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Research Article

Fixed Point, Data Dependence, and Well-Posed Problems for Multivalued Nonlinear Contractions

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The aim of the paper is to discuss data dependence, existence of fixed points, strict fixed points, and well posedness of some multivalued generalized contractions in the setting of complete metric spaces. Using auxiliary functions, we introduce Wardowski type multivalued nonlinear operators that satisfy a novel class of contractive requirements. Furthermore, the existence and data dependence findings for these multivalued operators are obtained. A nontrivial example is also provided to support the results. The results generalize, improve, and extend existing results in the literature.

1. Introduction and Preliminaries

Let $(\mathfrak{Z}, d)$ be a metric space (in short MS). The set of all nonempty subsets of $\mathfrak{Z}$ is denoted by $P(\mathfrak{Z})$, the set of all nonempty closed subsets of $\mathfrak{Z}$ is denoted by $CL(\mathfrak{Z})$, the set of all nonempty closed and bounded subsets of $\mathfrak{Z}$ is denoted by $CB(\mathfrak{Z})$, and the set of all nonempty compact subsets of X is denoted by $K(\mathfrak{Z})$. It is obvious that $CB(\mathfrak{Z})$ includes $K(\mathfrak{Z})$. For $U, V \in CB(\mathfrak{Z})$, define $H : CB(\mathfrak{Z}) \times CB(\mathfrak{Z}) \longrightarrow [0, \infty)$ by

$$H(U, V) = \max\left\{\sup_{u\in U} D(u, V), \sup_{v\in V} D(v, U)\right\}, \qquad (1)$$

where $D(u, V) = \inf\{d(u, v): v \in V\}$. Such a function H is called the Pompei-Hausdorff metric induced by d, for more details, see, e.g., [1].

Lemma 1 [2]. *Let $(\mathfrak{Z}, d)$ be a MS and $A, B \in CL(\mathfrak{Z})$ with $H(A, B) > 0$. Then, for each $h > 1$ and for each $a \in A$, there exists $b = b(a) \in B$ such that $d(a, b) < hH(A, B)$.*

If $\Omega : \mathfrak{Z} \longrightarrow P(\mathfrak{Z})$ is a multivalued operator, then an element $\varpi \in \mathfrak{Z}$ is called a fixed point for Ω if $x \in \Omega\varpi$. The symbol $\operatorname{fix}\Omega = \{\varpi \in \mathfrak{Z} : x \in \Omega\varpi\}$ denotes the fixed point set of Ω. On the other hand, a strict fixed point for Ω is an element $\varpi \in \mathfrak{Z}$ with the property $\{x\} = \Omega\varpi$. The set of all strict fixed points of Ω is denoted by $\operatorname{SFix}\Omega$.

Banach's contraction principle [3] is the most fundamental result in metric fixed point theory. Since then, many authors have extended and generalized Banach's contraction principle in many ways. Extensions of Banach's contraction principle have spawned a wealth of literature. (see [13, 29]). One of an attractive and important generalization is given by Wardowski in [10]. He introduced a new type of contraction called F-contraction and proved a new fixed point theorem concerning F-contraction.

Definition 2 [10]. Let $(\mathfrak{Z}, d)$ be a *MS*. A mapping $\Omega : \mathfrak{Z} \longrightarrow \mathfrak{Z}$ is said to be F -contraction if there exists $\tau > 0$ such that

$$d(\Omega\varpi, \Omega\omega) > 0 \text{ implies } \tau + F(d(\Omega\varpi, \Omega\omega)) \le F(d(\varpi, \omega)), \qquad (2)$$

for all $x, y \in X$, where $F : (0,\infty) \longrightarrow \mathbb{R}$ is a function satisfying

$(F1)$ F is strictly increasing

$(F2)$ For all sequence $\{t_n\} \subseteq (0,\infty)$, $\lim_{n\longrightarrow\infty} t_n = 0$, if and only if $\lim_{n\longrightarrow\infty} F(t_n) = -\infty$

$(F3)$ There exists $0 < k < 1$ such that $\lim_{k\longrightarrow 0^+} t^k F(t) = 0$

We denote by $\Delta(F)$ the collection of all functions $F : (0,\infty) \longrightarrow \mathbb{R}$ satisfying $(F1)$, $(F2)$, and $(F3)$. Also, define

$$\Delta(\bigcirc *) = \{F \in \Delta(F) \mid F \text{ satifies } (F4)\}, \tag{3}$$

where

$(F4)$ $F(\inf A) = \inf F(A)$ for all $A \subset (0,\infty)$ with $\inf A > 0$.

Theorem 3 [10]. *Let $(\mathfrak{Z}, d)$ be a complete MS and $\Omega : \mathfrak{Z} \longrightarrow \mathfrak{Z}$ be a F-contraction. Then, Ω has a unique fixed point $\varpi^* \in \mathfrak{Z}$ and for every $\varpi_0 \in \mathfrak{Z}$, a picard sequence $\{T^n \varpi_0\}_{n\in\mathbb{N}}$ converges to ϖ^*.*

Further, Turinici [11] is replaced $(F2)$ by the following condition: $(F2')$ $\lim_{t\longrightarrow 0^+} F(t) = -\infty$.

Note that, in general, $F \in \Delta(F)$ is not continuous. However, by $(F1)$ and the properties of the monotone functions, we have the following proposition.

