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Research Article

Infinitely Many Solutions for Discrete Boundary Value Problems with the (p, q)-Laplacian Operator

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In this paper, we consider the existence and multiplicity of solutions for a discrete Dirichlet boundary value problem involving the (p, q)-Laplacian. By using the critical point theory, we obtain the existence of infinitely many solutions under some suitable assumptions on the nonlinear term. Also, by our strong maximum principle, we can obtain the existence of infinitely many positive solutions.

1. Introduction

Let N be a positive integer and denote with $[1, N]$ the discrete set $\{1, \cdots, N\}$. In this paper, we consider the existence of infinitely many solutions for the following discrete Dirichlet boundary value problem

$$\begin{cases} -\Delta_p u(j-1) - \Delta_q u(j-1) + \alpha(j)\phi_p(u(j)) + \beta(j)\phi_q(u(j)) = \lambda g(j, u(j)), \forall j \in [1, N], \\ u(0) = u(N+1) = 0, \end{cases} \tag{1}$$

where $\Delta_r u(j) := \Delta(\phi_r(\Delta u(j)))$ is the discrete r-Laplacian, $\phi_r(u) = |u|^{r-2}u$ with $u \in \mathbb{R}$, $\Delta u(j) = u(j+1) - u(j)$ is the forward difference operator, $g(j, \cdot)$: $\mathbb{R} \longrightarrow \mathbb{R}$ is continuous for each $j \in [1, N]$, $1 < q \le p < +\infty$, λ is a positive parameter, and $\alpha(j), \beta(j) \ge 0$ for all $j \in [1, N]$.

In the past decades, there has been tremendous interest in the study of difference equations, with the development of engineering, physics, economy, and so on (see [1–4]). Most results about the boundary value problems of difference equations are obtained by using the method of upper and lower solutions and fixed point methods (see [5–7]). In 2003, Guo and Yu [8] first applied the critical point theory to study the existence of periodic and subharmonic solutions for a second-order difference equation. Since then, the critical point theory has been employed to study difference equations, and many meaningful results have been obtained, concerning periodic solutions [9, 10], homoclinic solutions [11–13], heteroclinic solutions [14], and especially in boundary value problems [15–20]. For example, Candito and Giovannelli [21] established the existence of multiple solutions of the following problem

$$\begin{cases} -\Delta_p u(j-1) = \lambda f(j, u(j)), j \in [1, N], \\ u(0) = u(N+1) = 0. \end{cases} \tag{2}$$

Later, Bonanno and Candito [22] established the existence of infinitely many solutions of the following problem

$$\begin{cases} -\Delta_p u(j-1) + q(k)\phi_p(u(j)) = \lambda f(j, u(j)), j \in [1, N], \\ u(0) = u(N+1) = 0, \end{cases} \tag{3}$$

where $q(j) \ge 0$ for all $j \in [1, N]$. Obviously, (2) is a special case $(q(j) = 0)$ of (3). After that, under different conditions, D'Aguì et al. [23] established the existence of at least two positive solutions of (3).

In [24], Li and Zhou considered the following discrete mixed boundary value problem

$$\begin{cases} -\Delta_p u(j-1) + s(j)\phi_q(u(j)) = \lambda f(k, u(k)), j \in [1, N], \\ u(0) = \Delta u(N) = 0, \end{cases} \tag{4}$$

where $s(j) \geq 0$ for all $j \in [1, N]$. By using the critical point theory, the authors obtained the existence of at least two positive solutions for (4).

The boundary value problems involving the sum of a p-Laplacian operator and of a q-Laplacian operator is more common, because this arises in the study of stationary solutions of reaction-diffusion systems (see [25]). For example, Mugnai and Papageorgiou [26] and Marano et al. [27] investigated the following Dirichlet problem

$$\begin{cases} -\Delta_p u - \mu\Delta_q u = f(x, u), & \text{in} \quad \Omega, \\ u = 0, & \text{on} \quad \partial\Omega, \end{cases} \tag{5}$$

where $f : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ satisfies Carathéodory's conditions, and they obtained the existence of multiple solutions of (5).

In [28], Nastasi et al. proved the existence of at least two positive solutions for problem (1). Compared with the discrete boundary value problem involving p-Laplacian operator, there are few results on the discrete boundary value problem with (p, q)-Laplacian operator except [28]. Inspired by the above results, we want to investigate the multiplicity of solutions for problem (1).

In this paper, under suitable assumptions, we use the critical point theory obtained in [29] to establish the existence of infinitely many solutions for discrete (p, q)-Laplacian equations with Dirichlet type boundary conditions. Moreover, by our strong maximum principle, we can obtain the existence of infinitely many positive solutions of (1).

The rest of this paper is organized as follows. In Section 2, we recall the critical point theory and show some basic lemmas. In Section 3, our main results and proofs are presented. After that, we have two examples to explain our main results. We conclude our results in the last section.

