u_m : m \in N)$, where $(u_m : m \in N)$ is an extended $\mathcal{J}_{\varphi;\Omega}$-convergent in U, we have $S^{\mathcal{J}_{\varphi;\Omega}}_{(u_m:m\in N)} \subset S^{\mathcal{J}_{\varphi;\Omega}}_{(v_m:m\in N)}$

(b) We observe that if $\varphi_\beta(u, v) = s$ for all $\beta \in \Omega$, where $s \geq 1$ and $\mathcal{J}_{\varphi;\Omega} = Q_{\varphi;\Omega}$, the above definitions of completeness reduce to the corresponding definitions in b-gauge spaces (see [28])

Definition 16. Let $(U, Q_{\varphi;\Omega})$ be an extended b-gauge space. The map $T^{[k]} : U \longrightarrow U$, where $k \in N$ is called to be an extended $Q_{\varphi;\Omega}$-closed map on U if for each sequence $(x_m : m \in N)$ in $T^{[k]}(U)$, which is extended $Q_{\varphi;\Omega}$-converging in U, i.e., $S^{Q_{\varphi;\Omega}}_{(x_m:m\in N)} \neq \varnothing$ and its subsequences $(v_m : m \in N)$ and $(u_m : m \in N)$ satisfy $\forall_{m\in N}\{v_m \in T^{[k]}(u_m)\}$ has the property that there exists $w \in S^{Q_{\varphi;\Omega}}_{(x_m:m\in N)}$ such that $w \in T^{[k]}(w)$.

Definition 17. Let $(U, Q_{\varphi;\Omega})$ be an extended b-gauge space, and let $\mathcal{J}_{\varphi;\Omega} = \{J_\beta : \beta \in \Omega\}$ be the extended $\mathcal{J}_{\varphi;\Omega}$-family on U. A set $Y \in 2^U$ is a $\mathcal{J}_{\varphi;\Omega}$-closed in U if $Y = cl_U^{\mathcal{J}_{\varphi;\Omega}}(Y)$, where $cl_U^{\mathcal{J}_{\varphi;\Omega}}(Y)$, is the $\mathcal{J}_{\varphi;\Omega}$-closure in U, which indicates the set of all $x \in U$ for which there exists a sequence $(x_m : m \in N)$ in Y which $\mathcal{J}_{\varphi;\Omega}$-converges to x.

Define $Cl^{\mathcal{J}_{\varphi;\Omega}}(U) = \{Y \in 2^U : Y = cl_U^{\mathcal{J}_{\varphi;\Omega}}(Y)\}$. Thus, $Cl^{\mathcal{J}_{\varphi;\Omega}}(U)$ denotes the class of all nonempty $\mathcal{J}_{\varphi;\Omega}$-closed subsets of U.

Definition 18. Let $(U, Q_{\varphi;\Omega})$ be an extended b-gauge space, let $\mathcal{J}_{\varphi;\Omega} = \{J_\beta : \beta \in \Omega\}$ be the extended $\mathcal{J}_{\varphi;\Omega}$-family on U, and let, for each $\beta \in \Omega$, $u \in U$ and for all $V \in Cl^{\mathcal{J}_{\varphi;\Omega}}(U)$,

$$J_\beta(u, V) = \inf \{J_\beta(u, z) : z \in V\}. \tag{11}$$

Define on $Cl^{\mathcal{J}_{\varphi;\Omega}}(U)$ the distance $D_\beta^{\mathcal{J}_{\varphi;\Omega}}$ of Hausdorff type, where $D_\beta^{\mathcal{J}_{\varphi;\Omega}} : Cl^{\mathcal{J}_{\varphi;\Omega}}(U) \times Cl^{\mathcal{J}_{\varphi;\Omega}}(U) \longrightarrow [0,\infty)$, $\beta \in \Omega$ as follows:

$$D_\beta^{\mathcal{J}_{\varphi;\Omega}}(U, V) = \begin{cases} \max\left\{\sup_{u\in U} J_\beta(u, V), \sup_{v\in V} J_\beta(v, U)\right\}, & \text{if the maximum exists;} \\ \infty, & \text{otherwise} \end{cases} \tag{12}$$

for each $\beta \in \Omega$ and for all $U, V \in Cl^{\mathcal{J}_{\varphi;\Omega}}(U)$.

In this paper, Ω is a directed set and $(U, Q_{\varphi;\Omega})$ be an extended b-gauge space enriched with the graph $G = (V, E)$ where the set of vertices coincides with set U and the set of edges E contains $\{(v, v) : v \in V\}$. Also, G is such that no two edges are parallel.

2. Periodic and Fixed Point Theorems

Our main results for multivalued mappings are now given below.

Theorem 19. *Let $(U, Q_{\varphi;\Omega})$ be an extended b-gauge space. Let $\mathcal{J}_{\varphi;\Omega} = \{J_\beta : \beta \in \Omega\}$, where $J_\beta : U \times U \longrightarrow [0,\infty)$, be the extended $\mathcal{J}_{\varphi;\Omega}$-family on U such that $(U, Q_{\varphi;\Omega})$ is extended $\mathcal{J}_{\varphi;\Omega}$-sequentially complete. Let $T : U \longrightarrow Cl^{\mathcal{J}_{\varphi;\Omega}}(U)$ be a multivalued edge preserving map and $\phi_\beta : U \longrightarrow [0,\infty)$, $\beta \in \Omega$ be a lower semicontinuous function such that for each $u \in U$ and $v \in Tu$ where $(u, v) \in E$, we have, for all $\beta \in \Omega$,*

$$J_\beta(v, Tv) \leq \phi_\beta(u) - \phi_\beta(v). \tag{13}$$

Assume, moreover, that the following conditions hold:

(i) There exist $z^0 \in U$ and $z^1 \in Tz^0$ such that $(z^0, z^1) \in E$

(ii) For each $\{r_\beta : r_\beta > 1\}_{\beta\in\Omega}$ and $u \in U$, there exists $v \in Tu$ such that

$$J_\beta(u, v) \leq r_\beta J_\beta(u, Tu), \tag{14}$$

for all $\beta \in \Omega$. Then, the following statements hold:

(I) For any $z^0 \in U$, $(z^m : m \in \{0\} \cup N)$ is extended $Q_{\varphi;\Omega}$-convergent sequence in U; thus, $\forall_{z^0\in U}\{S^{Q_{\varphi;\Omega}}_{(z^m:m\in\{0\}\cup N)} \neq \varnothing\}$

(II) Furthermore, assume that $T^{[k]}$ for some $k \in N$ is an extended $Q_{\varphi;\Omega}$-closed map on U. Then,

(a_1) $Fix(T^{[k]}) \neq \varnothing$

(a_2) $\forall_{z^0\in U}\exists_{z\in Fix(T^{[k]})}\{z \in S^{Q_{\varphi;\Omega}}_{(z^m:m\in\{0\}\cup N)}\}$

Proof. (I) We first show that $(z^m : m \in \{0\} \cup N)$ is an extended $\mathcal{J}_{\varphi;\Omega}$-Cauchy sequence in U.

By assumption (i), there exists $z^0 \in U$ and $z^1 \in Tz^0$ such that $(z^0, z^1) \in E$. Now using (13), we can write, for each $\beta \in \Omega$,

$$J_\beta(z^1, Tz^1) \leq \phi_\beta(z^0) - \phi_\beta(z^1). \tag{15}$$

Now by using assumption (ii) and (15), we have $r_\beta > 1$ for each $\beta \in \Omega$ and $z^2 \in Tz^1$ such that

$$J_\beta(z^1, z^2) \le r_\beta J_\beta(z^1, Tz^1) \le r_\beta\left\{\phi_\beta(z^0) - \phi_\beta(z^1)\right\}. \quad (16)$$

As T is edge preserving, we can write $(z^1, z^2) \in E$. Proceeding in the same manner, we have a sequence $\{z^m : m \in \{0\} \cup N\}$ such that $(z^m, z^{m+1}) \in E$ and for each $m \in N$ and for all $\beta \in \Omega$, we have

$$J_\beta(z^m, z^{m+1}) \le r_\beta J_\beta(z^m, Tz^m) \le r_\beta\left\{\phi_\beta(z^{m-1}) - \phi_\beta(z^m)\right\}. \quad (17)$$

This implies that the sequence $\{\phi_\beta(z^m)\}$ is a nonincreasing sequence; hence, there exits $l_\beta \ge 0$ such that $\{\phi_\beta(z^m)\} \longrightarrow l_\beta$ as $m \longrightarrow \infty$. Now for $m, p \in N$ and each $\beta \in \Omega$, we have

$$\begin{aligned} J_\beta(z^m, z^{m+p}) \le{}& \varphi_\beta(z^m, z^{m+p}) J_\beta(z^m, z^{m+1}) \\ &+ \varphi_\beta(z^m, z^{m+p})\varphi_\beta(z^{m+1}, z^{m+p}) J_\beta(z^{m+1}, z^{m+2}) \\ &+ \varphi_\beta(z^m, z^{m+p})\varphi_\beta(z^{m+1}, z^{m+p})\varphi_\beta \\ &\cdot (z^{m+2}, z^{m+p}) J_\beta(z^{m+2}, z^{m+3}) \\ &+ \cdots + \varphi_\beta(z^m, z^{m+p})\varphi_\beta(z^{m+1}, z^{m+p}) \cdots \varphi_\beta \\ &\cdot (z^{m+p-1}, z^{m+p}) J_\beta(z^{m+p-1}, z^{m+p}) \\ \le{}& \varphi_\beta(z^m, z^{m+p}) r_\beta\left\{\phi_\beta(z^{m-1}) - \phi_\beta(z^m)\right\} \\ &+ \varphi_\beta(z^m, z^{m+p})\varphi_\beta(z^{m+1}, z^{m+p}) r_\beta \\ &\cdot \left\{\phi_\beta(z^m) - \phi_\beta(z^{m+1})\right\} \\ &+ \varphi_\beta(z^m, z^{m+p})\varphi_\beta(z^{m+1}, z^{m+p})\varphi_\beta \\ &\cdot (z^{m+2}, z^{m+p}) r_\beta\left\{\phi_\beta(z^{m+1}) - \phi_\beta(z^{m+2})\right\} \\ &+ \cdots + \varphi_\beta(z^m, z^{m+p})\varphi_\beta(z^{m+1}, z^{m+p}) \cdots \varphi_\beta \\ &\cdot (z^{m+p-1}, z^{m+p}) r_\beta\left\{\phi_\beta(z^{m+p-2}) - \phi_\beta(z^{m+p-1})\right\}. \end{aligned} \quad (18)$$

Letting $m \longrightarrow \infty$, we have $\{\phi_\beta(z^m)\} \longrightarrow l_\beta$. This implies that $(z^m : m \in \{0\} \cup N)$ is an extended $\mathcal{J}_{\varphi;\Omega}$-cauchy sequence in U, i.e., for all $\beta \in \Omega$ and for each $z^0 \in U$,

$$\forall_{\varepsilon>0} \exists_{k\in N} \forall_{n,m\in N; n>m\ge k}\left\{J_\beta(z^m, z^n) < \varepsilon\right\}. \quad (19)$$