Proposition 4 [11]. *Let $F : (0,\infty) \longrightarrow \mathbb{R}$ be a function satisfying $(F1)$ and $(F2)$, and then there exists a countable subset $\Lambda(F) \subseteq (0, 1)$ such that*

$$F(t-0) = F(t) = F(t+0) \text{foreach } t \in (0,1) \setminus \Lambda(F). \tag{4}$$

Lemma 5 [11]. *Let $F : (0,\infty) \longrightarrow \mathbb{R}$ be a function satisfying $(F1)$ and $(F2')$. Then, for each sequence $\{t_n\}$ in $(0, 1)$,*

$$F(t_n) \longrightarrow -\infty \Rightarrow t_n \longrightarrow 0. \tag{5}$$

After this, many authors generalized the F-contraction in several ways (see [12–22] and references therein). In 2015, Klim and Wardowski [23] extended the concept of F-contractive mappings to the case of nonlinear F-contractions and proved fixed point theorems via the dynamic processes. In 2017, Wardowski [24] omitted one of the conditions of F-contraction and introduced nonlinear F-contraction).

Definition 6 [24]. A mapping $\Omega : \mathfrak{Z} \longrightarrow \mathfrak{Z}$ is said to be a (φ, F)-contraction (or nonlinear F-contraction), if there exists $F \in \mathcal{F}$ and a function $\varphi : (0,\infty) \longrightarrow (0,\infty)$ satisfying

(H1) $\liminf_{s\longrightarrow t^+} \varphi(s) > 0$, forall $t \geq 0$.

(H2) $\varphi(d(\varpi, \omega)) + F(d(\Omega\varpi, \Omega\omega)) \leq F(d(\varpi, \omega))$, for all $\varpi, \omega \in \mathfrak{Z}$ such that $\Omega\varpi \neq \Omega\omega$

Theorem 7 [24]. *Let $(\mathfrak{Z}, d)$ be a complete MS and let $\Omega : \mathfrak{Z} \longrightarrow \mathfrak{Z}$ be a (φ, F)-contraction. Then, Ω has a unique fixed point in $\mathfrak{Z}$.*

Very recently, Iqbal and Rizwan [25] considered a rich class of functions and generalized Definition 6 to obtain some new fixed point theorems for nonlinear F-contractions involving generalized distance. On unifying the concept of Wardowski [10], Nadler [9] and Altun et al. [26] gave the concept of multivalued F-contraction as follows.

Definition 8 [26]. Let $(\mathfrak{Z}, d)$ be a complete MS and $\Omega : \mathfrak{Z} \longrightarrow CB(\mathfrak{Z})$ be a mapping. Then, Ω is a multivalued F-contraction, if there exists $\tau > 0$ and $F \in \Delta(F)$ such that for all $\varpi, \omega \in \mathfrak{Z}$,

$$H(\Omega\varpi, \Omega\omega) > 0 \Rightarrow \tau + F(H(\Omega\varpi, \Omega\omega)) \leq F((d(\varpi, \omega)). \tag{6}$$

Theorem 9 [26]. *Let $(\mathfrak{Z}, d)$ be a complete MS and $\Omega : \mathfrak{Z} \longrightarrow K(\mathfrak{Z})$ be a multivalued F-contraction, and then Ω has a fixed point in $\mathfrak{Z}$.*

Afterwards, Olgun et al. [27] proved the nonlinear case of Theorem 9 as follows.

Theorem 10 [27]. *Let $(\mathfrak{Z}, d)$ be a complete MS and $\Omega : \mathfrak{Z} \longrightarrow K(\mathfrak{Z})$, if there exists $F \in \Delta(F)$ and $\varphi : (0, 1) \longrightarrow (0, 1)$ satisfying*

$$\begin{gathered} \liminf_{s\longrightarrow t^+} \varphi(s) > 0 \text{ for all} \varpi, \omega \in \mathfrak{Z}, \\ \varphi(d(\varpi, \omega)) + F(H(\Omega\varpi, \Omega\omega)) \leq F(d(\varpi, \omega)). \end{gathered} \tag{7}$$

Then, Ω has a fixed point in $\mathfrak{Z}$.

For more directions for nonlinear F-contractions, consult [28, 29] and references there in. Next, we denote by $\mathcal{P}$ the set of all continuous mappings $\rho : [0,\infty)^5 \longrightarrow [0,\infty)$ satisfying the following conditions:

(ρ_1) $\rho(1, 1, 1, 2, 0) \in (0, 1]$;

(ρ_2) ρ is subhomogeneous; that is, for all $(\varpi_1, \varpi_2, \varpi_3, \varpi_4, \varpi_5) \in [0,\infty)^5$ and $\alpha \geq 0$, we have $\rho(\alpha\varpi_1, \alpha\varpi_2, \alpha\varpi_3, \alpha\varpi_4, \alpha\varpi_5) \leq \alpha\rho(\varpi_1, \varpi_2, \varpi_3, \varpi_4, \varpi_5)$

(ρ_3) ρ is nondecreasing function; that is, for $\varpi_i, \omega_i \in \mathbb{R}^+$, $\varpi_i \leq \omega_i$, $i = 1, \cdots, 5$, we have $\rho(\varpi_1, \varpi_2, \varpi_3, \varpi_4, \varpi_5) \leq \rho(\omega_1, \omega_2, \omega_3, \omega_4, \omega_5)$

If $\varpi_i, \omega_i \in \mathbb{R}^+$, $\varpi_i < \omega_i$, $i = 1, \cdots, 4$, then $\rho(\varpi_1, \varpi_2, \varpi_3, \varpi_4, 0) < \rho(\omega_1, \omega_2, \omega_3, \omega_4, 0)$ and $\rho(\varpi_1, \varpi_2, \varpi_3, 0, \varpi_5) < \rho(\omega_1, \omega_2, \omega_3, 0, \omega_5)$.

