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Research Article

On Extended Convex Functions via Incomplete Gamma Functions

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Convex functions play an important role in many areas of mathematics. They are especially important in the study of optimization problems where they are distinguished by a number of convenient properties. In this paper, firstly we introduce the notion of h-exponential convex functions. This notion can be considered as generalizations of many existing definitions of convex functions. Then, we establish some well-known inequalities for the proposed notion via incomplete gamma functions. Precisely speaking, we established trapezoidal, midpoint, and He's inequalities for h-exponential and harmonically exponential convex functions via incomplete gamma functions. Moreover, we gave several remarks to prove that our results are more generalized than the existing results in the literature.

1. Introduction

Convex optimization contributed largely in many areas of pure and applied mathematics during recent years, and convex analysis provides main foundation for convex optimization [1, 2]. Due to huge applications of convex analysis, the researchers always show interest to generalization the notion of convexity. In literature, there exist many versions of convex functions, for example, h-convex function, see [3], r-convex functions, see [4], harmonic convex function, see [5], exponentially convex functions, see [6], etc. [7, 8].

Since convex function is a class of very important functions which is widely used in pure mathematics, functional analysis, optimization theory, and mathematical economics, so to study properties of certain classes of convex functions and establish different inequalities like trapezoidal, midpoint, He's Hermite-Hadamard, Fejér, etc., type inequality is an important area of research. A lot of work is devoted to establish different kinds of inequalities for different classes of convex functions, for example, Iscan [9] established Hermite-Hadamard type inequalities for harmonically convex functions. Bai et al. [10] presented Hermite-Hadamard type inequalities for the m and (α, m)-logarithmically convex functions. Özdemir et al. [11] developed Hermite–Hadamard-type inequalities via (α, m)-convex functions. Chu et al. [12] gave generalizations of Hermite-Hadamard type inequalities for MT-convex functions.

It is always appreciable to derive more version of inequalities for generalized convexities. For some important generalization, we refer [13, 14]. Fractional calculus also provides some broader variety to deal real-world problems. Just like other fields, fractional calculus also sets new trends in inequalities of convex analysis. For more details on fractional integral inequalities, we refer to the readers [15–18]. Many interesting controversies are also part of history of fractional calculus. Some famous definitions of fractional derivative are Riemann-Liouville [19], Caputo-Fabrizio [20], etc. [21–24]. In the present paper, we will deal with incomplete gamma functions. Firstly, we introduce the notions of h-exponential convex functions and harmonically exponential convex functions. Then, we establish some well-known inequalities for the proposed notions via incomplete gamma functions. Precisely speaking, we established trapezoidal, midpoint, and He's inequalities for h-exponential and harmonically exponential convex functions via incomplete gamma functions. Moreover, we gave several remarks to prove that our results are more generalized than the existing results in the literature.

The breakup of this paper is as follows: In Section 2, we present basic definitions and known results. Section 3 contains trapezoidal type inequalities via incomplete gamma function. Midpoint inequalities via incomplete gamma function are presented in Section 4, and He's inequality via the incomplete gamma functions is presented in Section 5. Last section contains concluding remarks and some future directions.

2. Preliminaries

Before starting the main findings, we review some definitions, notations, and theorems which are necessary to proceed. Throughout this paper, L^1 denotes space of all locally integrable functions.

Definition 1 [19]. For any L^1 function $z(u)$ on an interval $[x, y]$ with $u \in [x, y]k$-th left-RL fractional integral of $z(u)$ is given by

$$ {}^{\mathrm{RL}}J^k_{a^+} z(u) = \frac{1}{\Gamma(k)} \int_x^u (u-t)^{k-1} z(t)dt, \tag{1}$$

for Re $(k) > 0$. Also, the k-th right- RL fractional integral of $z(u)$ is given by

$$ {}^{RL}J^k_x z(u) = \frac{1}{\Gamma(k)} \int_u^y (t-u)^{k-1} z(t)dt. \tag{2}$$

Definition 2 [6]. We say that the function $z : M \subseteq R \longrightarrow R$ is exponential type convex on M if

$$ z(tx + (1-t)y) \le \left(e^t - 1\right) z(x) + \left(e^{1-t} - 1\right) z(y), \tag{3}$$

holds for every $x, y \in M$ and $t \in [0, 1]$.