Now, since $(U, Q_{\varphi;\Omega})$ is extended $\mathcal{J}_{\varphi;\Omega}$-sequentially complete b-gauge space, we have $(z^m : m \in \{0\} \cup N)$ extended $\mathcal{J}_{\varphi;\Omega}$-convergent in U, i.e., for all $z \in S^{\mathcal{J}_{\varphi;\Omega}}_{(z^m : m\in\{0\}\cup N)}$, we have, for all $\beta \in \Omega$ and for each $\varepsilon > 0$,

$$\exists_{k\in N} \forall_{m\in N; m\ge k}\left\{J_\beta(z, z^m) < \varepsilon\right\}. \quad (20)$$

Thus, from (19) and (20), fixing $z \in S^{\mathcal{J}_{\varphi;\Omega}}_{(z_m : m\in\{0\}\cup N}$, defining $(u_m = z^m : m \in \{0\} \cup N)$ and $(v_m = z : m \in \{0\} \cup N)$, and applying $(\mathcal{J}2)$ to these sequences, we get, for all $\beta \in \Omega$ and for each $\varepsilon > 0$,

$$\exists_{k\in N} \forall_{m\in N; m\ge k}\left\{q_\beta(z, z^m) < \varepsilon\right\}. \quad (21)$$

This implies $S^{Q_{\varphi;\Omega}}_{(z^m : m\in\{0\}\cup N)} \ne \varnothing$.

(II) To prove (a_1), let $z^0 \in U$ be arbitrary and fixed. Since $S^{Q_{\varphi;\Omega}}_{(z^m : m\in\{0\}\cup N)} \ne \varnothing$ and we have

$$z^{(m+1)k} \in T^{[k]}\left(z^{mk}\right), \text{for } m \in \{0\} \cup N, \quad (22)$$

thus defining $(z_m = z^{m-1+k} : m \in N)$, we can write

$$\begin{gathered} (z_m : m \in N) \subset T^{[k]}(U), \\ S^{Q_{\varphi;\Omega}}_{(z_m : m\in\{0\}\cup N)} = S^{Q_{\varphi;\Omega}}_{(z^m : m\in\{0\}\cup N)} \ne \varnothing. \end{gathered} \quad (23)$$

Also, its subsequences

$$\begin{gathered} \left(y_m = z^{(m+1)k} : m \in N\right) \subset T^{[k]}(U), \\ \left(x_m = z^{mk} : m \in N\right) \subset T^{[k]}(U), \end{gathered} \quad (24)$$

satisfy, for all $m \in N$,

$$y_m = T^{[k]}(x_m) \quad (25)$$

and are extended $Q_{\varphi;\Omega}$-convergent to each point $z \in S^{Q_{\varphi;\Omega}}_{(z^m : m\in\{0\}\cup N)}$. Now, using the fact below,

$$S^{Q_{\varphi;\Omega}}_{(z_m : m\in N)} \subset S^{Q_{\varphi;\Omega}}_{(y_m : m\in N)} \text{ and } S^{Q_{\varphi;\Omega}}_{(z_m : m\in N)} \subset S^{Q_{\varphi;\Omega}}_{(x_m : m\in N)}. \quad (26)$$

And the supposition that $T^{[k]}$ for some $k \in N$ is an extended $Q_{\varphi;\Omega}$-closed map on U, we have

$$\exists_{z\in S^{Q_{\varphi;\Omega}}_{(z_m : m\in\{0\}\cup N)} = S^{Q_{\varphi;\Omega}}_{(z^m : m\in\{0\}\cup N)}}\left\{z \in T^{[k]}(z)\right\}. \quad (27)$$

Thus, (a_1) holds. The assertion (a_2) follows from (a_1) and the fact that $S^{Q_{\varphi;\Omega}}_{(z^m : m\in\{0\}\cup N)} \ne \varnothing$. Hence, the theorem is proved.□

Theorem 20. *Let $(U, Q_{\varphi;\Omega})$ be an extended b-gauge space. Let $\mathcal{J}_{\varphi;\Omega} = \{J_\beta : \beta \in \Omega\}$, where $J_\beta : U \times U \longrightarrow [0,\infty)$, be the extended $\mathcal{J}_{\varphi;\Omega}$-family on U such that $(U, Q_{\varphi;\Omega})$ is extended $\mathcal{J}_{\varphi;\Omega}$-sequentially complete. Let $T : U \longrightarrow Cl^{\mathcal{J}_{\varphi;\Omega}}(U)$ be a multivalued edge preserving map and $\phi_\beta : U \longrightarrow [0,\infty)$, $\beta \in \Omega$ be a lower semicontinuous function such that for each $u \in$*

U and $v \in Tu$ *where* $(u, v) \in E$, *we have, for each* $\beta \in \Omega$,

$$J_\beta(u, v) \le \phi_\beta(u) - \phi_\beta(v). \tag{28}$$

Assume, moreover, that the following condition holds:

(i) There exist $z^0 \in U$ *and* $z^1 \in Tz^0$ *such that* $(z^0, z^1) \in E$

Then, the following statements hold:

(I) For any $z^0 \in U$, $(z^m : m \in \{0\} \cup N)$ *is extended* $Q_{\varphi;\Omega}$*-convergent sequence in U; thus,* $\forall_{z^0 \in U}\{S^{Q_{\varphi;\Omega}}_{(z^m : m\in\{0\}\cup N)} \neq \varnothing\}$

(II) Furthermore, assume that $T^{[k]}$ *for some* $k \in N$ *is an extended* $Q_{\varphi;\Omega}$*-closed map on U. Then,*

(b₁) $Fix(T^{[k]}) \neq \varnothing$

(b₂) $\forall_{z^0 \in U} \exists_{z \in Fix(T^{[k]})}\{z \in S^{Q_{\varphi;\Omega}}_{(z^m : m\in\{0\}\cup N)}\}$