Also, define

$$\mathbb{P} = \{\rho \in \mathcal{P} | \rho(1, 0, 0, 1, 1) \in (0, 1]\}. \tag{8}$$

Note that $\mathbb{P} \subseteq \mathcal{P}$.

Example 1. Define $\rho_1 : [0,\infty)^5 \longrightarrow [0,\infty)$ by

$$\rho_1(\varpi_1, \varpi_2, \varpi_3, \varpi_4, \varpi_5) = \varpi_1 + \xi x_5, \tag{9}$$

where $\xi \in (0, 1)$. Then, $\rho_1 \in \mathcal{P}$. Since $\rho_1(1, 0, 0, 1, 1) = 1 + \xi > 1$, so $\rho_1 \notin \mathbb{P}$. Also, note that $\rho_1(1, 1, 1, 0, 2) = 1 + 2\xi > 1$, so $\rho_1 \notin \mathfrak{R}$ [30].

Example 2. Define $\rho_2 : [0,\infty)^5 \longrightarrow [0,\infty)$ by

$$\rho_2(\varpi_1, \varpi_2, \varpi_3, \varpi_4, \varpi_5) = g \max\left\{\varpi_1, \frac{1}{2}(\varpi_2 + \varpi_3), \frac{1}{2}(\varpi_4 + \varpi_5)\right\}, \tag{10}$$

where $g \in (0, 1)$. Then, $\rho_2 \in \mathbb{P}$.

Example 3. Define $\rho_3 : [0,\infty)^5 \longrightarrow [0,\infty)$ by

$$\rho_3(\varpi_1, \varpi_2, \varpi_3, \varpi_4, \varpi_5) = a\varpi_1 + b(\varpi_2 + \varpi_3) + c(\varpi_4 + \varpi_5), \tag{11}$$

where $a + 2b + 2c < 1$. Then, $\rho_3 \in \mathscr{P}$.

Now, we prove the following Lemma.

Lemma 11. *If $\rho \in \mathscr{P}$ and $u, v \in [0,\infty)$ are such that*

$$u \le \max\{\rho(v, v, u, v + u, 0), \rho(v, v, u, 0, v + u), \rho(v, u, v, v + u, 0), \rho(v, u, v, 0, v + u)\}, \tag{12}$$

then $u \le v$.

Proof. Without loss of generality, we can suppose that $u \le \rho(v, v, u, v + u, 0)$. If $v < u$, then

$$u \le \rho(v, v, u, v + u, 0) < \rho(u, u, u, 2u, 0) \le u\rho(1, 1, 1, 2, 0) \le u, \tag{13}$$

a contradiction. Thus, $u \le v$. □

Now, consider following examples.

Example 4. Let $F : (0,\infty) \longrightarrow \mathbb{R}$ be a function defined by

$$F(t) = \frac{-1}{t} \text{ for } all\ t \in (0,\infty). \tag{14}$$

Then, F satisfies $(F1)$, $(F2')$, and F that is continuous but does not satisfies $(F3)$.

Example 5. Let $F : (0,\infty) \longrightarrow \mathbb{R}$ be a function defined by

$$F(t) = \begin{cases} \dfrac{-1}{t} & \text{if } t \in (0, 1) \\ 0 & \text{otherwise.} \end{cases} \tag{15}$$

Then, F satisfies $(F1)$ and $(F2')$ but F is not continuous.

Example 6. [31] Let $F : (0,\infty) \longrightarrow \mathbb{R}$ *be* a function defined by

$$F(t) = -\frac{1}{(t + [t])^{\ell}}, \tag{16}$$

where $[t]$ denotes the integral part of Ω and $\ell \in (0, 1/a)$, $a > 1$. Then, F satisfies $(F1)$, $(F2')$, and $(F3)$ but F is not continuous.

Examples 4–6 clearly show that there exist some functions $F : (0,\infty) \longrightarrow \mathbb{R}$ which does not satisfy the condition of continuity, $(F1)$, $(F2)$, and $(F3)$ at a time. By getting inspiration from this, in this paper, we prove fixed point results for contractive conditions involving functions F, not necessarily continuous and belongs to $\Delta(F)$ by taking support of a continuous function from $\mathscr{P}$. Our results generalize many results appearing recently in the literature including Altun et al. [32], Olgun et al., [27] Sgroi and Vetro [33], Vetro [34], Wardowski [24], and Wardowski and Dung [35].