2. Preliminaries

Let X be a reflexive real Banach space and let $I_\lambda : X \longrightarrow \mathbb{R}$ be a function satisfying the following structure hypothesis:

(H) $I_\lambda(u) = \Phi(u) - \lambda\Psi(u)$ for all $u \in X$, where $\Phi, \Psi : X \longrightarrow \mathbb{R}$ are two functions of class C^1 on X with Φ coercive, i.e., $\lim_{\|u\| \longrightarrow \infty} \Phi(u) = +\infty$, and λ is a real positive parameter

Provided that $\inf_X \Phi < r$, put

$$\varphi(r) = \inf_{u \in \Phi^{-1}(]-\infty, r[)} \frac{\left(\sup_{v \in \Phi^{-1}(]-\infty, r[)} \Psi(v) \right) - \Psi(u)}{r - \Phi(u)}, \tag{6}$$

and

$$\gamma = \liminf_{r \longrightarrow +\infty} \varphi(r), \delta = \liminf_{r \longrightarrow \left(\inf_X \Phi \right)^+} \varphi(r). \tag{7}$$

There is no doubt that $\gamma \geq 0$ and $\delta \geq 0$. When $\gamma = 0$ (or $\delta = 0$), in the sequel, we agree to regard $1/\gamma$ (or $1/\delta$) as $+\infty$.

Now, we recall Theorem 2.1 of [29], which is our main tool for investigating problem (1).

Lemma 1. *Assume that the condition (H) holds. We have*

(*a*) *For every $r > \inf_X \Phi$ and every $\lambda \in]0, 1/\varphi(r)[$, the restriction of the functional $I_\lambda = \Phi - \lambda\Psi$ to $\Phi^{-1}(]-\infty, r[)$ admits a global minimum, which is a critical point (local minimum) of I_λ in X.*

(*b*) *If $\gamma < +\infty$ then, for each $\lambda \in]0, 1/\gamma[$, the following alternative holds: either*

(b_1)*I_λ possesses a global minimum, or*

(b_2) *There is a sequence $\{u_n\}$ of critical points (local minimum) of I_λ such that $\lim_{n \longrightarrow +\infty} \Phi(u_n) = +\infty$*

(*c*) *If $\delta < +\infty$ then, for each $\lambda \in]0, 1/\delta[$, the following alternative holds: either*

(c_1) *There is a global minimum of Φ which is a local minimum of I_λ, or*

(c_2) *There is a sequence $\{u_n\}$ of pairwise distinct critical points (local minima) of I_λ, with $\lim_{n \longrightarrow +\infty} \Phi(u_n) = \inf_X \Phi$, which weakly converges to a global minimum of Φ*

Here, we consider the N-dimensional Banach space

$$X_d = \{u : [0, N+1] \longrightarrow \mathbb{R} \quad \text{such} \quad \text{that} \quad u(0) = u(N+1) = 0\}, \tag{8}$$

and define the norm

$$\|u\|_{r,h} := \left(\sum_{j=0}^{N} |\Delta u(j)|^r + \sum_{j=1}^{N} h(j)|u(j)|^r \right)^{1/r}, \tag{9}$$

where $h : [1, N] \longrightarrow \mathbb{R}$, with $h(j) \geq 0$ for all $j \in [1, N]$, and $r \in]1, +\infty[$. Then, let X_d be endowed with the norm $\|u\| = \|u\|_{p,\alpha} + \|u\|_{q,\beta}$. We denote the usual sup-norm by $\|u\|_\infty = \max_{j \in [1,N]} | u(j) |$, and then we consider the inequality (see ([30], Lemma 2.2)):

$$\|u\|_\infty \leq \frac{(N+1)^{(r-1)/r}}{2} \|u\|_{r,h} \quad \text{for} \quad \text{all} \quad u \in X_d. \tag{10}$$

Lemma 2. *Let $h = \sum_{j=1}^{N} h(j)$. The following inequalities hold*

$$\frac{2}{(N+1)^{(r-1)/r}} \|u\|_\infty \leq \|u\|_{r,h} \leq (2^r N + h)^{1/r} \|u\|_\infty. \tag{11}$$

Proof. The left-hand side of (11) follows by [30]. Consider the right-hand inequality,

$$\begin{aligned}\|u\|_{r,h}^r &= \sum_{j=0}^{N} |\Delta u(j)|^r + \sum_{j=1}^{N} h(j)|u(j)|^r \\ &= |\Delta u(0)|^r + |\Delta u(N)|^r + \sum_{j=1}^{N-1} |\Delta u(j)|^r + \sum_{j=1}^{N} h(j)|u(j)|^r \\ &\le 2\|u\|_\infty^r + \sum_{j=1}^{N-1} (2\|u\|_\infty)^r + \sum_{j=1}^{N} h(j)\|u\|_\infty^r \\ &\le (2^r N + h)\|u\|_\infty^r.\end{aligned} \tag{12}$$

□

Put

$$\begin{aligned}&A_1(u) = \frac{1}{p}\|u\|_{p,\alpha}^p, A_2(u) = \frac{1}{q}\|u\|_{q,\beta}^q \quad \text{and} \\ &\Psi(u) = \sum_{j=1}^{N} G(j, u(j)), \quad \text{for all} \quad u \in X_d,\end{aligned} \tag{13}$$

where the function $G : [1, N] \times \mathbb{R} \longrightarrow \mathbb{R}$ is given by $G(j, t) = \int_0^t g(j, s)ds$, for all $t \in \mathbb{R}$, $j \in [1, N]$.