Proof. (I) We first show that $(z^m : m \in \{0\} \cup N)$ is an extended $\mathcal{J}_{\varphi;\Omega}$-Cauchy sequence in U. By assumption (i), there exist $z^0 \in U$ and $z^1 \in Tz^0$ such that $(z^0, z^1) \in E$. Now using (28), we can write, for each $\beta \in \Omega$,

$$J_\beta(z^0, z^1) \le \phi_\beta(z^0) - \phi_\beta(z^1). \tag{29}$$

As T is edge preserving, we can write $(z^1, z^2) \in E$. Proceeding in the same manner, we have a sequence $\{z^m : m \in \{0\} \cup N\}$ such that $(z^m, z^{m+1}) \in E$ and for each $m \in N$ and for all $\beta \in \Omega$, we have

$$J_\beta(z^m, z^{m+1}) \le \phi_\beta(z^m) - \phi_\beta(z^{m+1}). \tag{30}$$

This implies that the sequence $\{\phi_\beta(z^m)\}$ is a nonincreasing sequence; hence, there exits $l_\beta \ge 0$ such that $\{\phi_\beta(z^m)\} \longrightarrow l_\beta$ as $m \longrightarrow \infty$. Now for $m, p \in N$ and each $\beta \in \Omega$, we have

$$\begin{aligned} J_\beta(z^m, z^{m+p}) \le{}& \varphi_\beta(z^m, z^{m+p}) J_\beta(z^m, z^{m+1}) + \varphi_\beta(z^m, z^{m+p})\varphi_\beta \\ &\cdot (z^{m+1}, z^{m+p}) J_\beta(z^{m+1}, z^{m+2}) + \varphi_\beta(z^m, z^{m+p})\varphi_\beta \\ &\cdot (z^{m+1}, z^{m+p})\varphi_\beta(z^{m+2}, z^{m+p}) J_\beta(z^{m+2}, z^{m+3}) \\ &+\cdots+\varphi_\beta(z^m, z^{m+p})\varphi_\beta(z^{m+1}, z^{m+p}) \cdots \varphi_\beta \\ &\cdot (z^{m+p-1}, z^{m+p}) J_\beta(z^{m+p-1}, z^{m+p}) \\ \le{}& \varphi_\beta(z^m, z^{m+p})\left\{\phi_\beta(z^m) - \phi_\beta(z^{m+1})\right\} \\ &+ \varphi_\beta(z^m, z^{m+p})\varphi_\beta(z^{m+1}, z^{m+p})\left\{\phi_\beta(z^{m+1}) - \phi_\beta(z^{m+2})\right\} \\ &+ \varphi_\beta(z^m, z^{m+p})\varphi_\beta(z^{m+1}, z^{m+p})\varphi_\beta(z^{m+2}, z^{m+p}) \\ &\cdot \left\{\phi_\beta(z^{m+2}) - \phi_\beta(z^{m+3})\right\} \\ &+\cdots+\varphi_\beta(z^m, z^{m+p})\varphi_\beta(z^{m+1}, z^{m+p}) \cdots \varphi_\beta(z^{m+p-1}, z^{m+p}) \\ &\cdot \left\{\phi_\beta(z^{m+p-1}) - \phi_\beta(z^{m+p})\right\}. \end{aligned} \tag{31}$$

Letting $m \longrightarrow \infty$, we have $\{\phi_\beta(z^m)\} \longrightarrow l_\beta$. This implies that $(z^m : m \in \{0\} \cup N)$ is an extended $\mathcal{J}_{\varphi;\Omega}$-cauchy sequence in U, i.e., for all $\beta \in \Omega$ and for each $z^0 \in U$,

$$\forall_{\varepsilon>0}\exists_{k\in N}\forall_{n,m\in N; n>m\ge k}\{J_\beta(z^m, z^n) < \varepsilon\}. \tag{32}$$

Now, since $(U, Q_{\varphi;\Omega})$ is extended $\mathcal{J}_{\varphi;\Omega}$-sequentially complete b-gauge space, we have $(z^m : m \in \{0\} \cup N)$ extended $\mathcal{J}_{\varphi;\Omega}$-convergent in U, i.e., for all $z \in S^{\mathcal{J}_{\varphi;\Omega}}_{(z^m : m\in\{0\}\cup N)}$, we have, for all $\beta \in \Omega$ and for each $\varepsilon > 0$,

$$\exists_{k\in N}\forall_{m\in N; m\ge k}\{J_\beta(z, z^m) < \varepsilon\}. \tag{33}$$

Thus, from (32) and (33), fixing $z \in S^{\mathcal{J}_{\varphi;\Omega}}_{(z_m : m\in\{0\}\cup N}$, defining $(u_m = z^m : m \in \{0\} \cup N)$ and $(v_m = z : m \in \{0\} \cup N)$, and applying $(\mathcal{J}2)$ to these sequences, we get, for all $\beta \in \Omega$ and for each $\varepsilon > 0$,

$$\exists_{k\in N}\forall_{m\in N; m\ge k}\left\{q_\beta(z, z^m) < \varepsilon\right\}. \tag{34}$$

This implies $S^{Q_{\varphi;\Omega}}_{(z^m : m\in\{0\}\cup N)} \neq \varnothing$.