For convenience, we set Φ, the collection of all functions $\chi : (0,\infty) \longrightarrow (0,\infty)$ satisfying

$$\lim_{s \longrightarrow t^+} \inf \chi(s) > 0 \text{ forall } t \ge 0. \tag{17}$$

Theorem 12. *Let $(\mathfrak{Z}, d)$ be a complete MS and $\Omega : \mathfrak{Z} \longrightarrow K(\mathfrak{Z})$ be a multivalued mapping. Assume that there exists $\chi \in \Phi$, a nondecreasing real valued function F_1 on $(0, \infty)$ and a real valued function F_2 on $(0, \infty)$ satisfying condition $(F2')$ and $(F3)$ such that the following conditions hold:*

(N1) $F_1(c) \le F_2(c)$ for all $c > 0$

(N2) For all $\varpi, \omega \in \mathfrak{Z}$ and $\rho \in \mathscr{P}$, $H(\Omega\varpi, \Omega\omega)) > 0$ implies

$$\begin{aligned} &\chi(d(\varpi, \omega)) + F_2(H(\Omega\varpi, \Omega\omega)) \\ &\quad \le F_1(\rho(d(\varpi, \omega), D(\varpi, \Omega\varpi), D(\omega, \Omega\omega), D(\varpi, \Omega\omega), D(\omega, \Omega\varpi))). \end{aligned} \tag{18}$$

Then, fix Ω is nonempty.

Proof. Let $\varpi_0 \in \mathfrak{Z}$ be an arbitrary point and $\varpi_1 \in \Omega\varpi_0$. Assume that $\varpi_1 \in \Omega\varpi_1$; otherwise, ϖ_1 is a fixed point of Ω, and the proof is complete. Then, $D(\varpi_1, \Omega\varpi_1) > 0$ and consequently $H(\Omega\varpi_0, \Omega\varpi_1) > 0$. Compactness of $\Omega\varpi_1$ ensures the existence of $\varpi_2 \in \Omega\varpi_1$, such that $d(\varpi_1, \varpi_2) = D(\varpi_1, \Omega\varpi_1)$. From (N1) and (N2), we get

$$\begin{aligned} F_1(d(\varpi_1, \varpi_2)) &= F_1(D(\varpi_1, \Omega\varpi_1)) \le F_1(H(\Omega\varpi_0, \Omega\varpi_1)) \le F_2(H(\Omega\varpi_0, \Omega\varpi_1)) \\ &\le F_1(\rho(d(\varpi_0, \varpi_1), D(\varpi_0, \Omega\varpi_0), D(\varpi_1, \Omega\varpi_1), D(x_0, \Omega\varpi_1), D(\varpi_1, \Omega\varpi_0))) \\ &\quad - \chi(d(\varpi_0, \varpi_1)) < F_1(\rho(d(\varpi_0, \varpi_1), d(x_0, \varpi_1), d(\varpi_1, \varpi_2), d(\varpi_0, \varpi_2), 0)). \end{aligned} \tag{19}$$

□

Since F_1 is an nondecreasing function, (19) with (ρ_3) implies that

$$\begin{aligned} d(\varpi_1, \varpi_2) &< \rho(d(\varpi_0, \varpi_1), d(x_0, \varpi_1), d(\varpi_1, \varpi_2), d(x_0, \varpi_1), 0) \\ &\le \rho(d(\varpi_0, \varpi_1), d(\varpi_0, \varpi_1), d(\varpi_1, \varpi_2), d(\varpi_0, \varpi_1) + d(\varpi_1, \varpi_2), 0). \end{aligned} \tag{20}$$

By using Lemma 11, (20) implies

$$d(\varpi_1, \varpi_2) < d(\varpi_0, \varpi_1). \tag{21}$$

Next, arguing as previous, we get $\varpi_3 \in \Omega\varpi_2$, such that $d(\varpi_2, \varpi_3) = D(\varpi_2, \Omega\varpi_2)$ with $D(x_2, \Omega\varpi_2) > 0$. Also, by using Lemma 11, from (N1) and (N2), we obtain

$$d(\varpi_2, \varpi_3) < d(\varpi_1, \varpi_2). \tag{22}$$

Continuing in the same manner, we get a sequence $\{\varpi_n\} \subset \mathfrak{Z}$ such that $\varpi_{n+1} \in \Omega\varpi_n$ satisfying $d(\varpi_n, \varpi_{n+1}) = D(\varpi_n, \Omega\varpi_n)$ with $D(\varpi_n, \Omega\varpi_n) > 0$ and

$$d(\varpi_n, \varpi_{n+1}) < d(\varpi_{n-1}, \varpi_n), \tag{23}$$

for all $n \in \mathbb{N}$. (23) implies that $\{d(\varpi_n, \varpi_{n+1})\}_{n\in\mathbb{N}}$ is a decreasing sequence of positive real numbers. Hence, from (N1) and (N2), we get

$$\begin{aligned}
&\chi(d(\varpi_n, \varpi_{n+1})) + F_2(H(\Omega\varpi_n, \Omega\varpi_{n+1})) \\
&\quad \le F_1(\rho(d(\varpi_n, \varpi_{n+1}), D(\varpi_n, \Omega\varpi_n), D(\varpi_{n+1}, \Omega\varpi_{n+1}), D(\varpi_n, \Omega\varpi_{n+1}), D(\varpi_{n+1}, \Omega\varpi_n))) \\
&\quad \le F_1(\rho(d(\varpi_n, \varpi_{n+1}), d(\varpi_n, \varpi_{n+1}), d(\varpi_{n+1}, \varpi_{n+2}), d(\varpi_n, \varpi_{n+1}) + d(\varpi_{n+1}, \varpi_{n+2}), 0)) \\
&\quad \le F_1(\rho(d(\varpi_n, \varpi_{n+1}), d(\varpi_n, \varpi_{n+1}), d(\varpi_n, \varpi_{n+1}), 2d(\varpi_n, \varpi_{n+1}), 0)) \\
&\quad \le F_1(d(\varpi_n, \varpi_{n+1})\rho(1, 1, 1, 2, 0)) \le F_1(d(\varpi_n, \varpi_{n+1})) = F_1(D(\varpi_n, \Omega\varpi_n)) \\
&\quad \le F_1(H(\Omega\varpi_{n-1}, \Omega\varpi_n)) \le F_2(H(\Omega\varpi_{n-1}, \Omega\varpi_n)).
\end{aligned} \tag{24}$$

