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Research Article

Inequalities on Generalized Sasakian Space Forms

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In this paper, we find the second variational formula for a generalized Sasakian space form admitting a semisymmetric metric connection. Inequalities regarding the stability criteria of a compact generalized Sasakian space form admitting a semisymmetric metric connection are established.

1. Introduction

The harmonic maps have aspects from both Riemannian's geometry and analysis. Harmonic mappings are considered a vast field, and because of the minimization of energy due to its dual nature, it has many applications in the field of mathematics, physics, relativity, engineering, geometry, crystal liquid, surface matching, and animation. Some particular examples of harmonic maps are geodesics, immersion, and solution of the Laplace equation. In physics, p-harmonic maps were studied in image processing. Exponential harmonic maps were discussed in the field of gravity. Due to generalized properties, F-harmonic maps have many applications in cosmology. Harmonic maps have played a significant role in Finsler's geometry. On complex manifolds, we have interesting and useful outcomes of harmonic maps (for details, see [1, 2]).

During the past years, harmonicity on almost contact metric manifolds has been considered a parallel to complex manifolds ([3–5]). The identity map on a Riemannian manifold with a compact domain becomes a trivial case of the harmonicity. However, the stability and second variation theory are complex and remarkable here. In [6], a Laplacian upon functions with its first eigenvalue is used to explain stability on Einstein's manifolds. From [7, 8], we know about the stability-based classification of a Riemannian that simply connected irreducible spaces with a compact domain.

From [6], we know a well-known result about the stability of S^{2n+1}. Further in [5], identity map stability upon a compact domain of the Sasakian space form was explained by Gherge et al. (see also [9]). Considering the generalization of Sasakian space forms, Alegre et al. presented the generalized Sasakian space forms [10]. Therefore, we naturally study the identity map stability upon a compact domain of generalized Sasakian space forms, as discussed in some results in [11]. One of the most important terms in differential geometry is connection. Research on manifolds is incomplete without the notion of connection. In manifold theory, from the relation of metric and connection, we have a very important notion known as curvature tensor. The concept of a semisymmetric metric connection was initiated by Friedmann and Schouten in 1932 [12, 13]. Semisymmetric metric connections have many applications in the field of Riemannian manifolds and are useful to study many physical problems. In the current paper, we compute the stability criteria of a generalized Sasakian space form admitting a semisymmetric metric connection.

After recollecting the essential facts about harmonic maps between Riemannian manifolds in Section 2, we explain generalized Sasakian space forms throughout Section 3. In Section 4, we give the main results for a second variational formula and establish the inequalities for the identity map stability criteria upon a compact domain generalized Sasakian space form admitting a semisymmetric metric connection.

2. Harmonic Maps on Riemannian Manifolds

We can view harmonic maps on Riemannian manifolds as the generalization of geodesics that is the case of a one-dimensional domain and range as Euclidean space. In common, a map is known as harmonic if its Laplacian becomes zero and is known as totally geodesic if its Hessian becomes zero. In this present section, the basic facts of the harmonic maps theory [14, 15] are provided. Consider a smooth map $\psi : (S, g) \longrightarrow (Q, h)$. Let the dimension of the Riemannian manifold (S, g) be s and the dimension of (Q, h) be q. The function $e(\psi)$: $S \longrightarrow [0,\infty)$ that is smooth can be considered as the energy density of ψ and is expressed as

$$e(\psi)_p = \frac{1}{2} Tr_g(\psi^* h)(p) = \frac{1}{2}\sum_{i=1}^{s} h\left(\psi_{*p} u_i, \psi_{*p} u_i\right), \quad (1)$$

at a point $p \in S$ and for any orthonormal basis $\{u_1, \cdots, u_s\}$ of T_pS. Considering the compact domain of a Riemannian manifold S, we take the energy density integral as the energy $E(\psi)$ of ψ; that is, we have

$$E(\psi) = \int_S e(\psi) v_g, \quad (2)$$

where the volume measure is represented by v_g that is related to the metric g on manifold S. In the set $C^\infty(S, Q)$ of all smooth maps from (S, g) to (Q, h), a critical point of the energy E is named as a harmonic map. That is, for any smooth variation $\psi_t \in C^\infty(S, Q)$ of $\psi(t \in (-\varepsilon, \varepsilon))$ with $\psi_0 = \psi$, we can take