Definition 3 [3]. We say that the function z: $M \subseteq R \longrightarrow$ is h-convex function on M if

$$ z(tx + (1-t)y) \le h(t)z(x) + h(1-t)z(y), \tag{4}$$

where $x, y \in M$ and $t \in [0, 1]$.

We are now ready to define some new convexity, called as h-exponential convex function.

Definition 4. We say that the function z: $M \subseteq R \longrightarrow$ is h-exponential type convex on M if

$$ z(tx + (1-t)y) \le h\left(e^t - 1\right) z(x) + h\left(e^{1-t} - 1\right) z(y), \tag{5}$$

where $x, y \in M$ and $t \in [0, 1]$.

Remark 5.

(1) By substituting $h(e^t - 1) = 1/(e^t - 1)$, $h(e^{1-t} - 1) = 1/(e^{1-t} - 1$ in Definition 3, we get harmonically exponential convex function

(2) By substituting $h(e^t - 1) = e^t - 1$, $h(e^{1-t} - 1) = e^{1-t} - 1$ in Definition 3, we get Definition 2 of exponential convex function

Now, the integral inequality of Hermite-Hadamard (HH) type for a convex function is give by

$$ z\left(\frac{x+y}{2}\right) \le \frac{1}{y-x} \int_x^y z(x)dx \le \frac{z(x) + z(y)}{2}. \tag{6}$$

Sarikaya et al. [25] generalized the HH-inequality (6) to fractional integrals of RL type which is given by

$$ z\left(\frac{x+y}{2}\right) \le \frac{\Gamma(k+1)}{2(y-x)^k} \left[{}^{RL}J^k_{x^+} z(y) + {}^{RL}J^k_{y^-} z(x) \right] \le \frac{z(x) + z(y)}{2}, \tag{7}$$

where $k > 0$ and $z[x, y] \longrightarrow R$ is let to be an L^1 convex function. After that, Sarikaya and Yildirim [26] found a new inequality of the above

$$ \begin{aligned} z\left(\frac{x+y}{2}\right) &\le \frac{2^{k-1}\Gamma(k+1)}{(y-x)^k} \left[{}^{\mathrm{RL}}J^k_{\left(\frac{x+y}{2}\right)^+} z(y) + {}^{RL}J^k_{\left(\frac{x+y}{2}\right)^-} z(x) \right] \\ &\le \frac{z(x+y)}{2}. \end{aligned} \tag{8}$$

The following facts will be needed in establishing our main results:

Remark 6 (21). For Re > 0, the following identities hold:

$$ \begin{aligned} &\int_0^1 t^{k-1} e^t dt = (-1)^k \gamma(k,-1); \\ &\int_0^1 t^{k-1} e^{1-t} dt = \gamma(k, 1), \\ &\gamma(k, x) = \int_0^x t^{k-1} e^{-t} dt, x \in C. \end{aligned} \tag{9}$$

Remark 7 (21). For Re > 0, the following identities hold:

$$ \int_0^1 t^{k-1} e^{t/2} dt = (-2)^k \gamma\left(k, \frac{-1}{2}\right); \tag{10}$$

$$ \int_0^1 t^{k-1} e^{1-(t/2)} dt = e2^k \gamma\left(k, \frac{1}{2}\right). \tag{11}$$

Lemma 8 [25]. *If $z:[x,y]\longrightarrow R$ is $L^1[x,y]$ with $0<x<y$ and $k>0$, then we have*

$$\frac{z(x)+z(y)}{2}-\frac{\Gamma(k+1)}{(y-x)^k}\left[{}^{RL}J_{x^+}z(y)+{}^{RL}J_{y^-}^{k}z(x)\right.\\ =\frac{y-x}{2}\int_0^1\left[(1-t)^k-t^k\right]z(tx+(1-t)y)dt. \quad (12)$$