Thus, for all $n \in \mathbb{N}$,

$$F_2(H(\Omega\varpi_n, \Omega\varpi_{n+1})) \le F_2(H(\Omega\varpi_{n-1}, \Omega\varpi_n)) - \chi(d(\varpi_n, \varpi_{n+1})). \tag{25}$$

Since $\chi \in \Phi$, there exists $h > 0$ and $n_0 \in \mathbb{N}$ such that $\chi(d(\varpi_n, \varpi_{n+1})) > h$, for all $n \ge n_0$. From (25), we obtain

$$\begin{aligned}
F_2(H(\Omega\varpi_n, \Omega\varpi_{n+1})) &\le F_2(H(\Omega\varpi_{n-1}, \Omega\varpi_n)) - \chi(d(\varpi_n, \varpi_{n+1})) \\
&\le F_2(H(\Omega\varpi_{n-2}, \Omega\varpi_{n-1})) - \chi(d(\varpi_{n-1}, \varpi_n)) \\
&\quad - \chi(d(\varpi_n, \varpi_{n+1})) \vdots \\
&\le F_2(H(\Omega\varpi_0, \Omega\varpi_1)) - \sum_{i=1}^{n} \chi(d(\varpi_i, \varpi_{i+1})) \\
&= F_2(H(\Omega\varpi_0, \Omega\varpi_1)) - \sum_{i=1}^{n_0-1} \chi(d(\varpi_i, \varpi_{i+1})) \\
&\quad - \sum_{i=n_0}^{n} \chi(d(\varpi_i, \varpi_{i+1})) \\
&\le F_2(H(\Omega\varpi_0, \Omega\varpi_1)) - (n - n_0)h, n \ge n_0.
\end{aligned} \tag{26}$$

Taking $n \longrightarrow \infty$ in (26), we get $F_2(H(\Omega\varpi_n, \Omega\varpi_{n+1})) \longrightarrow -\infty$ and by $(F2')$, we have

$$\lim_{n\longrightarrow\infty} H(\Omega\varpi_n, \Omega\varpi_{n+1}) = 0, \tag{27}$$

which further implies that

$$\lim_{n\longrightarrow\infty} d(\varpi_n, \varpi_{n+1}) = \lim_{n\longrightarrow\infty} D(\varpi_n, \Omega\varpi_n) \le \lim_{n\longrightarrow\infty} H(\Omega\varpi_{n-1}, \Omega\varpi_n) = 0. \tag{28}$$

Now from $(F3)$, there exists $k \in (0, 1)$ such that

$$\lim_{n\longrightarrow\infty} (H(\Omega\varpi_n, \Omega\varpi_{n+1}))^k F_2(H(\Omega\varpi_n, \Omega\varpi_{n+1})) = 0. \tag{29}$$

Then, from (26), for all $n \in \mathbb{N}$, we have

$$\begin{aligned}
&(H(\Omega\varpi_n, \Omega\varpi_{n+1}))^k F_2(H(\Omega\varpi_n, \Omega\varpi_{n+1})) \\
&\quad - (H(\Omega\varpi_n, \Omega\varpi_{n+1}))^k F_2(H(\Omega\varpi_0, \Omega\varpi_1)) \\
&\le (H(\Omega\varpi_n, \Omega\varpi_{n+1}))^k (F_2(H(\Omega\varpi_0, \Omega\varpi_1)) - (n - n_0)h) \\
&\quad - (H(\Omega\varpi_n, \Omega\varpi_{n+1}))^k F_2(H(\Omega\varpi_0, \Omega\varpi_1)) \\
&= -(H(\Omega\varpi_n, \Omega\varpi_{n+1}))^k (n - n_0)h \le 0.
\end{aligned} \tag{30}$$

Taking limit $n \longrightarrow \infty$, in (30) and using (27) and (29), we have

$$\lim_{n\longrightarrow\infty} n(H(\Omega\varpi_n, \Omega\varpi_{n+1}))^k = 0. \tag{31}$$

Observe that from (31), there exist $sn_1 \in \mathbb{N}$ such that $n(H(\Omega\varpi_n, \Omega\varpi_{n+1}))^k \le 1$ for all $n \ge n_1$. Thus, for all $n \ge n_1$, we have

$$H(\Omega\varpi_n, \Omega\varpi_{n+1}) \le \frac{1}{n^{1/k}} \text{ for } all\, n \ge n_1, \tag{32}$$

which further implies that

$$d(\varpi_n, \varpi_{n+1}) = D(\varpi_n, \Omega\varpi_n) \le H(\Omega\varpi_{n-1}, \Omega\varpi_n) \le \frac{1}{n^{1/k}} \text{for all } n \ge n_1. \tag{33}$$