$$\left.\frac{d}{dt} E(\psi_t)\right|_{t=0} = 0. \quad (3)$$

Now, we consider (S, g) as a compact Riemannian manifold and take a map $\psi : (S, g) \longrightarrow (Q, h)$ that is harmonic. We consider smooth variation $\psi_{r,t}$ through constraints $r, t \in (-\varepsilon, \varepsilon)$ satisfying $\psi_{0,0} = \psi$. Respective variational vector fields are represented through W and Z. Therefore, we can define Hessian H_ψ for a harmonic map ψ through the following relation:

$$H_\psi(W, Z) = \left.\frac{\partial^2}{\partial r \partial t}\left(E\left(\psi_{r,t}\right)\right)\right|_{(r,t)=(0,0)}. \quad (4)$$

The expression regarding the second variation of E is as follows ([6, 16]):

$$H_\psi(W, Z) = \int_P h\left(J_\psi(W), Z\right) v_g, \quad (5)$$

where J_ψ is the second order operator that is self-adjoint upon the space $\Gamma(\psi^{-1}(TQ))$ of variation vector fields and is represented as

$$J_\psi(U) = -\sum_{i=1}^{s}\left(\nabla^{\sim}{}_{u_i}\nabla^{\sim}{}_{u_i} - \nabla^{\sim}{}_{\nabla_{u_i}u_i}\right)U - \sum_{i=1}^{s} R^Q(U, d\psi(u_i))d\psi(u_i), \quad (6)$$

for $U \in \Gamma(\psi^{-1}(TQ))$ and any local orthonormal frame $\{u_1, \cdots, u_s\}$ on S. Here, R^Q shows the curvature tensor of (Q, h), and $\tilde{\nabla}$ illustrates the pull-back connection of ψ along with the Levi-Civita connection of Q.

We compute the dimension of the biggest subspace of $\Gamma(\psi^{-1}(TQ))$ where the Hessian H_ψ has values that are negative definite known as the index of a harmonic map $\psi : (S, g) \longrightarrow (Q, h)$. Therefore, if the index of harmonic map ψ is zero, then it is stable; otherwise, it is unstable.

An operator $\bar{\Delta}_\psi$ is represented by

$$\bar{\Delta}_\psi U = -\sum_{i=1}^{s}\left(\nabla^{\sim}{}_{u_i}\nabla^{\sim}{}_{u_i} - \nabla^{\sim}{}_{\nabla_{u_i}u_i}\right)U, \quad V \in \Gamma\left(\phi^{-1}(TQ)\right). \quad (7)$$

It is named the rough Laplacian. We consider the spectra of J_ψ; because of the Hodge de Rham Kodaira theory, this spectra is constructed as a discrete set of infinite number of eigenvalues with finite multiplicities with no accumulation points.

3. Generalized Sasakian Space Forms

Generalized Sasakian space forms have the generalized curvature expression that combines the curvature expessions of Sasakian, Kenmotsu, and Cosymplectic space forms. Due to a generalized curvature expression, generalized Sasakian space forms have very useful and interesting properties. The current unit presents basics of almost contact metric manifolds particularly of generalized Sasakian space forms [17].

A Riemannian manifold P^{2n+1} with odd dimensions is known as an almost contact manifold if a $(1,1)$-tensor field φ exists on P and ξ and a vector field η and a 1-form exist so that

$$\varphi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1. \tag{8}$$

Further, φ and η satisfy $\varphi(\xi) = 0$ and $\eta o \varphi = 0$. A compatible metric g on any almost contact manifold is defined as

$$g(\varphi W_1, \varphi W_2) = g(W_1, W_2) - \eta(W_1)\eta(W_2), \tag{9}$$

for any vector fields W_1, W_2 on manifold P known as an almost contact metric manifold. An almost contact metric manifold becomes a contact metric manifold if for a fundamental 2-form Ω, we have $d\eta = \Omega$, and $\Omega(W_1, W_2) = g(W_1, \varphi W_2)$ for $W_1, W_2 \in \Gamma(TP)$. Like the parallel condition of integrability for almost complex manifolds, the almost contact metric structure on P becomes normal when