Clearly, $A_1, A_2, \Psi \in C^1(X_d, \mathbb{R})$ and we have the following Gâteaux derivatives at the point $u \in X_d$:

$$\left\langle A_1'(u), v\right\rangle = \sum_{j=0}^{N} \phi_p(\Delta u(j))\Delta v(j) + \sum_{j=1}^{N} \alpha(j)\phi_p(u(j))v(j), \tag{14}$$

$$\left\langle A_2'(u), v\right\rangle = \sum_{j=0}^{N} \phi_q(\Delta u(j))\Delta v(j) + \sum_{j=1}^{N} \beta(j)\phi_q(u(j))v(j), \tag{15}$$

$$\left\langle \Psi'(u), v\right\rangle = \sum_{j=1}^{N} g(j, u(j))v(j), \tag{16}$$

for all $v \in X_d$. Now, for $r \in]1,+\infty[$,

$$\begin{aligned}&\sum_{j=0}^{N} \phi_r(\Delta u(j))\Delta v(j) \\ &\quad= \sum_{j=0}^{N} [\phi_r(\Delta u(j))v(j+1) - \phi_r(\Delta u(j))v(j)] \\ &\quad= \sum_{j=1}^{N} \phi_r(\Delta u(j-1))v(j) - \sum_{j=1}^{N} \phi_r(\Delta u(j))v(j) \\ &\quad= -\sum_{j=1}^{N} \Delta\phi_r(\Delta u(j-1))v(j).\end{aligned} \tag{17}$$

If we plug this result back into the calculation of Gâteaux derivatives above, then

$$\left\langle A_1'(u), v\right\rangle = \sum_{j=1}^{N} \left[-\Delta\phi_p(\Delta u(j-1)) + \alpha(j)\phi_p(u(j))\right] v(j), \tag{18}$$

$$\left\langle A_2'(u), v\right\rangle = \sum_{j=1}^{N} \left[-\Delta\phi_q(\Delta u(j-1)) + \beta(j)\phi_q(u(j))\right] v(j), \tag{19}$$

for all $u, v \in X_d$. Let

$$\Phi(u) = A_1(u) + A_2(u). \tag{20}$$

Consider the functional $I_\lambda : X_d \longrightarrow \mathbb{R}$ given as

$$I_\lambda(u) = \Phi(u) - \lambda\Psi(u), \quad \text{for all} \quad u \in X_d. \tag{21}$$

We have

$$\begin{aligned}&\left\langle I_\lambda'(u), v\right\rangle \\ &\quad= \sum_{j=1}^{N} \left[-\Delta_p u(j-1) - \Delta_q u(j-1) + \alpha(j)\phi_p(u(j)) + \beta(j)\phi_q(u(j)) - \lambda g(j, u(j))\right] v(j),\end{aligned} \tag{22}$$

for all $u, v \in X_d$. Thus, $u \in X_d$ is a solution of problem (1) if and only if u is a critical point of I_λ.

Lemma 3. *Fix $u \in X_d$ such that either*

$$\begin{aligned}&u(j) > 0 or - \Delta_p u(j-1) - \Delta_q u(j-1) \\ &\quad+ \alpha(j)\phi_p(u(j)) + \beta(j)\phi_q(u(j)) \ge 0,\end{aligned} \tag{23}$$

for all $j \in [1, N]$. Then, either $u > 0$ in $[1, N]$ or $u \equiv 0$.

Proof. Fix $u \in X_d \setminus \{0\}$ and $Z = \{j \in [1, N]: u(j) \le 0\}$. If $Z = \varnothing$, then, $u > 0$. Now, if $\min Z = 1$, we can get

$$-\Delta_p u(0) - \Delta_q u(0) + \alpha(1)\phi_p(u(1)) + \beta(1)\phi_q(u(1)) \ge 0, \tag{24}$$

which implies that

$$\begin{aligned}&\Delta\left(\phi_p(\Delta u(0))\right) + \Delta\left(\phi_q(\Delta u(0))\right) \\ &\quad\le \alpha(1)\phi_p(u(1)) + \beta(1)\phi_q(u(1)) \le 0.\end{aligned} \tag{25}$$

Thus,

$$\phi_p(\Delta u(1)) + \phi_q(\Delta u(1)) \le \phi_p(\Delta u(0)) + \phi_q(\Delta u(0)). \tag{26}$$

Since ϕ_p and ϕ_q are both strictly increasing, we have $\Delta u(1) \le \Delta u(0)$, which implies $u(2) - u(1) \le u(1) - 0 \le 0$. It follows that $u(2) \le 0$, then $\Delta(\phi_p(\Delta u(1))) + \Delta(\phi_q(\Delta u(1))) \le$

$\alpha(2)\phi_p(u(2)) + \beta(2)\phi_q(u(2)) \le 0$. An easy induction gives

$$0 = u(N+1) \le u(N) \le \cdots \le u(1) \le 0. \quad (27)$$

That is $u \equiv 0$, and this is absurd. Next, we assume that $\min Z = z \in [2, N]$,

$$\Delta\left(\phi_p(\Delta u(z-1))\right) + \Delta\left(\phi_q(\Delta u(z-1))\right) \le \alpha(z)\phi_p(u(z)) + \beta(z)\phi_q(u(z)) \le 0. \quad (28)$$