Now, in order to show that $\{\varpi_n\}_{n\in\mathbb{N}}$ is Cauchy sequence, consider $m, n \in \mathbb{N}$ such that $m > n > n_1$. From (33), we get

$$d(\varpi_m, \varpi_n) \le \sum_{i=n}^{m-1} d(\varpi_i, \varpi_{i+1}) \le \sum_{i=n}^{m-1} \frac{1}{i^{1/k}} \le \sum_{i=n}^{\infty} \frac{1}{i^{1/k}}. \tag{34}$$

As a result of the above and the series' convergence, $\sum_{i=n}^{\infty} (1/i^{1/k})$, we receive that $\{\varpi_n\}_{n\in\mathbb{N}}$ is Cauchy sequence. Since $\mathfrak{Z}$ is a complete space, so there exists $\varpi^* \in \mathfrak{Z}$ such that

$$\lim_{n\longrightarrow\infty} \varpi_n = \varpi^*. \tag{35}$$

Now,

$$\begin{aligned}
F_1(H(\Omega\varpi, \Omega\omega)) &\le F_2(H(\Omega\varpi, \Omega\omega)) < \phi(d(\varpi, \omega)) + F_2(H(\Omega\varpi, \Omega\omega) \\
&\le F_1(\rho(d(\varpi, \omega), D(\varpi, \Omega\varpi), D(\omega, \Omega\omega), D(\varpi, \Omega\omega), D(\omega, \Omega\varpi))).
\end{aligned} \tag{36}$$

Since F_1 is nondecreasing function, we obtain for all ϖ

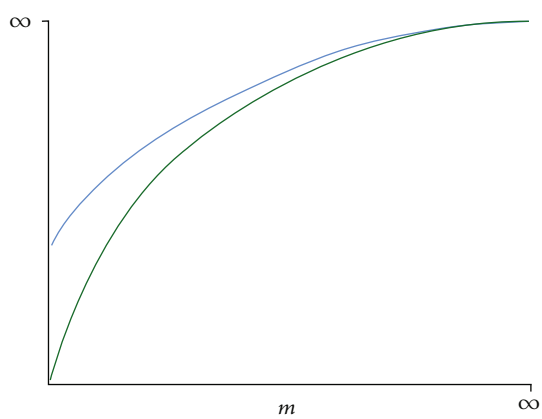

Figure 1: Graph of Inequality (44).

$, \omega \in \mathfrak{Z}$.

$$H(\Omega\varpi, \Omega\omega) \leq \rho(d(\varpi, \omega), D(\varpi, \Omega\varpi), D(\omega, \Omega\omega), D(\varpi, \Omega\omega), D(\omega, \Omega\varpi)). \tag{37}$$

We claim that ϖ^* is fixed point of $\mathfrak{Z}$. On contrary, suppose that $D(\varpi^*, \Omega\varpi^*) > 0$ and by equation (37), we have

$$\begin{aligned} D(\varpi^*, \Omega\varpi^*) &\leq d(\varpi^*, \varpi_{n+1}) + D(\varpi_{n+1}, \Omega\varpi^*) \\ &< d(\varpi^*, \varpi_{n+1}) + H(\Omega\varpi_n, \Omega\varpi^*) \\ &< d(\varpi^*, \varpi_{n+1}) + \rho(d(\varpi_n, \varpi^*), D(\varpi_n, \Omega\varpi_n), \\ &\quad D(\varpi^*, \Omega\varpi^*), D(\varpi_n, \Omega\varpi^*), D(x^*, \Omega\varpi_n)) \\ &\leq d(\varpi^*, \varpi_{n+1}) + \rho(d(\varpi_n, \varpi^*), d(\varpi_n, \varpi_{n+1}), \\ &\quad D(\varpi^*, \Omega\varpi^*), d(\varpi_n, \varpi^*) \\ &\quad + D(\varpi^*, \Omega\varpi^*), d(\varpi^*, \varpi_{n+1})). \end{aligned} \tag{38}$$

Passing to limit as $n \longrightarrow \infty$ in the above inequality, we obtain

$$D(\varpi^*, \Omega\varpi^*) < \rho(0, 0, D(\varpi^*, \Omega\varpi^*), 0 + D(\varpi^*, \Omega\varpi^*), 0), \tag{39}$$

which implies by Lemma 1 that

$$0 < D(\varpi^*, \Omega\varpi^*) < 0, \tag{40}$$

which is a contradiction. Hence, $D(\varpi^*, \Omega\varpi^*) = 0$. Since $\Omega\varpi^*$ is closed, therefore, $\varpi^* \in \Omega\varpi^*$.

Remark 13. By defining $F_1 = F_2 = F$ and $\rho(\varpi_1, \varpi_2, \varpi_3, \varpi_4, \varpi_5) = \varpi_1$ in Theorem 12, we get back Theorem 2.3 of [27].

Example 7. Consider $\mathfrak{Z} = \{u_n = n(n+1)/2 : n \in \mathbb{N}\}$, and then $\mathfrak{Z}$ is complete MS with metric $d(u, v) = |u - v|$. Define functions $F_1, F_2 : (0,\infty) \longrightarrow \mathbb{R}$ by

$$F_1(u) = \begin{cases} -\dfrac{1}{u} & \text{if } u \in (0, 1) \\ u & \text{if } u \in [1,\infty) \end{cases} \tag{41}$$

and $F_2(u) = \ln u + u$ for all $u \in (0,\infty)$, respectively. Then F_1 is nondecreasing, F_2 satisfy the conditions $(F2')$ and $(F3)$ and $F_1(u) \leq F_2(u)$ for all $u > 0$.