(II) To prove (b_1), let $z^0 \in U$ be arbitrary and fixed. Since $S^{Q_{\varphi;\Omega}}_{(z^m : m\in\{0\}\cup N)} \neq \varnothing$ and we have

$$z^{(m+1)k} \in T^{[k]}\left(z^{mk}\right), \text{for } m \in \{0\} \cup N, \tag{35}$$

thus defining $(z_m = z^{m-1+k} : m \in N)$, we can write

$$\begin{gathered} (z_m : m \in N) \subset T^{[k]}(U), \\ S^{Q_{\varphi;\Omega}}_{(z_m : m\in\{0\}\cup N)} = S^{Q_{\varphi;\Omega}}_{(z^m : m\in\{0\}\cup N)} \neq \varnothing. \end{gathered} \tag{36}$$

Also, its subsequences

$$\begin{gathered} \left(y_m = z^{(m+1)k} : m \in N\right) \subset T^{[k]}(U), \\ \left(x_m = z^{mk} : m \in N\right) \subset T^{[k]}(U), \end{gathered} \tag{37}$$

satisfy, for all $m \in N$,

$$y_m = T^{[k]}(x_m), \tag{38}$$

and are extended $Q_{\varphi;\Omega}$-convergent to each point $z \in S^{Q_{\varphi;\Omega}}_{(z^m : m\in\{0\}\cup N)}$. Now, using the fact below,

$$S^{Q_{\varphi;\Omega}}_{(z_m : m\in N)} \subset S^{Q_{\varphi;\Omega}}_{(y_m : m\in N)} \text{ and } S^{Q_{\varphi;\Omega}}_{(z_m : m\in N)} \subset S^{Q_{\varphi;\Omega}}_{(x_m : m\in N)}. \tag{39}$$

And the supposition that $T^{[k]}$ for some $k \in N$ is an extended $Q_{\varphi;\Omega}$-closed map on U, we have

$$\exists_{z\in S^{Q_{\varphi;\Omega}}_{(z_m:m\in\{0\}\cup N)}=S^{Q_{\varphi;\Omega}}_{(z^m:m\in\{0\}\cup N)}}\left\{z\in T^{[k]}(z)\right\}. \quad (40)$$

Thus, (b_1) holds. The assertion (b_2) follows from (b_1) and the fact that $S^{Q_{\varphi;\Omega}}_{(z^m:m\in\{0\}\cup N)}\neq\varnothing$. Hence, the theorem is proved. □

Theorem 21. *Let $(U,Q_{\varphi;\Omega})$ be an extended b-gauge space. Let $\mathcal{J}_{\varphi;\Omega}=\{J_\beta:\beta\in\Omega\}$, where $J_\beta:U\times U\longrightarrow[0,\infty)$, be the extended $\mathcal{J}_{\varphi;\Omega}$-family on U such that $(U,Q_{\varphi;\Omega})$ is extended $\mathcal{J}_{\varphi;\Omega}$-sequentially complete. Let $T:U\longrightarrow Cl^{\mathcal{J}_{\varphi;\Omega}}(U)$ be a multivalued edge preserving map and $\psi_\beta:U\longrightarrow[0,\infty)$, $\beta\in\Omega$ be a upper semicontinuous function such that for each $u\in U$ and $v\in Tu$ where $(u,v)\in E$, we have, for each $\beta\in\Omega$,*

$$J_\beta(v,Tv)\leq\psi_\beta(u)-\psi_\beta(v). \quad (41)$$

Assume, moreover, that the following conditions hold:

(i) There exist $z^0\in U$ and $z^1\in Tz^0$ such that $(z^0,z^1)\in E$

(ii) For each $\{r_\beta:r_\beta>1\}_{\beta\in\Omega}$ and $x\in U$, there exists $y\in Tx$ such that for each $\beta\in\Omega$,

$$J_\beta(x,y)\leq r_\beta J_\beta(x,Tx). \quad (42)$$

Then, the following statements hold:

(I) For any $z^0\in U$, $(z^m:m\in\{0\}\cup N)$ is extended $Q_{\varphi;\Omega}$-convergent sequence in U; thus, $\forall_{z^0\in U}\{S^{Q_{\varphi;\Omega}}_{(z^m:m\in\{0\}\cup N)}\neq\varnothing\}$

(II) Furthermore, assume that $T^{[k]}$ for some $k\in N$ is an extended $Q_{\varphi;\Omega}$-closed map on U. Then,

(c_1) $Fix(T^{[k]})\neq\varnothing$

(c_2) $\forall_{z^0\in U}\exists_{z\in Fix(T^{[k]})}\{z\in S^{Q_{\varphi;\Omega}}_{(z^m:m\in\{0\}\cup N)}\}$

Proof. (I) We first show that $(z^m:m\in\{0\}\cup N)$ is an extended $\mathcal{J}_{\varphi;\Omega}$-Cauchy sequence in U. By assumption (i), there exist $z^0\in U$ and $z^1\in Tz^0$ such that $(z^0,z^1)\in E$. Now using (41), we can write, for each $\beta\in\Omega$,

$$J_\beta(z^1,Tz^1)\leq\psi_\beta(z^0)-\psi_\beta(z^1). \quad (43)$$

Now by using assumption (ii) and (43), we have $r_\beta>1$ for each $\beta\in\Omega$ and $z^2\in Tz^1$ such that

$$J_\beta(z^1,z^2)\leq r_\beta J_\beta(z^1,Tz^1)\leq r_\beta\left\{\psi_\beta(z^0)-\psi_\beta(z^1)\right\}. \quad (44)$$

As T is edge preserving, we can write $(z^1,z^2)\in E$. Proceeding as above, we have a sequence $\{z^m:m\in\{0\}\cup N\}$ such that $(z^m,z^{m+1})\in E$, and for each $m\in N$ and for all $\beta\in\Omega$, we have

$$J_\beta(z^m,z^{m+1})\leq r_\beta J_\beta(z^m,Tz^m)\leq r_\beta\left\{\psi_\beta(z^{m-1})-\psi_\beta(z^m)\right\}. \quad (45)$$