Due to the monotonicity of ϕ_p and ϕ_q, $\Delta u(z) \le \Delta u(z-1)$, which means $u(z+1) - u(z) \le u(z) - u(z-1)$. Because $u(z-1) > 0$, we have $u(z+1) < u(z) \le 0$. By repeating this argument, it is easy to see

$$0 = u(N+1) < u(N) < \cdots < u(z) \le 0, \quad (29)$$

which leads to a contradiction. □

Now, consider the function $G^+ : [1, N] \times \mathbb{R} \longrightarrow \mathbb{R}$ given as

$$G^+(j, t) = \int_0^t g(j, s^+)ds, \quad \text{for all} \quad t \in \mathbb{R}, j \in [1, N], \quad (30)$$

where $s^+ = \max\{s, 0\}$. Now, we define $I_\lambda^+(u) = \Phi(u) - \lambda\Psi^+(u)$, for all $u \in X_d$, where $\Psi^+(u) = \sum_{j=1}^N G^+(j, u(j))$. Similarly, the critical points of I_λ^+ are the solutions of the following problem

$$\begin{cases} -\Delta_p u(j-1) - \Delta_q u(j-1) + \alpha(j)\phi_p(u(j)) + \beta(j)\phi_q(u(j)) = \lambda g(j, u^+(j)), \forall j \in [1, N], \\ u(0) = u(N+1) = 0. \end{cases} \quad (31)$$

Lemma 4. *If $g(j, 0) \ge 0$ for all $j \in [1, N]$, then each nonzero critical point of I_λ^+ is a positive solution of (1).*

Proof. We note that each positive solution $u \in X_d$ of (31) is a positive solution of (1). By an application of Lemma 3, we conclude that $u > 0$. It follows that the nonzero solutions of (31) are positive and hence are positive solutions of (1). □

3. Main Results

Let

$$\alpha = \sum_{j=1}^N \alpha(j), \beta = \sum_{j=1}^N \beta(j), L_\infty(j) = \liminf_{t \longrightarrow +\infty} \frac{G(j, t)}{t^p} \text{and} L_\infty = \min_{j \in [1,N]} L_\infty(j). \quad (32)$$

The main results are as follows.

Theorem 5. *Assume that $L_\infty > 0$, and there are two real sequences $\{a_n\}$ and $\{b_n\}$, with $\lim_{n \longrightarrow +\infty} a_n = +\infty$, such that*

$$|b_n| < \min\left\{\frac{2a_n}{(\alpha+2)^{1/p}(N+1)^{(p-1)/p}}, \frac{2a_n}{(\beta+2)^{1/q}(N+1)^{(q-1)/q}}\right\}, \quad \text{for every} \quad n \in \mathbb{N}, \quad (33)$$

$$A_\infty := \liminf_{n \longrightarrow +\infty} \frac{\sum_{j=1}^N \max_{|t| \le a_n} G(j, t) - \sum_{j=1}^N G(j, b_n)}{\left((2a_n)^p/p(N+1)^{p-1}\right) + \left((2a_n)^q/q(N+1)^{q-1}\right) - [(2+\alpha)/p]|b_n|^p - [(2+\beta)/q]|b_n|^q} < \frac{qL_\infty}{(2^p+2^q)N + \alpha + \beta}. \quad (34)$$

Then for each $\lambda \in][(2^p+2^q)N + \alpha + \beta]/qL_\infty, 1/A_\infty[$, problem (1) admits an unbounded sequence of solutions.

Proof. Fixλin$][(2^p+2^q)N + \alpha + \beta]/qL_\infty, 1/A_\infty[$, then, we can take the real Banach space X_d as defined in Section 2, and the definitions of Φ, Ψ, I_λ are the same as before. We will prove Theorem 5 by applying Lemma 1 part (b) to function I_λ. Since (H) is trivial to prove, it suffices to prove $\gamma < +\infty$ and I_λ turns out to be unbounded from below. To this end, let

$$\rho_n := \frac{(2a_n)^p}{p(N+1)^{p-1}} \quad \text{and} \quad \sigma_n := \frac{(2a_n)^q}{q(N+1)^{q-1}}, \quad \text{for every} \quad n \in \mathbb{N}. \quad (35)$$

Since, owing to (10), if $\|u\|_{p,\alpha} \le (p\rho_n)^{1/p}$ then $\|u\|_\infty \le a_n$, and if $\|u\|_{q,\beta} \le (q\sigma_n)^{1/q}$ then $\|u\|_\infty \le a_n$. So, let $r_n = \rho_n + \sigma_n$. From $\Phi(u) \le r_n$, we have $\|u\|_\infty \le a_n$.

We obtain

$$\varphi(r_n) \le \inf_{\Phi(u) \le r_n} \frac{\sum_{j=1}^N \max_{|t| \le a_n} G(j, t) - \sum_{j=1}^N G(j, u(j))}{r_n - \Phi(u)}. \quad (36)$$

Then, we define $w(j)$ such that $w_n(j) = b_n$ for every $j \in [1, N]$, $w_n(0) = w_n(N+1) = 0$. Clearly $w_n(j) \in X_d$ and $\Phi(w_n) < r_n$ owing to (33). One has

$$\varphi(r_n) \le \frac{\sum_{j=1}^{N} \max_{|t|\le a_n} G(j,t) - \sum_{j=1}^{N} G(j,c)}{\left((2a_n)^p/p(N+1)^{p-1}\right) + \left((2a_n)^q/q(N+1)^{q-1}\right) - [(2+\alpha)/p]|b_n|^p - [(2+\beta)/q]|b_n|^q}. \quad (37)$$

Therefore, $\gamma \le \liminf_{n\longrightarrow+\infty} \varphi(r_n) \le A_\infty < +\infty$. It remains to show that I_λ is unbounded from below.