Next, define $\Omega : \mathfrak{Z} \longrightarrow K(\mathfrak{Z})$, $\rho : [0,\infty)^5 \longrightarrow [0,\infty)$ and $\chi : (0,\infty) \longrightarrow (0,\infty)$ by

$$\Omega\varpi = \begin{cases} \{u_1\} & \text{if } \varpi = u_1, \\ \{u_1, u_2\} & \text{if } \varpi = u_n, n \geq 2, \end{cases} \tag{42}$$
$$\rho(\varpi_1, \varpi_2, \varpi_3, \varpi_4, \varpi_5) = \varpi_1 + \xi\varpi_5, \xi \in (0, 1)$$

and $\chi(t) = 1/t$ for all $t \in (0,\infty)$, respectively. Then $\chi \in \Phi$ and $\rho \in \mathscr{P}$ (see Example 1). Observe that

$$m, n \in \mathbb{N}, H(\Omega u_m, \Omega u_n) > 0 \Leftrightarrow (m > 2 \text{and } n = 1). \tag{43}$$

Assume that $H(\Omega\varpi, \Omega\omega) > 0$, and then $m > 2$ and $n = 1$. From Figure 1, it is clear that

$$\frac{2}{|m^2 + m - 2|} + \ln\left|\frac{|m^2 - m - 2|}{2}\right| + \frac{|m^2 - m - 2|}{2} \leq \frac{|m^2 + m - 2|}{2}, \tag{44}$$

$H(\Omega u_m, \Omega u_1) = |u_{m-1} - 1|$ and $D(u_1, \Omega u_m) = 0$. Which further implies that

$$\begin{aligned} \chi(d(u_m, u_1)) + F_2(H(\Omega u_m, \Omega u_1)) &= \frac{1}{|u_m - u_1|} + F_2(|u_{m-1} - 1|) \\ &= \frac{2}{|m^2 + m - 2|} + \ln\left|\frac{|m^2 - m - 2|}{2}\right| + \frac{|m^2 - m - 2|}{2} \\ &\leq \frac{|m^2 + m - 2|}{2} = d(u_m, u_1) + \xi D(u_1, \Omega u_m) \\ &= F_1(d(u_m, u_1) + \xi D(u_1, \Omega u_m)) \\ &= F_1(\rho(d(u_m, u_1), D(u_1, \Omega u_1), D(u_m, \Omega u_m), D(u_m, \Omega u_1), \\ &\quad D(u_1, \Omega u_m))). \end{aligned} \tag{45}$$

All hypothesis of Theorem 12 are satisfied and fix $\Omega = \{u_1, u_2\}$.

Observe the following in Example 7:

(i) F_1 is not continuous at 1

(ii) $F_1 \neq F_2$

(iii) $\rho \notin \mathfrak{R}$

(iv) $\rho \notin \mathbb{P}$.

Corollary 14. *Let $(\mathfrak{Z}, d)$ be a complete MS and $\Omega : \mathfrak{Z} \longrightarrow K(\mathfrak{Z})$ be a multivalued mapping. Assume that there exists $\chi \in \Phi$, a nondecreasing real valued function F_1 on $(0, \infty)$ and a real valued function F_2 on $(0, \infty)$ satisfying condition $(F2')$ and $(F3)$ such that (N1) and the following condition holds:*

$H(\Omega\varpi, \Omega\omega)) > 0$ implies $\chi(d(\varpi, \omega)) + F_2(H(\Omega\varpi, \Omega\omega)) \leq F_1(M(\varpi, \omega))$ for all $\varpi, \omega \in \mathfrak{Z}$, where

$$M(\varpi, \omega) = \max\left\{d(\varpi, \omega), D(\varpi, \Omega\varpi), D(\omega, \Omega\omega), \frac{D(x, \Omega\omega) + D(\omega, \Omega\varpi)}{2}\right\}. \tag{46}$$

Then, fix Ω is nonempty.

Proof. Define $\rho : [0,\infty)^5 \longrightarrow [0,\infty)$ by

$$\rho(\varpi_1, \varpi_2, \varpi_3, \varpi_4, \varpi_5) = \max\left\{\varpi_1, \varpi_2, \varpi_3, \frac{\varpi_4 + \varpi_5}{2}\right\}. \tag{47}$$

Then, $\rho \in \mathscr{P}$ and result follow from Theorem 12. □

Remark 15. Corollary 14 generalizes and improves Theorem 2.4 of [35]. In fact, by taking $F_1 = F_2$ and by defining $\Omega\varpi = \{\varpi\}$ for all $\varpi, \omega \in \mathfrak{Z}$ and $\chi(t) = \tau > 0$ for all $t \in (0,\infty)$ in Corollary 14, then we find Theorem 2.4 of [35]. Corollary 14 shows that condition $(F2)$ can be replaced by $(F2')$ and the strictness of the monotonicity of F is not necessary.