Lemma 9 [26]. *If $z:[x,y]\longrightarrow R$ is $L^1[x,y]$ with $0<x<y$ and $k>0$, then we have*

$$\frac{2^{k-1}\Gamma(k+1)}{(y-x)^k}\left[{}^{RL}J^k_{((x+y)/2)^+}z(y)+{}^{RL}J^k_{((x+y)/2)^-}z(x)-z\left(\frac{x+y}{2}\right)\right]\\ =\frac{y-x}{4}\left[\int_0^1 e^t z\left(\frac{t}{2}x+\frac{2-t}{2}y\right)dt-\int_0^1 t^k z\left(\frac{2-t}{2}x+\frac{t}{2}y\right)dt\right]. \quad (13)$$

3. Trapezoidal Type Inequalities via Incomplete Gamma Function

In this section, we present trapezoidal type inequalities via incomplete gamma function.

Theorem 10. *Suppose that $z:[x,y]\longrightarrow R$ is $L^1[x,y]$ and h-exponential convex function, then we have for $k>0$,*

$$z\left(\frac{x+y}{2}\right)\le\frac{kh\left(e^{1/2}-1\right)\Gamma(k+1)}{(y-x)^k}\left[{}^{RL}J^k_x z(y)+{}^{RL}J^k_y z(x)\right]\\ \le h\left(e^{1/2}-1\right)M[z(x)+z(y)], \quad (14)$$

where $h(e^t+e^{1-t}-2)\le M$.

Proof. Let $z:I\longrightarrow R$ is h-exp convex function and $k>0$ then by definition

$$z\left(\frac{x+y}{2}\right)=z\left[\frac{(tx+(1-t)y)+(1-t)x+ty}{2}\right]\\ \le h\left(e^{1/2}-1\right)z(tx+(1-t)y)\\ +h\left(e^{1/2}-1\right)z((1-t)x+ty)). \quad (15)$$

Multiplying t^{k-1} on both sides and then integrating on $[0,1]$, we get

$$\frac{1}{k}z\left(\frac{x+y}{2}\right)\le h\left(e^{1/2}-1\right)\int_0^1 t^{k-1}z(tx+(1-t)y))dt\\ +h\left(e^{1/2}-1\right)\times\int_0^1 t^{k-1}z((1-t)x+ty))dt. \quad (16)$$

Again by small substitution, we have

$$z\left(\frac{x+y}{2}\right)\le kh\left(e^{(1/2)-1}\right)\frac{1}{x-y}\left[\int_y^x\left(\frac{y-u}{y-x}\right)^{k-1}z(u)d(u)\right.\\ \left.+\int_x^y\left(\frac{v-x^{k-1}}{y-x}\right)z(v)d(v)\right]\\ \le kh\left(e^{1/2}-1\right)\left[\frac{1}{x-y}\int_y^x\left(\frac{y-u}{y-x}\right)^{k-1}z(u)d(u)\right.\\ \left.+\frac{1}{x-y}\int_x^y\left(\frac{v-x}{y-x}\right)^{k-1}z(v)d(v)\right]\\ \le\frac{\Gamma(k+1)h\left(e^{1/2}-1\right)}{(x-y)^k}\left[{}^{\mathrm{RL}}J^k_{x^+}z(y)+{}^{\mathrm{RL}}J^k_{y^-}z(x)\right]. \quad (17)$$

For other inequalities, take

$$z(tx+(1-t)y)\le h\left(e^t-1\right)z(x)+h\left(e^{1-t}-1\right)z(y)\,;\\ z((1-t)x+ty)\le h\left(e^{1-t}-1\right)z(x)+h\left(e^t-1\right)z(y). \quad (18)$$