$$[\varphi, \varphi] + 2d\eta \otimes \xi = 0. \tag{10}$$

The Nijenhuis torsion of φ is represented by $[\varphi, \varphi]$ and is defined as

$$\begin{aligned} [\varphi, \varphi](Y_1, Y_2) = \varphi^2[Y_1, Y_2] &+ [\varphi Y_1, \varphi Y_2] \\ &- \varphi[\varphi Y_1, Y_2] - \varphi[Y_1, \varphi Y_2]. \end{aligned} \tag{11}$$

A Sasakian manifold is a normal contact metric manifold, and if $d\eta = 0$, a normal almost contact metric manifold is known as the Kenmotsu manifold with

$$d\Omega(Y_1, Y_2, Y_3) = \frac{2}{3}\sigma_{(Y_1,Y_2,Y_3)}\{\eta(Y_1)\phi(Y_2, Y_3)\}, \quad Y_1, Y_2, Y_3 \in \Gamma(TP), \tag{12}$$

where the cyclic sum is represented by σ. A real space form is a Riemannian manifold with a constant sectional curvature c, and its curvature tensor is represented by the following relation:

$$R(Y_1, Y_2)Y_3 = c\{g(Y_2, Y_3)Y_1 - g(Y_1, Y_3)Y_2\}, \tag{13}$$

where Y_1, Y_2, and Y_3 are vector fields on P. An almost contact metric manifold $P(\varphi, \xi, \eta, g)$ can be identified as a generalized Sasakian space form provided that there are three functions f_1, f_2, f_3 upon P so as the curvature tensor on P is represented with the following relation:

$$\begin{aligned} R(V_1, V_2)V_3 = f_1\{&g(V_2, V_3)V_1 - g(V_1, V_3)V_2\} \\ &+ f_2\{g(V_1, \phi V_3)\phi V_2 - g(V_2, \phi V_3)\phi V_1 \\ &+ 2g(V_1, \phi V_2)\phi V_3\} + f_3\{\eta(V_1)\eta(V_3)V_2 \\ &- \eta(V_2)\eta(V_3)V_1 + g(V_1, V_3)\eta(V_2)\xi \\ &- g(V_2, V_3)\eta(V_1)\xi\}, \end{aligned} \tag{14}$$

provided that vector fields V_1, V_2, and V_3 are on P, see [10]. In particular, if $f_1 = (c+3)/4$ and $f_2 = f_3 = (c-1)/4$, then P can be identified as a Sasakian space form. $f_1 = (c-3)/4$ and $f_2 = f_3 = (c+1)/4$ can lead to a Kenmotsu-space form [10, 18].

The semisymmetric metric connection ∇' and the Levi Civita connection ∇ defined on contact metric manifold (P^{2m+1}, g) are related by the following expression that is obtained by Yano [19] and is represented as

$$\nabla'_{W_1}W_2 = \nabla_{W_1}W_2 + \eta(W_2)W_1 - g(W_1, W_2)\xi, \tag{15}$$

where W_1 and W_2 are vector fields on P. As mentioned in [20], we have the following relation of the curvature tensor R with respect to the Levi-Civita connection ∇ and the curvature tensor R' regarding the semisymmetric metric connection ∇' of the generalized Sasakian space form.

$$\begin{aligned} R'(V_1, V_2)V_3 = R(V_1, V_2)V_3 &\\ &+ \{g(\phi V_2, V_3)V_1 - g(\phi V_1, V_3)V_2 \\ &+ g(V_2, V_3)\phi V_1 - g(V_1, V_3)\phi V_2\} \\ &+ \{\eta(V_2)V_1 - \eta(V_1)V_2\}\eta(V_3) \\ &+ \{g(V_2, V_3)\eta(V_1) - g(V_1, V_3)\eta(V_2)\}\xi, \end{aligned} \tag{16}$$

taking vector fields V_1, V_2, V_3, on P.