Let $\{u_n\} \subset X_d$ be a sequence with $u_n(j) \ge 1$ for $j \in [1,N]$ such that $\lim_{n\longrightarrow\infty} \|u_n\| = +\infty$. Because $L_\infty > 0$, fix L such that $L_\infty > L > [(2^p + 2^q)N + \alpha + \beta]/q\lambda$, and we deduce that there is $\delta_j > 0$ such that $G(j,t) > Lt^p$ for all $t > \delta_j$. Moreover, since $G(j,t)$ is a continuous function, there exists a constant $C(j) \ge 0$ such that $G(j,t) \ge Lt^p - C(j)$ for all $t \in [0,\delta_j]$. Thus, $G(j,t) \ge Lt^p - C(j)$ for all $t \ge 0$ and $j \in [1,N]$. It follows that

$$\begin{aligned}\Psi(u_n) &= \sum_{j=1}^{N} G(j,u_n(j)) \ge \sum_{j=1}^{N} \left[L(u_n(j))^p - C(j)\right] \\ &\ge L\|u_n\|_\infty^p - C, \quad \text{for all} \quad n \in \mathbb{N},\end{aligned} \quad (38)$$

where $C = \sum_{j=1}^{N} C(j)$. Since $\|u_n\|_\infty \ge 1$, one has

$$\begin{aligned}I_\lambda(u_n) &= \frac{\|u_n\|_{p,\alpha}^p}{p} + \frac{\|u_n\|_{q,\beta}^q}{q} - \lambda \sum_{j=1}^{N} G(j,u_n(j)) \\ &\le \frac{2^pN+\alpha}{p}\|u_n\|_\infty^p + \frac{2^qN+\beta}{q}\|u_n\|_\infty^q - \lambda L\|u_n\|_\infty^p + \lambda C \\ &\le \left[\frac{(2^p+2^q)N+\alpha+\beta}{q} - \lambda L\right]\|u_n\|_\infty^p + \lambda C.\end{aligned} \quad (39)$$

As $[(2^p+2^q)N+\alpha+\beta]/q - \lambda L < 0$, it is obvious that $\lim_{n\longrightarrow+\infty} I_\lambda(u_n) = -\infty$. Hence, I_λ is unbounded from below and the proof is complete. □

Let

$$B^\infty = \limsup_{t\longrightarrow+\infty} \frac{\sum_{j=1}^{N} G(j,t)}{t^p}. \quad (40)$$

The following theorem can be obtained if we change some of the conditions.

Theorem 6. *Assume that there are two real sequences $\{a_n\}$ and $\{b_n\}$, with $\lim_{n\longrightarrow+\infty} a_n = +\infty$, such that (33) holds and*

$$\begin{aligned}A_\infty &:= \liminf_{n\longrightarrow+\infty} \frac{\sum_{j=1}^{N} \max_{|t|\le a_n} G(j,t) - \sum_{j=1}^{N} G(j,b_n)}{\left((2a_n)^p/p(N+1)^{p-1}\right) + \left((2a_n)^q/q(N+1)^{q-1}\right) - [(2+\alpha)/p]|b_n|^p - [(2+\beta)/q]|b_n|^q} \\ &< \frac{B^\infty}{4+\alpha+\beta}.\end{aligned} \quad (41)$$

Then, for each $\lambda \in](4+\alpha+\beta)/qB^\infty, 1/A_\infty[$, problem (1) admits an unbounded sequence of solutions.

Proof. The first half of the argument is analogous to that in Theorem 5, and put $\Phi, \Psi, I_\lambda, r_n$ as above. So, we have $\gamma \le \liminf_{n\longrightarrow+\infty} \varphi(r_n) \le A_\infty < +\infty$.

Our task now is to verify that I_λ is unbounded from below. First, we assume that $B^\infty = +\infty$. Fix M such that $B^\infty > M > (4+\alpha+\beta)/q\lambda$, and let $\{t_n\}$ be a sequence with $t_n \ge 1$ and $\lim_{n\longrightarrow+\infty} t_n = +\infty$, such that

$$\sum_{j=1}^{N} G(j,t_n) > Mt_n^p, \quad \text{for all} \quad n \in \mathbb{N}. \quad (42)$$

Taking the sequence x_n in X_d defined by $x_n(j) = t_n$ for every $j \in [1,N]$, $x_n(0) = x_n(N+1) = 0$, we have

$$\begin{aligned}I_\lambda(x_n) &= \frac{\|x_n\|_{p,\alpha}^p}{p} + \frac{\|x_n\|_{q,\beta}^q}{q} - \lambda \sum_{j=1}^{N} G(j,x_n(j)) \\ &= \frac{2+\alpha}{p}t_n^p + \frac{2+\beta}{q}t_n^q - \lambda \sum_{j=1}^{N} G(j,t_n) \\ &< \frac{2+\alpha}{p}t_n^p + \frac{2+\beta}{q}t_n^q - \lambda Mt_n^p \\ &< \left(\frac{\alpha+\beta+4}{q} - \lambda M\right)t_n^p.\end{aligned} \quad (43)$$

It is easy to see $\lim_{n\longrightarrow+\infty} I_\lambda(x_n) = -\infty$.