This implies that the sequence $\{\psi_\beta(z^m)\}$ is a nonincreasing sequence; hence, there exits $l_\beta\geq 0$ such that $\{\psi_\beta(z^m)\}\longrightarrow l_\beta$ as $m\longrightarrow\infty$. Now for $m,p\in N$ and each $\beta\in\Omega$, we have

$$\begin{aligned}J_\beta(z^m,z^{m+p})\leq{}&\varphi_\beta(z^m,z^{m+p})J_\beta(z^m,z^{m+1})+\varphi_\beta(z^m,z^{m+p})\varphi_\beta\\&\cdot(z^{m+1},z^{m+p})J_\beta(z^{m+1},z^{m+2})\\&+\varphi_\beta(z^m,z^{m+p})\varphi_\beta(z^{m+1},z^{m+p})\varphi_\beta\\&\cdot(z^{m+2},z^{m+p})J_\beta(z^{m+2},z^{m+3})\\&+\cdots+\varphi_\beta(z^m,z^{m+p})\varphi_\beta(z^{m+1},z^{m+p})\cdots\varphi_\beta\\&\cdot(z^{m+p-1},z^{m+p})J_\beta(z^{m+p-1},z^{m+p})\\\leq{}&\varphi_\beta(z^m,z^{m+p})r_\beta\left\{\psi_\beta(z^{m-1})-\psi_\beta(z^m)\right\}\\&+\varphi_\beta(z^m,z^{m+p})\varphi_\beta(z^{m+1},z^{m+p})r_\beta\\&\cdot\left\{\psi_\beta(z^m)-\psi_\beta(z^{m+1})\right\}+\varphi_\beta(z^m,z^{m+p})\varphi_\beta\\&\cdot(z^{m+1},z^{m+p})\varphi_\beta(z^{m+2},z^{m+p})r_\beta\\&\cdot\left\{\psi_\beta(z^{m+1})-\psi_\beta(z^{m+2})\right\}\\&+\cdots+\varphi_\beta(z^m,z^{m+p})\varphi_\beta(z^{m+1},z^{m+p})\cdots\varphi_\beta\\&\cdot(z^{m+p-1},z^{m+p})r_\beta\left\{\psi_\beta(z^{m+p-2})-\psi_\beta(z^{m+p-1})\right\}.\end{aligned} \quad (46)$$

Letting $m\longrightarrow\infty$, we have $\{\psi_\beta(z^m)\}\longrightarrow l_\beta$. This implies that $(z^m:m\in\{0\}\cup N)$ is an extended $\mathcal{J}_{\varphi;\Omega}$-cauchy sequence in U, i.e., for all $\beta\in\Omega$ and for each $z^0\in U$,

$$\forall_{\varepsilon>0}\exists_{k\in N}\forall_{n,m\in N;n>m\geq k}\{J_\beta(z^m,z^n)<\varepsilon\}. \quad (47)$$

Now, since $(U,Q_{\varphi;\Omega})$ is extended $\mathcal{J}_{\varphi;\Omega}$-sequentially complete b-gauge space, we have $(z^m:m\in\{0\}\cup N)$ extended $\mathcal{J}_{\varphi;\Omega}$-convergent in U, i.e., for all $z\in S^{\mathcal{J}_{\varphi;\Omega}}_{(z^m:m\in\{0\}\cup N)}$, we have, for all $\beta\in\Omega$ and for each $\varepsilon>0$,

$$\exists_{k\in N}\forall_{m\in N;m\geq k}\{J_\beta(z,z^m)<\varepsilon\}. \quad (48)$$

Thus, from (47) and (48), fixing $z\in S^{\mathcal{J}_{\varphi;\Omega}}_{(z_m:m\in\{0\}\cup N}$, defining $(u_m=z^m:m\in\{0\}\cup N)$ and $(v_m=z:m\in\{0\}\cup N)$, and applying $(\mathcal{J}2)$ to these sequences, we get, for all $\beta\in\Omega$ and for each $\varepsilon>0$,

$$\exists_{k\in N}\forall_{m\in N;m\geq k}\left\{q_\beta(z,z^m)<\varepsilon\right\}. \quad (49)$$

This implies $S^{Q_{\varphi;\Omega}}_{(z^m:m\in\{0\}\cup N)}\neq\varnothing$.

(II) To prove (c_1), let $z^0 \in U$ be arbitrary and fixed. Since $S^{Q_{\varphi;\Omega}}_{(z^m:m\in\{0\}\cup N)} \neq \varnothing$ and we have

$$z^{(m+1)k} \in T^{[k]}\left(z^{mk}\right), \text{ for } m \in \{0\} \cup N, \tag{50}$$

thus defining $(z_m = z^{m-1+k} : m \in N)$, we can write

$$\begin{gathered}(z_m : m \in N) \subset T^{[k]}(U), \\ S^{Q_{\varphi;\Omega}}_{(z_m:m\in\{0\}\cup N)} = S^{Q_{\varphi;\Omega}}_{(z^m:m\in\{0\}\cup N)} \neq \varnothing.\end{gathered} \tag{51}$$

Also, its subsequences

$$\begin{gathered}\left(y_m = z^{(m+1)k} : m \in N\right) \subset T^{[k]}(U), \\ \left(x_m = z^{mk} : m \in N\right) \subset T^{[k]}(U),\end{gathered} \tag{52}$$

satisfy, for all $m \in N$,

$$y_m = T^{[k]}(x_m), \tag{53}$$

and are extended $Q_{\varphi;\Omega}$-convergent to each point $z \in S^{Q_{\varphi;\Omega}}_{(z^m:m\in\{0\}\cup N)}$. Now, using the fact below,

$$S^{Q_{\varphi;\Omega}}_{(z_m:m\in N)} \subset S^{Q_{\varphi;\Omega}}_{(y_m:m\in N)} \text{ and } S^{Q_{\varphi;\Omega}}_{(z_m:m\in N)} \subset S^{Q_{\varphi;\Omega}}_{(x_m:m\in N)}. \tag{54}$$

And the supposition that $T^{[k]}$ for some $k \in N$ is an extended $Q_{\varphi;\Omega}$-closed map on U, we have

$$\exists_{z\in S^{Q_{\varphi;\Omega}}_{(z_m:m\in\{0\}\cup N)}=S^{Q_{\varphi;\Omega}}_{(z^m:m\in\{0\}\cup N)}}\left\{z \in T^{[k]}(z)\right\}. \tag{55}$$

Thus, (c_1) holds. The assertion (c_2) follows from (c_1) and the fact that $S^{Q_{\varphi;\Omega}}_{(z^m:m\in\{0\}\cup N)} \neq \varnothing$. Hence, the theorem is proved. □

Remark 22. (a) The fixed point results concerning Caristi-type contractions in gauge space in [51] require the completeness of the space (U, d). Therefore, our main theorems for Caristi-type G-contractions in the extended b-gauge space are a new generalization of the results in [51] in which assumptions are weaker and assertions are stronger.