4. Stability on Generalized Sasakian Space Forms with Semisymmetric Metric Connection

Identity maps are always harmonic maps, but here, the second variational formula is not a trivial case. In this section, with the help of the second variational formula, we derive the inequalities for the stability criteria on the generalized Sasakian space forms with a semisymmetric metric connection. Consider the identity map on a compact generalized Sasakian space form $M(\varphi, \xi, \eta, g)$ that is $(\phi = 1_M)$. Then, the second variation formula is ([2]) as follows:

$$H_{1_M}(V, V) = \int_M h(\bar{\Delta}V, V)v_g - \sum_{i=1}^{2n+1}\int_M h(R(V, u_i)u_i, V)v_g, \tag{17}$$

where $V \in \Gamma(TM)$ and $\{u_1, \cdots, u_{2n+1}\}$ represents the local orthonormal frame on TM.

The rough Laplacian defined by (7) upon a generalized Sasakian manifold M^{2n+1} admitting a semisymmetric metric connection can be computed by the following lemma.

Lemma 1. *For a generalized Sasakian space form admitting semisymmetric metric connection, the rough Laplacian in* *the adopted frame field* $\{e_1, \cdots, e_n, \phi e_1, \cdots, \phi e_n, \xi\}$ *is given by*

$$\acute{\bar{\Delta}} Y = \bar{\Delta} Y + 2trB_Y - g(tr\nabla, Y)\xi - (2\,\mathrm{div}\,Y)\xi + 2\eta(Y)\xi - 2Y + \phi Y + \sum g(e_i, Y)\phi e_i + \sum g(e_i, \phi Y)e_i, \tag{18}$$

where $B_Y(V, W) = \eta(\nabla_V Y)W$.

Proof. Let ∇ and ∇ represent the semisymmetric connection and the Levi Civita connection on the generalized Sasakian space form, respectively. Therefore, it can be computed as

$$\begin{aligned}\acute{\nabla}_V \acute{\nabla}_V Y &= \nabla_V \acute{\nabla}_V Y + \eta\left(\acute{\nabla}_V Y\right)V - g\left(V, \acute{\nabla}_V Y\right)\xi = \nabla_V \nabla_V Y + \nabla_V(\eta(Y)V) \\ &\quad - \nabla_V(g(V, Y)\xi) + \eta(\nabla_V Y)V + \eta(V)\eta(Y)V - g(V, Y)V \\ &\quad - g(Y, \nabla_V V)\xi - g(V, \nabla_V Y)\xi - g(V, Y)\nabla_V \xi.\end{aligned} \tag{19}$$

We have $\nabla_V(\eta(Y)V) = \nabla_V(g(\xi, Y)V)$. Then, from equation (19), we have

$$\begin{aligned}\acute{\nabla}_V \acute{\nabla}_V Y &= \nabla_V \nabla_V Y + g(\nabla_V \xi, Y)V - \nabla_V(g(V, Y)\xi) + \eta(Y)\nabla_V V \\ &\quad + 2\eta(\nabla_V Y)V + \eta(V)\eta(Y)V - g(V, Y)V - g(\nabla_V V, Y)\xi \\ &\quad - g(V, \nabla_V Y)\xi - g(V, Y)\nabla_V \xi = \nabla_V \nabla_V Y + g(\nabla_V \xi, Y)V \\ &\quad + \eta(Y)\nabla_V V + 2\eta(\nabla_V Y)V + \eta(V)\eta(Y)V - g(V, Y)V \\ &\quad - 2g(\nabla_V V, Y)\xi - 2g(V, \nabla_V Y)\xi - 2g(V, Y)\nabla_V \xi \\ &= \nabla_V \nabla_V Y + g(V, \phi Y)V + \eta(Y)\nabla_V V + 2\eta(\nabla_V Y)V \\ &\quad - 2g(\nabla_V V, Y)\xi - 2g(V, \nabla_V Y)\xi - 2g(V, Y)\nabla_V \xi \\ &= \nabla_V \nabla_V Y + g(V, \phi Y)V + \eta(Y)\nabla_V V + 2\eta(\nabla_V Y)V \\ &\quad - 2g(\nabla_V V, Y)\xi - 2g(V, \nabla_V Y)\xi - 2g(V, Y)\nabla_V \xi \\ &= \nabla_V \nabla_V Y + g(V, \phi Y)V + \eta(Y)\nabla_V V + 2\eta(\nabla_V Y)V \\ &\quad - 2g(\nabla_V V, Y)\xi - 2g(V, \nabla_V Y)\xi + 2g(V, Y)\phi V \\ &\quad - 2g(V, Y)V + 2\eta(V)g(V, Y)\xi.\end{aligned} \tag{20}$$