Adding both inequalities, we get

$$z(tx+(1-t)y)+z((1-t)x+ty)\le h\left(e^t+e^{1-t}-2\right)[z(x)+z(y)]. \quad (19)$$

Multiplying both sides by t^{k-1} and integrating on [0,1], we have

$$\int_0^1 t^{k-1}z(tx+(1-t)y)dt+\int_0^1 t^{k-1}z((1-t)x+ty)dt\\ \le\int_0^1 t^{k-1}h\left(e^t+e^{1-t}-2\right)dt[z(x)+z(y)]. \quad (20)$$

By making the change of variables

$$\frac{1}{x-y}\int_y^x\left(\frac{y-u}{y-x}\right)^{k-1}z(u)d(u)+\frac{1}{x-y}\int_y^x\left(\frac{v-x}{y-x}\right)^{k-1}z(v)d(v)\\ \le[z(x)+z(y)]\int_0^1 t^{k-1}h\left(e^t+e^{1-t}-2\right)dt,\\ \frac{\Gamma(k)}{(x-y)^k}\left[{}^{\mathrm{RL}}J^k_{x^+}z(y)+{}^{\mathrm{RL}}J^k_{y^-}z(x)\right]\\ \le[z(x)+z(y)]\int_0^1 t^{k-1}h\left(e^t+e^{1-t}-2\right)dt. \quad (21)$$

Multiplying by $k>0$ and $h(e^{1/2}-1)$ on both sides, we get.

$$\frac{\Gamma(k+1)h(e^{1/2}-1)}{(x-y)^k}\left[{}^{\mathrm{RL}}J^k_{x^+}z(y)+{}^{\mathrm{RL}}J^k_{y^-}z(x)\right] \leq kh\left(e^{1/2}-1\right)[z(x)+z(y)]\int_0^1 t^{k-1}h\left(e^t+t^{1-t}-2\right)dt, \tag{22}$$

$$\frac{\Gamma(k+1)h\left(e^{1/2}-1\right)}{(x-y)^k}\left[{}^{\mathrm{RL}}J^k_{x^+}z(y)+{}^{\mathrm{RL}}J^k_{y^-}z(x)\right] \leq h\left(e^{1/2}-1\right)M[z(x)+z(y)]. \tag{23}$$

□

Corollary 11. *If we substitute $h(e^t+e^{1-t}-2)=(e^t+e^{1-t}-2)$ in (22) and use Remark 6, then both of inequalities (17) and (22) become (7) of [27].*

Remark 12. For $h(e^t+e^{1-t}-2)=1/(e^t+e^{1-t}-2)$, (22) yields trapezoidal type inequalities via the incomplete gamma function for harmonically exponential convex function.

Theorem 13. *Let $z:[x,y]\longrightarrow R$ be $L^1[x,y]$ with $0<x<y$ and $k>0$. If $|z|$ is an h-exp convex function, then we*

$$\begin{aligned} z\left(\frac{x+y}{2}\right) &\leq \frac{h\left(e^{1/2}-1\right)\Gamma(k+1)}{(x-y)^k}\left[{}^{RL}J^k_{x^+}z(y)+{}^{RL}J^k_{y^-}z(x)\right] \\ &\leq \frac{y-x}{2}\left([\delta_0(k,h_0,h)]+\delta_1(k,h_1,h)]|z(x)|\right. \\ &\quad+[\delta_0(k,h_1,h)+\delta_1(k,h_0,h)]|z(y)| \\ &\quad+[\delta_0(k,h_0,h)+\delta_0(k,h_1,h)]|z(x)| \\ &\quad+[\delta_1(k,h_0,h)+\delta_0(k,h_0,h)]|z(y)|, \end{aligned} \tag{24}$$