Then, we assume that $B^\infty < +\infty$ and fix $\varepsilon > 0$ such that $\varepsilon < B^\infty - (4+\alpha+\beta)/q\lambda$. Let $\{t_n\}$ be a sequence with $t_n \ge 1$, such that $\lim_{n\longrightarrow+\infty} t_n = +\infty$ and

$$(B^\infty + \varepsilon)t_n^p > \sum_{j=1}^{N} G(j,t_n) > (B^\infty - \varepsilon)t_n^p, \forall n \in \mathbb{N}. \quad (44)$$

Let the sequence $\{x_n\}$ in X_d be the same as the case where $B^\infty = +\infty$, such that

$$I_\lambda(x_n) < \left[\frac{4+\alpha+\beta}{q} - \lambda(B^\infty - \varepsilon)\right]b_n^p, \quad (45)$$

which implies that $\lim_{n\longrightarrow+\infty} I_\lambda(x_n) = -\infty$.

So, in both cases, I_λ is unbounded from below, which completes the proof of Theorem 6.

Let

$$B^0 := \limsup_{t\longrightarrow0^+} \frac{\sum_{j=1}^{N} G(j,t)}{t^q}. \quad (46)$$

Applying part (c) of Lemma 1, we get the following theorem. □

Theorem 7. *Assume that there exist two real sequences* $\{c_n\}$ *and* $\{d_n\}$, *with* $\lim_{n\longrightarrow+\infty} d_n = 0$, *such that*

$$|c_n| < \min\left\{\frac{2d_n}{(\alpha+2)^{1/p}(N+1)^{(p-1)/p}}, \frac{2d_n}{(\beta+2)^{1/q}(N+1)^{(q-1)/q}}\right\}, \quad \textit{for every} \quad n \in \mathbb{N}, \tag{47}$$

$$A_0 := \liminf_{n\longrightarrow+\infty} \frac{\sum_{j=1}^N \max_{|t|\le d_n} G(j,t) - \sum_{j=1}^N G(j,c_n)}{\left((2d_n)^p/p(N+1)^{p-1}\right) + \left((2d_n)^q/q(N+1)^{q-1}\right) - [(2+\alpha)/p]|c_n|^p - [(2+\beta)/q]|c_n|^q} < \frac{B^0}{4+\alpha+\beta}. \tag{48}$$

Then, for each $\lambda \in](4+\alpha+\beta)/qB^0, 1/A_0[$, *problem (1) admits a sequence of nonzero solutions which converges to zero.*

Proof. Fix λ in $](4+\alpha+\beta)/qB^0, 1/A_0[$, and we can take the real Banach space X_d and functional Φ, Ψ, I_λ as defined in Section 2. Our aim is to apply Lemma 1 part (c) to function I_λ. To this end, let

$$\rho_n := \frac{(2d_n)^p}{p(N+1)^{p-1}} \quad \text{and} \quad \sigma_n := \frac{(2d_n)^q}{q(N+1)^{q-1}}, \quad \text{for every} \quad n \in \mathbb{N}. \tag{49}$$

Owing to (10), if $\|u\|_{p,\alpha} \le (p\rho_n)^{1/p}$ then $\|u\|_\infty \le d_n$, and if $\|u\|_{q,\beta} \le (q\sigma_n)^{1/q}$ then $\|u\|_\infty \le d_n$. So, let $r_n = \rho_n + \sigma_n$. It follows that if $\Phi(u) \le r_n$, then $\|u\|_\infty \le d_n$. We obtain

$$\varphi(r_n) \le \inf_{\Phi(u)\le r_n} \frac{\sum_{j=1}^N \max_{|t|\le d_n} G(j,t) - \sum_{j=1}^N G(j,u(j))}{r_n - \|u\|_{p,\alpha}^p/p - \|u\|_{q,\beta}^q/q}. \tag{50}$$

Now, for each $n \in \mathbb{N}$, let $v_n(j)$ be defined by $v_n(j) = c_n$ for every $j \in [1,N]$, $v_n(0) = v_n(N+1) = 0$. Clearly $v_n(j) \in X_d$, and $\Phi(v_n) \le r_n$ from (47). We have

$$\varphi(r_n) \le \frac{\sum_{j=1}^N \max_{|t|\le d_n} G(j,t) - \sum_{j=1}^N G(j,c_n)}{\left((2d_n)^p/p(N+1)^{p-1}\right) + \left((2d_n)^q/q(N+1)^{q-1}\right) - [(2+\alpha)/p]|c_n|^p - [(2+\beta)/q]|c_n|^q}. \tag{51}$$

Hence, $\delta \le \liminf_{n\longrightarrow+\infty} \varphi(r_n) \le A_0 < +\infty$ follows.

In fact, $\inf_{X_d} \Phi = 0$, so our task now is to verify that the 0 is not a local minimum of I_λ. First, assume that $B^0 = +\infty$. Fix M such that $B^0 > M > (4+\alpha+\beta)/q\lambda$, and let $\{s_n\}$ be a sequence of positive numbers, with $s_n \le 1$ and $\lim_{n\longrightarrow+\infty} s_n = 0$, such that

$$\sum_{j=1}^N G(j,s_n) > M s_n^q, \quad \text{for all} \quad n \in \mathbb{N}. \tag{52}$$

Thus, taking the sequence $\{y_n\}$ in X_d, let $y_n(j) = s_n$ for every $j \in [1,N]$, $y_n(0) = y_n(N+1) = 0$. Some tedious manipulation yields

$$I_\lambda(y_n) < \left(\frac{4+\alpha+\beta}{q} - \lambda M\right) s_n^q, \tag{53}$$

which implies that $I_\lambda(y_n) < 0$.