Also, we have

$$\begin{aligned}\acute{\nabla}_V \acute{\nabla}_V Y - \acute{\nabla}_{\nabla_V V} Y &= \nabla_V \nabla_V Y - \nabla_{\nabla_V V} Y + g(V, \phi Y)V \\ &\quad + 2\eta(\nabla_V Y)V - g(\nabla_V V, Y)\xi - 2g(V, \nabla_V Y)\xi \\ &\quad + 2g(V, Y)\phi V - 2g(V, Y)V + 2\eta(V)g(V, Y)\xi.\end{aligned} \tag{21}$$

Take into account that $B_Y(V, W) = \eta(\nabla_V Y)W$. Then, in an adopted frame field $\{e_1, \cdots, e_n, \phi e_1, \cdots, \phi e_n, \xi\}$, we arrived at

$$\acute{\bar{\Delta}} Y = \bar{\Delta} Y + 2trB_Y - g(tr\nabla, Y)\xi - (2divY)\xi + 2\eta(Y)\xi - 2Y + \phi Y + \sum g(e_i, Y)\phi e_i + \sum g(e_i, \phi Y)e_i. \tag{22}$$

□

Theorem 2. *The second variation formula for the identity map on the generalized Sasakian space form admitting a semisymmetric connection is expressed as*

$$\begin{aligned}H_{1_M}(Y, Y) &= \int_M h(\bar{\Delta} Y, Y)v_g - (3f_2 + 2nf_1 - f_3 - 2n + 3)\int_M h(Y, Y)v_g \\ &\quad + (3f_2 + (2n-1)f_3 - 2n + 3)\int_M \eta(Y)\eta(Y)v_g.\end{aligned} \tag{23}$$

Proof.

$$H_{1_M}(Y, Y) = \int_M h\left(\bar{\Delta}' Y, Y\right)v_g - \sum_{i=1}^{2n+1}\int_M h\left(R'(Y, u_i)u_i, Y\right)v_g,$$

$$\begin{aligned}h\left(\acute{\Delta} Y, Y\right) &= h\left(\bar{\Delta} Y, Y\right) + 2h(tr\bar{B}_Y, Y) - h(tr\nabla, Y)h(\xi, Y) \\ &\quad - (2divY)h(\xi, Y) + 2\eta(Y)h(\xi, Y) - 2h(Y, Y) + h(\phi Y, Y) \\ &\quad + \sum h(e_i, Y)h(\phi e_i, Y) + \sum h(e_i, \phi Y)h(e_i, Y), \left(\bar{\Delta}' Y, Y\right) \\ &= h\left(\bar{\Delta} Y, Y\right) + 2h(trB_Y, Y) - h(tr\nabla, Y)h(\xi, Y) \\ &\quad - (2divY)h(\xi, Y) + 2\eta(Y)h(\xi, Y) - 2h(Y, Y) + h(\phi Y, Y) \\ &\quad + \sum h(e_i, Y)h(\phi e_i, Y) + \sum h(e_i, \phi Y)h(e_i, Y),\end{aligned} \tag{24}$$

since $\int_M \mathrm{div}\,(Y) = 0$, over a compact domain M, by Green's formula and $\eta(\nabla_{e_i} Y) = h(\nabla_{e_i} Y, \xi) = e_i h(Y, \xi) - h(Y, \nabla_{e_i}\xi) = 0$, similarly, $h(tr\nabla, Y)h(\xi, Y) = 0$. Therefore, we have

$$\int_M h\left(\bar{\Delta}' Y, Y\right)v_g = \int_M h(\bar{\Delta} Y, Y)v_g + 2\int_M \eta^2(Y)v_g - 2\int_M h(Y, Y)v_g. \tag{25}$$