where

$$\begin{cases} \delta_0(k.h_0,h)=\int_0^{\frac{1}{2}}(1-t)^k h\left(e^t-1\right)dt=\int_{\frac{1}{2}}^1 t^k h\left(e^{1-t}-1\right)dt\,; \\ \delta_0(k,h_1,h)=\int_0^{\frac{1}{2}}(1-t)^k h\left(e^{1-t}-1\right)dt=\int_{\frac{1}{2}}^1 t^k h\left(e^t-1\right)dt\,; \\ \delta_1(k,h_0,h)=-\int_0^{\frac{1}{2}} t^k h\left(e^{1-t}-1\right)dt=-\int_{\frac{1}{2}}^1(1-t)^k h\left(e^t-1\right)dt\,; \\ \delta_1(k,h_1,h)=-\int_0^{\frac{1}{2}} t^k h\left(e^t-1\right)dt=-\int_{\frac{1}{2}}^1(1-t)^k h\left(e^{1-t}-1\right)dt. \end{cases} \tag{25}$$

Proof. From Lemma 8, we have

$$\begin{aligned} &\left|\frac{z(x)+z(y)}{2}-\frac{\Gamma(k+1)}{2(y-x)^k}\left[{}^{\mathrm{RL}}J^k_{x^+}z(y)+{}^{\mathrm{RL}}J^k_{y^-}z(x)\right]\right| \\ &\quad\leq \frac{y-x}{2}\int_0^1\left((1-t)^k-t^k\right)|z(tx+(1-t)y|dt \\ &\quad= \frac{y-x}{2}\left[\int_0^{\frac{1}{2}}\left((1-t)^k-t^k\right)|z(tx+(1-t))y|dt\right] \\ &\qquad+\left[\int_{\frac{1}{2}}^1\left(t^k-(1-t)^k\right)|z(tx+(1-t))y|dt\right]. \end{aligned} \tag{26}$$

By using the h-exp convexity of $|z|$

$$\begin{aligned} &\left|\frac{z(x)+z(y)}{2}-\frac{\Gamma(k+1)}{2(y-x)^k}\left[{}^{\mathrm{RL}}J^k_{x^+}z(y)+{}^{\mathrm{RL}}J^k_{y^-}z(x)\right]\right| \\ &\quad\leq \frac{y-x}{2}\left[\int_0^{\frac{1}{2}}\left((1-t)^k-t^k\right)\left[h\left(e^t-1\right)|z(x)|\right.\right. \\ &\qquad\left.+h\left(e^{1-t}-1\right)|z(y)|\right]dt \\ &\qquad+\int_{\frac{1}{2}}^1\left(t^k-(1-t)^k\right)\left[h\left(e^t-1\right)|z(x)|\right. \\ &\qquad\left.\left.+h\left(e^{1-t}-1\right)|z(y)|\right]dt\right]. \end{aligned} \tag{27}$$

By using identities (25), we get required result. □

4. Midpoint Inequalities via Incomplete Gamma Function

This section contains midpoint inequalities via incomplete gamma function.

Theorem 14. *Let $z:[x,y]\longrightarrow R$ be $L^1[x,y]$ with $0<x<y$ and $k>0$. If $|z|$ is an h -exp convex function, then we*

$$\begin{aligned} &z\left(\frac{x+y}{2}\right)\frac{h\left(e^{1/2}-1\right)\Gamma(k+1)}{(x-y)^k}\left[{}^{RL}J^k_{((x+y)/2)^+}z(y)+{}^{RL}J^k_{((x+y)/2)^-}z(x)\right] \\ &\quad\leq h\left(e^{1/2}-1\right)M[z(x)+z(y)], \end{aligned} \tag{28}$$

where $h(e^{t/2}+e^{1-(t/2)}-2)\leq M$.