Then, we assume that $B^0 < +\infty$ and fix $\varepsilon > 0$ such that $\varepsilon < B^0 - (4+\alpha+\beta)/q\lambda$. Let $\{s_n\}$ be a sequence of positive numbers, with $s_n \le 1$, such that $\lim_{n\longrightarrow+\infty} s_n = 0$ and

$$\left(B^0 + \varepsilon\right) s_n^q > \sum_{j=1}^N G(j,s_n) > \left(B^0 - \varepsilon\right) s_n^q, \forall n \in \mathbb{N}. \tag{54}$$

Choosing the same $\{y_n\}$ in X_d as the case $B^0 = +\infty$, one has

$$I_\lambda(y_n) < \left[\frac{4+\alpha+\beta}{q} - \lambda\left(B^0 - \varepsilon\right)\right] s_n^q. \tag{55}$$

That is $I_\lambda(y_n) < 0$. Since 0 is the global minimum of Φ, in both cases, $u = 0$ is not a local minimum of I_λ and the proof is complete. □

By setting

$$A_* := \liminf_{n\longrightarrow+\infty} \frac{\sum_{j=1}^N \max_{|t|\le a_n} G(j,t)}{\left((2a_n)^p/p(N+1)^{p-1}\right) + \left((2a_n)^q/q(N+1)^{q-1}\right)}, \bar{A}_\infty := \liminf_{t\longrightarrow+\infty} \frac{\sum_{j=1}^N \max_{|\xi|\le t} G(j,\xi)}{t^q + t^p}, \tag{56}$$

we get the following consequences.

Corollary 8. *Assume that*

$$\bar{A}_\infty < \frac{2^q}{p(N+1)^{p-1}(4+\alpha+\beta)} B^\infty. \tag{57}$$

Then, for each $\lambda \in](4+\alpha+\beta)/qB^\infty, 2^q/p(N+1)^{p-1}\bar{A}_\infty[$, *problem (1) admits an unbounded sequence of solutions.*

Proof. Let $\{a_n\}$ be a sequence of positive numbers with $\lim_{n\longrightarrow\infty} a_n = +\infty$, such that

$$\bar{A}_\infty = \liminf_{n\longrightarrow+\infty} \frac{\sum_{j=1}^N \max_{|\xi|\le a_n} G(j,\xi)}{a_n^q + a_n^p}. \tag{58}$$

After simple scaling and calculation, we have

$$A_* \le \frac{p(N+1)^{p-1}}{2^q} \bar{A}_\infty. \tag{59}$$

Taking $b_n = 0$ for each $n \in \mathbb{N}$, from Theorem 6, the conclusion follows. □

If $g(j,0)$ satisfies the nonnegative condition, we have the following conclusion.

Corollary 9. *Assume that $g(j,0) \geq 0$ for all $j \in [1,N]$, and*

$$\bar{A}_\infty < \frac{2^q}{p(N+1)^{p-1}(4+\alpha+\beta)} B^\infty. \quad (60)$$

Then, for each $\lambda \in](4+\alpha+\beta)/qB^\infty, 2^q/p(N+1)^{p-1}\bar{A}_\infty[$, problem (1) admits an unbounded sequence of positive solutions.

Proof. Let

$$g^+(j,t) = \begin{cases} g(j,t), & \text{if } t > 0, \\ g(j,0), & \text{if } t \leq 0. \end{cases} \quad (61)$$

Since $g(j,0) \geq 0$,

$$\max_{0\leq s\leq t} \int_0^s g^+(j,\xi)d\xi = \max_{0\leq s\leq t} \int_0^s g(j,\xi)d\xi, \quad (62)$$

for all $t \geq 0$. From Corollary 8, we know that problem (1) with g replaced by g^+ admits an unbounded sequence of solutions for each $\lambda \in](4+\alpha+\beta)/qB^\infty, 2^q/p(N+1)^{p-1}\bar{A}_\infty[$. Then, all these solutions are positive solutions of problem (1) by Lemma 4.

Let

$$\bar{A}_0 := \liminf_{t\longrightarrow 0^+} \frac{\sum_{j=1}^N \max_{|\xi|\leq t} G(j,\xi)}{t^q + t^p}. \quad (63)$$

Arguing as in the proof of Corollary 8 and taking $c_n = 0$ for each $n \in [1,N]$, by Theorem 7, we have the following corollary. □

Corollary 10. *Assume that*

$$\bar{A}_0 < \frac{2^q}{p(N+1)^{p-1}(4+\alpha+\beta)} B^0. \quad (64)$$

Then, for each $\lambda \in](4+\alpha+\beta)/qB^0, 2^q/p(N+1)^{p-1}\bar{A}_0[$, problem (1) admits a sequence of nonzero solutions which converges to zero.

Arguing as in Corollary 9, we have the following result.

Corollary 11. *Assume that $g(j,0) \geq 0$ for all $j \in [1,N]$, and*

$$\bar{A}_0 < \frac{2^q}{p(N+1)^{p-1}(4+\alpha+\beta)} B^0. \quad (65)$$

Then, for each $\lambda \in](4+\alpha+\beta)/qB^0, 2^q/p(N+1)^{p-1}\bar{A}_0[$, problem (1) admits a sequence of positive solutions which converges to zero.