(b) Our results for Caristi-type G-contractions in extended b-gauge space tell about periodic points as well, hence improve the results in [51]

(c) We observe that by taking $\forall_{\beta\in\Omega}\{\varphi_\beta(u, v) = s \geq 1\}$ in this paper, we obtain the results in b-gauge space.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

References

[1] J. Caristi, "Fixed point theorems for mapping satisfying inwardness conditions," *Transactions of the American Mathematical Society*, vol. 215, pp. 241–251, 1976.

[2] W. A. Kirk, "Caristis fixed point theorem and metric convexity," *Colloquium Mathematicum*, vol. 36, no. 1, pp. 81–86, 1976.

[3] J. D. Weston, "A characterization of metric completeness," *Proceedings of the American Mathematical Society*, vol. 64, no. 1, pp. 186–188, 1977.

[4] E. Karapnar, F. Khojasteh, and Z. D. Mitrović, "A proposal for revisiting Banach and Caristi type theorems in b-metric spaces," *Mathematics*, vol. 7, no. 4, p. 308, 2019.

[5] W. S. Du and E. Karapnar, "A note on Caristi-type cyclic maps: related results and applications," *Fixed Point Theory and Applications*, vol. 2013, Article ID 344, 2013.

[6] E. Karapnar, "Generalizations of Caristi Kirk's theorem on partial metric spaces," *Fixed Point Theory and Applications*, vol. 2011, Article ID 4, 2011.

[7] F. Khojasteh, E. Karapnar, and H. Khandani, "Some applications of Caristi's fixed point theorem in metric spaces," *Fixed Point Theory and Applications*, vol. 2016, Article ID 16, 2016.

[8] O. Alqahtani and E. Karapnar, "A bilateral contraction via simulation function," *Univerzitet u Nišu*, vol. 33, no. 15, pp. 4837–4843, 2019.

[9] A. Pant, R. P. Pant, and V. Rakocević, "Meir-Keeler type and Caristi type fixed point theorems," *Applicable Analysis and Discrete Mathematics*, vol. 13, no. 3, pp. 849–858, 2019.

[10] E. Karapnar, F. Khojasteh, and W. Shatanawi, "Revisiting Círić-type contraction with Caristi's approach," *Symmetry*, vol. 11, no. 6, p. 726, 2019.

[11] J. Jachymski, "The contraction principle for mappings on a metric space with a graph," *Proceedings of the American Mathematical Society*, vol. 136, pp. 1359–1373, 2008.

[12] M. Samreen and T. Kamran, "Fixed point theorems for integral G-contractions," *Fixed Point Theory and Applications*, vol. 2013, 11 pages, 2013.

[13] T. Kamran, M. Samreen, and N. Shahzad, "Probabilistic G -contractions," *Fixed Point Theory and Applications*, vol. 2013, 14 pages, 2013.

[14] M. Samreen, T. Kamran, and N. Shahzad, "Some fixed point theorems in b-metric space endowed with graph," *Abstract and applied analysis*, vol. 2013, 9 pages, 2013.

[15] J. Tiammee and S. Suantai, "Coincidence point theorems for graph-preserving multi-valued mappings," *Fixed Point Theory and Applications*, vol. 2014, 11 pages, 2014.

[16] A. Nicolae, D. O. Regan, and A. Petrusel, "Fixed point theorems for single-valued and multivalued generalized contractions in metric spaces endowed with a graph," *Georgian Mathematical Journal*, vol. 18, no. 2, pp. 307–327, 2011.

[17] D. Tataru, "Viscosity solutions of Hamilton-Jacobi equations with unbounded nonlinear terms," *Journal of Mathematical Analysis and Applications*, vol. 163, no. 2, pp. 345–392, 1992.

[18] F. Bojor, "Fixed point of ϕ-contraction in metric spaces endowed with a graph," *Annals of the University of Craiova-*

Mathematics and Computer Science Series, vol. 37, pp. 85–92, 2010.

[19] F. Bojor, "Fixed point theorems for Reich type contractions on metric spaces with a graph," *Nonlinear Analysis*, vol. 75, no. 9, pp. 3895–3901, 2012.

[20] F. Bojor, "Fixed points of Kannan mappings in metric spaces endowed with a graph," *Analele Universitatii" Ovidius" Constanta-Seria Matematica*, vol. 20, pp. 31–40, 2012.

[21] J. H. Asl, B. Mohammadi, S. Rezapour, and S. M. Vaezpour, "Some fixed point results for generalized quasi-contractive multifunctions on graphs," *Univerzitet u Nišu*, vol. 27, pp. 311–315, 2013.

[22] S. M. A. Aleomraninejad, S. Rezapour, and N. Shahzad, "Some fixed point results on a metric space with a graph," *Topology and its Applications*, vol. 159, no. 3, pp. 659–663, 2012.

[23] O. Kada, T. Suzuki, and W. Takahashi, "Nonconvex minimization theorems and fixed point theorems in complete metric spaces," *Mathematica Japonica*, vol. 44, no. 2, pp. 381–391, 1996.

[24] T. Suzuki, "Generalized distance and existence theorems in complete metric spaces," *Journal of Mathematical Analysis and Applications*, vol. 253, no. 2, pp. 440–458, 2001.