Proof. Let $z:I\longrightarrow R$ is h-exponential convex function and $k>0$, then by definition

$$\begin{aligned} z\left(\frac{x+y}{2}\right) &= z\left(\frac{[(t/2)x+((2-t)/2)y]+[((2-t)/2)x+(t/2)y]}{2}\right) \\ &\leq h\left(e^{1/2}-1\right)z\left(\frac{t}{2}x+\frac{2-t}{2}y\right)+h\left(e^{1/2}-1\right)z\left(\frac{2-t}{2}x+\frac{t}{2}y\right). \end{aligned} \tag{29}$$

Multiplying by t^{k-1} on both sides and integrating w.r.t "t" from [0,1], we get

$$\frac{1}{k}z\left(\frac{x+y}{2}\right) \le h\left(e^{1/2}-1\right)\int_0^1 t^{k-1}z\left(\frac{t}{2}x+\left(1-\frac{t}{2}\right)y\right)dt + h\left(e^{1/2}-1\right)\int_0^1 t^{k-1}z\left(\frac{2-t}{2}x+\frac{t}{2}y\right)dt. \tag{30}$$

Again by small substitution, we have

$$\begin{aligned}\frac{1}{k}z\left(\frac{x+y}{2}\right) &\le h\left(e^{1/2}-1\right)\frac{1}{x-y}\int_y^{\frac{x+y}{2}} 2^k\left(\frac{y-u}{y-x}\right)^{k-1} z(u)d(u) \\ &\quad + h\left(e^{1/2}-1\right)\frac{1}{x-y}\int_{\frac{x+y}{2}}^x 2^k\left(\frac{v-x}{y-x}\right)^{k-1} z(v)d(v) \\ &\le 2^k h\left(e^{1/2}-1\right)\frac{1}{x-y}\int_y^{\frac{x+y}{2}}\left(\frac{y-u}{y-x}\right)^{k-1} z(u)d(u) \\ &\quad + 2^k h\left(e^{1/2}-1\right)\frac{1}{x-y}\int_{\frac{x+y}{2}}^x\left(\frac{v-x}{y-x}\right)^{k-1} z(v)d(v).\end{aligned} \tag{31}$$

Implies

$$z\left(\frac{x+y}{2}\right) = \frac{h\left(e^{1/2}-1\right)2^k\Gamma(k+1)}{(x-y)^k}\cdot\left[{}^{\mathrm{RL}}J^k_{((x+y)/2)^+}z(y) + {}^{\mathrm{RL}}J^k_{((x+y)/2)^-}z(x)\right]. \tag{32}$$

For other inequalities, take

$$\begin{aligned}z\left(\frac{t}{2}x+\frac{2-t}{2}y\right) &\le h\left(e^{t/2}-1\right)z(x) + h\left(e^{1-(t/2)}-1\right)z(y); \\ z\left(\frac{2-t}{2}x+\frac{t}{2}y\right) &\le h\left(e^{(1-t)/2}-1\right)z(x) + h\left(e^{t/2}-1\right)z(y).\end{aligned} \tag{33}$$

Adding both inequalities, we have

$$\left(\frac{t}{2}x+\frac{2-t}{2}y\right) + z\left(\frac{2-t}{2}x+\frac{t}{2}y\right)z \le h\left(e^{t/2}+e^{1-(t/2)}-2\right)[z(x)+z(y)]. \tag{34}$$

Multiplying by t^{k-1} on both sides and integrating on [0,1], we get

$$\int_0^1 t^{k-1}z\left(\frac{t}{2}x+\frac{2-t}{2}y\right)dt + \int_0^1 t^{k-1}z\left(\frac{2-t}{2}x+\frac{t}{2}y\right)dt \le [z(x)+z(y)]\int_0^1 t^{k-1}h\left(e^{t/2}+e^{1-(t/2)}-2\right)dt. \tag{35}$$

By making the change of variables, we get

$$\begin{aligned}&2^k\left[\frac{1}{x-y}\int_y^{\frac{x+y}{2}}\left(\frac{y-u}{y-x}\right)^{k-1}z(u)d(u) + \frac{1}{x-y}\int_{\frac{x+y}{2}}^x\left(\frac{v-x}{y-x}\right)^{k-1}z(v)d(v)\right] \\ &\quad\le [z(x)+z(y)]\int_0^1 t^{k-1}h\left(e^{t/2}+e^{1-(t/2)}-2\right)dt, \\ &\frac{2^k\Gamma(k)}{(x-y)^k}\left[{}^{\mathrm{RL}}J^k_{((x+y)/2)^+}z(y) + {}^{\mathrm{RL}}J^k_{((x+y)/2)^-}z(x)\right] \\ &\quad\le [z(x)+z(y)]\int_0^1 t^{k-1}h\left(e^{t/2}+e^{1-(t/2)}-2\right)dt.\end{aligned} \tag{36}$$