Finally, we give two easy examples to illustrate our results.

Example 1. Let $\alpha = \beta = 0$, $q = 2$, $p = 3$,

$$g(j,x) = g(x) = \begin{cases} 3x^2 \sin\left(\frac{1}{2}\ln|x|\right) + \frac{1}{2}x^2\cos\left(\frac{1}{2}\ln|x|\right) + \frac{25}{8}x^2, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases} \quad (66)$$

for each $j \in [1,N]$. Then,

$$\liminf_{t\longrightarrow+\infty} \frac{\max_{|\xi|\leq t}\int_0^\xi \left[3x^2 \sin(\ln x/2) + x^2\cos(\ln x/2)/2 + 25x^2/8\right]dx}{t^2+t^3} = \liminf_{t\longrightarrow+\infty} \frac{t^3 \sin(\ln t/2) + 25t^3/24}{t^2+t^3} = \frac{1}{24}, \quad (67)$$

and

$$\limsup_{t\longrightarrow+\infty} \frac{\int_0^t \left[3x^2 \sin(\ln x/2) + x^2\cos(\ln x/2)/2 + 25x^2/8\right]dx}{t^3} = \limsup_{t\longrightarrow+\infty} \frac{t^3 \sin(\ln t/2) + 25t^3/24}{t^3} = \frac{49}{24}. \quad (68)$$

By choosing $N = 3$, we have

$$\frac{2^q}{p(N+1)^{p-1}(4+\alpha+\beta)} = \frac{1}{48}. \quad (69)$$

From the above calculation, we obtain

$$\bar{A}_\infty = \liminf_{t\longrightarrow+\infty} \frac{\sum_{j=1}^3 \max_{|\xi|\leq t}\int_0^\xi x^2[3\sin(\ln x/2) + \cos(\ln x/2)/2 + 25/8]dx}{t^2+t^3} = \frac{1}{8}, \quad (70)$$

$$B^\infty = \limsup_{t\longrightarrow+\infty} \frac{\sum_{j=1}^3 \int_0^t x^2[3\sin(\ln x/2) + \cos(\ln x/2)/2 + 25/8]dx}{t^3} = \frac{49}{8}. \quad (71)$$

It is clear that $\bar{A}_\infty < 2^q B^\infty/p(N+1)^{p-1}(4+\alpha+\beta)$, by Corollary 9, the problem

$$\begin{cases} -(|\Delta u(j)|+1)\Delta u(j) + (|\Delta u(j-1)|+1)\Delta u(j-1) = \frac{1}{2}g(u(j)), \forall j \in [1,3], \\ u(0) = u(4) = 0, \end{cases} \quad (72)$$

admits an unbounded sequence of positive solutions.

Example 2. Let $q = 2, p > 2$ and

$$g(j, x) = g(x) = \begin{cases} x(2 + 2\varepsilon + 2\cos(\varepsilon \ln |x|) - \varepsilon \sin(\varepsilon \ln |x|)), & \text{if} \quad x \neq 0, \\ 0, & \text{if} \quad x = 0, \end{cases} \tag{73}$$

for each $j \in [1, N]$. Then,

$$G(j, x) = G(x) = \int_0^x g(s)ds = x^2[1 + \varepsilon + \cos(\varepsilon \ln x)], \tag{74}$$

for $x > 0$. Since $g(x) \geq 0$ for $x \geq 0$, $G(x)$ is increasing. We have

$$\bar{A}_0 = \liminf_{t \longrightarrow 0^+} \frac{\sum_{j=1}^N \max_{0 \leq \xi \leq t} G(j, \xi)}{t^q + t^p} = N \liminf_{t \longrightarrow 0^+} \frac{t^2[1 + \varepsilon + \cos(\varepsilon \ln t)]}{t^2 + t^p} = N\varepsilon, \tag{75}$$

$$B^0 = \limsup_{t \longrightarrow 0^+} \frac{\sum_{j=1}^N G(j, t)}{t^q} = N \limsup_{t \longrightarrow 0^+} \frac{t^2[1 + \varepsilon + \cos(\varepsilon \ln t)]}{t^2} = N(2 + \varepsilon). \tag{76}$$

Let ε be a sufficiently small constant, such that

$$N\varepsilon < \frac{2^q}{p(N+1)^{p-1}(4 + \alpha + \beta)} N(2 + \varepsilon). \tag{77}$$

Then, by Corollary 11, for each $\lambda \in](4 + \alpha + \beta)/qB^0, 2^q/p(N+1)^{p-1}\bar{A}_0[$, problem (1) admits a sequence of positive solutions which converges to zero.

4. Conclusions

In this paper, we consider a discrete Dirichlet boundary value problem involving the (p, q)-Laplacian. Unlike the existing result in [28], which is the existence of at least two positive solutions, we consider the existence of infinitely many solutions for problem (1) for the first time. In fact, by using Theorem 2.1 of [29], we show that problem (1) admits a sequence of pairwise distinct solutions under some appropriate assumptions on the nonlinear term near at infinity and at the origin. Moreover, we prove the existence of infinitely many positive solutions through our strong maximum principle. It seems that we can use the method in this paper to study other similar problems, such as the existence and multiplicity of solutions for difference equations with different boundary value conditions. This will be left as our future work.

Data Availability

No data were used to support the study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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