Now, we consider a φ-adapted orthonormal local frame $\{e_i, \varphi e_i, \xi\}$. After that, we have

$$\begin{aligned}\sum_{i=1}^{2n+1} h(R(e_i, Y)e_i, Y) &= (f_1 - 3f_2)\sum_{i=1}^{n}\left\{h(Y, e_i)^2 + h(Y, \varphi e_i)^2\right\} \\ &\quad - [(2n+1)f_1 - f_3]h(Y, Y) \\ &\quad + [(2n-1)f_3 + f_1]h(Y, \xi)^2,\end{aligned} \tag{26}$$

and thus, we have

$$\begin{aligned}\sum_{i=1}^{2n+1} h(R(e_i, Y)e_i, Y) &= -[3f_2 + 2nf_1 - f_3]h(Y, Y) \\ &\quad + [3f_2 + (2n-1)f_3]h(Y, \xi)^2,\end{aligned} \tag{27}$$

and with semisymmetric metric connection, it can be written as

$$\sum_{i=1}^{2n+1} h\Big(R'(Y,e_i)e_i, Y\Big) = \sum_{i=1}^{2n+1} h(R(Y,e_i)e_i, Y) - (2n-1)h(Y,Y) + (2n-1)\eta^2(Y) = [3f_2 + 2nf_1 - f_3]h(Y,Y) - [3f_2 + (2n-1)f_3]h(Y,\xi)^2 - (2n-1)h(Y,Y) + (2n-1)\eta^2(Y). \tag{28}$$

From (24) and (28), we have acquired the result of ((24)). □

Proposition 3. *Consider a compact generalized Sasakian space form M admitting a semisymmetric metric connection. The identity map 1_M is weakly stable, if $(3f_2 + 2nf_1 - f_3 - 2n + 3) \leq 0$ and $(3f_2 + (2n-1)f_3 - 2n + 3) \geq 0$.*

Proof. We can easily prove that

$$\int_M h\big(\bar{\Delta} V, V\big)v_g = \int_M h\big(\nabla^{\sim} V, \nabla^{\sim} V\big)v_g, \quad V \in \Gamma(TM). \tag{29}$$

□ □

Now, the second variation formula with respect to a semisymmetric connection becomes

$$H_{1_M}(Y,Y) = \int_M h\Big(\nabla^{\sim} Y, \nabla^{\sim} Y) - (3f_2 + 2nf_1 - f_3 - 2n + 3)\int_M h(Y,Y)v_g + (3f_2 + (2n-1)f_3 - 2n + 3)\int_M \eta(Y)\eta(Y)v_g. \tag{30}$$

Therefore, for the inequalities $(3f_2 + 2nf_1 - f_3 - 2n + 3) \leq 0$ and $(3f_2 + (2n-1)f_3 - 2n + 3) \geq 0$, the identity map is weakly stable.

Corollary 4. *Let M be the Kenmotsu space form admitting a semisymmetric metric connection; then, the identity map on its compact domain is stable if $(3n - 7/n + 1) \leq c \leq ((7(n-1))/(n+1))$.*

On the Kenmotsu space form M, $f_1 = ((c-3)/4)$, $f_2 = f_3 = ((c+1)/4)$ [10]. And $(3f_2 + 2nf_1 - f_3 - 2n + 3) \leq 0$ implies $c \leq ((7(n-1))/(n+1))$, and $(3f_2 + (2n-1)f_3 - 2n + 3) \geq 0$ implies $c \geq ((3n-7)/(n+1))$. Then, by the above results, the identity of the 1_M map becomes stable for the values of $c \in [((3n-7)/(n+1)), ((7(n-1))/(n+1))]$.

5. Conclusion

The 2nd variational formula for a generalized Sasakian space form admitting a semisymmetric metric connection has been successfully obtained in this work. All results in this work are novel where inequalities concerning the stability criteria of a compact generalized Sasakian space form admitting a semisymmetric metric connection have been established. Further research works can be conducted depending on all our obtained results in this paper.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

The authors declare that the study was realized in collaboration with equal responsibility. All authors read and approved the final manuscript.

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