Multiplying $k>0$ and $h(e^{1/2}-1)>0$ on both sides, we have

$$\begin{aligned}&\frac{h\left(e^{1/2}-1\right)2^k\Gamma(k+1)}{(x-y)^k}\left[{}^{\mathrm{RL}}J^k_{((x+y)/2)^+}z(y) + {}^{\mathrm{RL}}J^k_{((x+y)/2)^-}z(x)\right] \\ &\quad\le kh\left(e^{1/2}-1\right)[z(x)+z(y)]\int_0^1 t^{k-1}h\left(e^{t/2}+e^{1-(t/2)}-2\right)dt \\ &\quad\le h\left(e^{1/2}-1\right)M[z(x)+z(y)].\end{aligned} \tag{37}$$

□

Corollary 15. *When we introduced $h(e^{1/2}+e^{1-(t/2)}-2) = e^t + e^{1-(t/2)}-2$ in (37) using Remark 7 and rearrange both inequalities (32) and (37), we get (10) of [27].*

Remark 16. For $h(e^{t/2}+e^{1-(t/2)}-2) = 1/(e^{t/2}+e^{1-(t/2)}-2)$, (37) yields midpoint type inequalities via the incomplete gamma function for harmonically exponential convex function.

5. He's Inequality via the Incomplete Gamma Functions

He's inequality via the incomplete gamma functions is presented in this section.

Definition 17. For any L^1 function z on interval $[0,s]$, the k-th He's fractional derivative of $z(x)$ is defined by

$$\begin{aligned}D^k_s w(s) &= \frac{1}{\Gamma(n-k)}\frac{d^n}{ds^n}\int_0^s (t-s)^{n-k-1}z(t)dt, \\ z\left(\frac{x+y}{2}\right) &= z\left(\frac{[tx+(1-t)y]+[(1-t)x+ty]}{2}\right).\end{aligned} \tag{38}$$

By using h-exponential convex function

$$z\left(\frac{x+y}{2}\right) \le h\left(e^{1/2}-1\right)z((tx+(1-t)y) + h\left(e^{1/2}-1\right)z((1-t)x+ty). \tag{39}$$

Taking $x = 0$ and $y > 0$ for all $s \in (0, 1)$ and multiplying by $(t-s)^{n-k-1}/\Gamma(n-k)$, we get

$$\begin{aligned}
&\frac{1}{\Gamma(n-k)} z\left(\frac{y}{2}\right)\int_0^s (t-s)^{n-k-1} dt \\
&\le \frac{h(e^{1/2}-1)}{\Gamma(n-k)} \left[\int_0^s \left((t-s)^{n-k-1} z(1-t)y\right) dt \right.\\
&\quad \left. + \int_0^s (t-s)^{n-k-1} z(ty) dt\right] \frac{(-1)^{n-k} s^{n-k}}{\Gamma(n-k)} z\left(\frac{y}{2}\right) \\
&\le \frac{h(e^{1/2}-1)}{\Gamma(n-k)} \left[\int_0^s \left((t-s)^{n-k-1} z(1-t)y\right) dt \right.\\
&\quad \left. + \int_0^s (t-s)^{n-k-1} z(ty) dt\right].
\end{aligned} \tag{40}$$

Hence

$$\frac{(-1)^{n-k} s^{n-k}}{\Gamma(n-k)} z\left(\frac{y}{2}\right) \le \frac{h(e^{1/2}-1)}{\Gamma(n-k)} \left[\int_0^s \left((t-s)^{n-k-1} z(1-t)y\right) dt + \int_0^s (t-s)^{n-k-1} z(ty) dt\right]. \tag{41}$$

After getting the n-th derivatives on both sides of (41) w.r.t to s and using Definition 17, we get

$$(-1)^{n-k} z\left(\frac{y}{2}\right) \le \left[D^k_{sb} z(sb) + (-1)^{n-k} D^k_{(1-s)b} z((1-s)b)\right]. \tag{42}$$

Remark 18. By putting $h(e^{1/2}-1) = (e^{1/2}-1)$ in (41), we get He's inequality (14) of [27].