[25] L. J. Lin and W. S. Du, "Ekeland's variational principle, minimax theorems and existence of nonconvex equilibria in complete metric spaces," *Journal of Mathematical Analysis and Applications*, vol. 323, no. 1, pp. 360–370, 2006.

[26] J. Dugundji, *Topology, Series in Advanced Mathematics*, Allyn and Bacon, Inc, Boston, 1966.

[27] I. L. Reily, "Quasi-gauge spaces," *Journal of the London Mathematical Society*, vol. 6, pp. 481–487, 1973.

[28] M. U. Ali, T. Kamran, and M. Postolache, "Fixed point theorems for multivalued G-contractions in Housdorff b-gauge space," *Journal of Nonlinear Sciences and Applications*, vol. 8, no. 5, pp. 847–855, 2015.

[29] R. P. Agarwal, Y. J. Cho, and D. O. Regan, "Homotopy invariant results on complete gauge spaces," *Bulletin of the Australian Mathematical Society*, vol. 67, no. 2, pp. 241–248, 2003.

[30] M. Frigon, "Fixed point results for generalized contractions in gauge spaces and applications," *Proceedings of the American Mathematical Society*, vol. 128, no. 10, pp. 2957–2966, 2000.

[31] A. Chis and R. Precup, "Continuation theory for general contractions in gauge spaces," *Fixed Point Theory and Applications*, vol. 3, 185 pages, 2004.

[32] C. Chifu and G. Petrusel, "Fixed point results for generalized contractions on ordered gauge spaces with applications," *Fixed Point Theory and Applications*, vol. 2011, 10 pages, 2011.

[33] T. Lazara and G. Petrusel, "Fixed points for non-self operators in gauge spaces," *Journal of Nonlinear Sciences and Applications*, vol. 6, no. 1, pp. 29–34, 2013.

[34] M. Cherichi, B. Samet, and C. Vetro, "Fixed point theorems in complete gauge spaces and applications to second ordernonlinear initial value problems," *Journal of Function Spaces and Applications*, vol. 2013, pp. 1–8, 2013.

[35] M. Cherichi and B. Samet, "Fixed point theorems on ordered gauge spaces with applications to nonlinear integral equations," *Fixed Point Theory and Applications*, vol. 2012, 19 pages, 2012.

[36] M. Jleli, E. Karapnar, and B. Samet, "Fixed point results for a $\alpha - \psi_\lambda$-contractions on gauge spaces and applications," *Abstract and Applied Analysis*, vol. 2013, 7 pages, 2013.

[37] K. Wlodarczyk and R. Plebaniak, "New completeness and periodic points of discontinuous contractions of Banach-type in quasi-gauge spaces without Hausdorff property," *Fixed Point Theory and Applications*, vol. 2013, 27 pages, 2013.

[38] I. A. Bakhtin, "The contraction mapping principle in almost metric spaces," *Functional Analysis*, vol. 30, pp. 26–37, 1989.

[39] S. Czerwik, "Contraction mappings in b-metric spaces," *Acta Mathematica et Informatica Universitatis Ostraviensis*, vol. 1, pp. 5–11, 1993.

[40] S. Gulyaz-Ozyurt, "On some alpha-admissible contraction mappings on Branciari b-metric spaces," *Advances in the Theory of Nonlinear Analysis and its Applications*, vol. 1, no. 1, pp. 1–13, 2017.

[41] E. Karapnar, A. Fulga, and A. Petrusel, "On Istratescu type contractions in b-metric spaces," *Mathematics*, vol. 8, no. 3, p. 388, 2020.

[42] M. A. Alghamdi, S. Gulyaz-Ozyurt, and E. Karapnar, "A note on extended Z-contraction," *Mathematics*, vol. 8, no. 2, p. 195, 2020.

[43] H. Aydi, E. Karapnar, M. F. Bota, and S. Mitrovic, "A fixed point theorem for set-valued quasi-contractions in b-metric spaces," *Fixed Point Theory and Applications*, vol. 2012, 8 pages, 2012.

[44] H. Aydi, M. F. Bota, E. Karapnar, and S. Moradi, "A common fixed point for weak phi-contractions on b-metric spaces," *Fixed Point Theory*, vol. 13, no. 2, pp. 337–346, 2012.

[45] H. Afshari, H. Aydi, and E. Karapnar, "On generalized α-ψ -Geraghty contractions on b-metric spaces," *Georgian Mathematical Journal*, vol. 27, no. 1, pp. 9–21, 2020.

[46] E. Karapnar and C. Chifu, "Results in wt-distance over b -metric spaces," *Mathematics*, vol. 8, no. 2, p. 220, 2020.

[47] H. Alsulami, S. Gulyaz, E. Karapnar, and I. Erhan, "An Ulam stability result on quasi-b-metric-like spaces," *Open Mathematics*, vol. 14, no. 1, pp. 1087–1103, 2016.

[48] A. Fulga, E. Karapnar, and G. Petrusel, "On hybrid contractions in the context of quasi-metric spaces," *Mathematics*, vol. 8, no. 5, p. 675, 2020.

[49] U. Aksoy, E. Karapnar, and I. M. Erhan, "Fixed points of generalized alpha-admissible contractions on b-metric spaces with an application to boundary value problems," *Journal of Nonlinear and Convex Analysis*, vol. 17, no. 6, pp. 1095–1108, 2016.

[50] T. Kamran, M. Samreen, and Q. U. Ain, "A generalization of b -metric space and some fixed point theorems," *Mathematics*, vol. 5, no. 2, pp. 1–7, 2017.

[51] M. U. Ali, T. Kamran, and Q. Kiran, "Fixed point theorems for set valued Caristi type contractions on gauge spaces," *Communications in Optimization Theory*, vol. 6, 2015.