5.1. He's Inequality for Harmonically Exponential Convex Function. From Definition 17 and by using definition of h-exponential convex function, we have

$$\begin{aligned}
z\left(\frac{x+y}{2}\right) &\le h\left(e^{1/2}-1\right) z((tx + (1-t)y) \\
&\quad + h\left(e^{1/2}-1\right) z((1-t)x + ty).
\end{aligned} \tag{43}$$

By harmonically exponential convex function, we have

$$z\left(\frac{x+y}{2}\right) \le \frac{1}{e^{1/2}-1} z((tx + (1-t)y) + \frac{1}{e^{1/2}-1} z((1-t)x + ty). \tag{44}$$

Taking $x = 0$ and $y > 0$ for all $s \in (0, 1)$, multiplying by $(t-s)^{n-k-1}/\Gamma(n-k)$

$$\begin{aligned}
&\frac{1}{\Gamma(n-k)} z\left(\frac{y}{2}\right)\int_0^s (t-s)^{n-k-1} dt \\
&\le \frac{1}{(e^{1/2}-1)\Gamma(n-k)} \left[\int_0^s \left((t-s)^{n-k-1} z(1-t)y\right) dt \right.\\
&\quad \left. + \int_0^s (t-s)^{n-k-1} z(ty) dt\right] \frac{(-1)^{n-k} s^{n-k}}{\Gamma(n-k)} z\left(\frac{y}{2}\right) \\
&\le \frac{1}{(e^{1/2}-1)\Gamma(n-k)} \left[\int_0^s \left((t-s)^{n-k-1} z(1-t)y\right) dt \right.\\
&\quad \left. + \int_0^s (t-s)^{n-k-1} z(ty) dt\right].
\end{aligned} \tag{45}$$

Hence

$$\begin{aligned}
\frac{(-1)^{n-k} s^{n-k}}{\Gamma(n-k)} z\left(\frac{y}{2}\right) &\le \frac{1}{(e^{1/2}-1)\Gamma(n-k)} \\
&\quad \cdot \left[\int_0^s \left((t-s)^{n-k-1} z(1-t)y\right) dt \right.\\
&\quad \left. + \int_0^s (t-s)^{n-k-1} z(ty) dt\right].
\end{aligned} \tag{46}$$

After getting the n-th derivatives on both sides of (41) w.r.t to s and using Definition 17, we get

$$(-1)^{n-k} z\left(\frac{y}{2}\right) \le \left[D^k_{sb} z(sb) + (-1)^{n-k} D^k_{(1-s)b} z((1-s)b)\right]. \tag{47}$$

Remark 19. For $1/(e^{1/2}-1) = h(e^{1/2}-1)$ in (47), we get (41). Again substitute $h(e^{1/2}-1) = e^t - 1$ in (41), we get He's inequality (14) of [27].

6. Conclusion

The inequalities in analysis play a vital role to study qualitative properties of functions and solutions of differential equations; we develop various Hermite-Hadamard type inequalities, midpoint inequalities, and trapezoidal and He's inequalities for$h$-exponential convex functions with appropriate substitutions; we may obtain the inequalities for harmonically$h$-exponential convex functions. To establish same inequalities for h-exponential convex stochastic processes is an interesting problem.

Data Availability

There is no additional data required for the finding of results of this paper.

Conflicts of Interest

It is declared that authors have no competing interests.

Authors' Contributions

All authors have equal contribution in this article.

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