Corollary 16. *Let $(\mathfrak{Z}, d)$ be a complete MS and $\Omega : \mathfrak{Z} \longrightarrow K(\mathfrak{Z})$ be a multivalued mapping. Assume that there exist $\chi \in \Phi$, a non decreasing real valued function F_1 on $(0, \infty)$ and a real valued function F_2 on $(0, \infty)$ satisfying condition $(F2')$ and $(F3)$ such that (N1) and the following condition holds:*

$H(\Omega\varpi, \Omega\omega)) > 0$ implies $\chi(d(\varpi, \omega)) + F_2(H(\Omega\varpi, \Omega\omega)) \leq F_1(N(\varpi, \omega))$ for all $\varpi, \omega \in \mathfrak{Z}$, where

$$N(\varpi, \omega) = ad(\varpi, \omega) + bD(\varpi, \Omega\varpi) + cD(\omega, \Omega\omega) + e[D(\varpi, \Omega\omega) + D(\omega, \Omega\varpi)], \tag{48}$$

$a, b, c, e \geq 0$ and $a + b + c + 2e < 1$. Then Fix Ω is nonempty.

Proof. Define $\rho : [0,\infty)^5 \longrightarrow [0,\infty)$ by

$$\rho(\varpi_1, \varpi_2, \varpi_3, \varpi_4, \varpi_5) = a\varpi_1 + b\varpi_2 + c\varpi_3 + e[\varpi_4 + \varpi_5], \tag{49}$$

where $a, b, c, e > 0$ and $a + b + c + 2e < 1$. Then $\rho \in \mathscr{P}$ and result follows from Theorem 12. □

Now, in the next Theorem, we replace the condition $(F3)$ of F_2 by continuity of F_1 in hypothesis of Theorem 12 and obtain another fixed point result.

Theorem 17. *Let $(\mathfrak{Z}, d)$ be a complete MS and $\Omega : \mathfrak{Z} \longrightarrow K(\mathfrak{Z})$ be a multivalued mapping. Assume that there exists $\chi \in \Phi$, a continuous, nondecreasing real-valued function F_1 on $(0, \infty)$ and a real valued function F_2 on $(0, \infty)$ satisfying condition $(F2')$ such that following conditions hold:*

(N1) $F_1(c) \leq F_2(c)$ for all $c > 0$

(N2) $H(\Omega\varpi, \Omega\omega)) > 0$ implies

$$\chi(d(\varpi, \omega)) + F_2(H(\Omega\varpi, \Omega\omega)) \leq F_1(\rho(d(\varpi, \omega), D(\varpi, \Omega\varpi), D(\omega, \Omega\omega), D(\varpi, \Omega\omega), D(\omega, \Omega\varpi))) \tag{50}$$

for all $\varpi, \omega \in \mathfrak{Z}$ and $\rho \in \mathbb{P}$.

Then, fix Ω is nonempty.

Proof. Let $\varpi_0 \in \mathfrak{Z}$ be an arbitrary point and $\varpi_1 \in \Omega\varpi_0$. Then, as in proof of Theorem 12, we get a sequence $\{\varpi_n\} \subset \mathfrak{Z}$ such that $\varpi_{n+1} \in \Omega\varpi_n$ satisfying $d(\varpi_n, \varpi_{n+1}) = D(\varpi_n, \Omega\varpi_n)$ with $D(\varpi_n, \Omega\varpi_n) > 0$,

$$d(\varpi_n, \varpi_{n+1}) < d(\varpi_{n-1}, \varpi_n), \text{ for } all\ n \in \mathbb{N}, \tag{51}$$

$$F_2(H(\Omega\varpi_n, \Omega\varpi_{n+1})) \leq F_2(H(\Omega\varpi_0, \Omega\varpi_1)) - (n - n_0)h, n \geq n_0. \tag{52}$$

Taking $n \longrightarrow \infty$ in (52), we get $F_2(H(\Omega\varpi_n, \Omega\varpi_{n+1})) \longrightarrow -\infty$ and by $(F2')$, we have

$$\lim_{n\longrightarrow\infty} H(\Omega\varpi_n, \Omega\varpi_{n+1}) = 0, \tag{53}$$

which further implies that

$$\lim_{n\longrightarrow\infty} d(\varpi_n, \varpi_{n+1}) = \lim_{n\longrightarrow\infty} D(\varpi_n, \Omega\varpi_n) \leq \lim_{n\longrightarrow\infty} H(\Omega\varpi_{n-1}, \Omega\varpi_n) = 0. \tag{54}$$

Next, we claim that

$$\lim_{n,m\longrightarrow\infty} d(\varpi_n, \varpi_m) = 0. \tag{55}$$

If (55) is not true, then there exists $\delta > 0$ such that for all $r \geq 0$, there exists $m_k > n_k > r$

$$d(\varpi_n, \varpi_m) > \delta. \tag{56}$$

Also, there exists $r_0 \in \mathbb{N}$ such that

$$\lambda_{r_0} = d(\varpi_{n-1}, \varpi_n) < \delta \text{ forall } n \geq r_0. \tag{57}$$

Consider two subsequences $\{\varpi_{n_k}\}$ and $\{\varpi_{m_k}\}$ of $\{\varpi_n\}$ satisfying

$$r_0 \leq n_k \leq m_k + 1 \text{ and } d\left(\varpi_{m_k}, \varpi_{n_k}\right) > \delta \text{ forall } k > 0. \tag{58}$$

Observe that

$$d\left(\varpi_{m_k-1}, \varpi_{n_k}\right) \leq \delta \text{ forall } k, \tag{59}$$

where m_k is chosen as minimal index for which (59) is

satisfied. Also, note that because of (58) and (59), the case $n_{k+1} \le n_k$ is impossible. Thus, $n_{k+2} \le m_k$ for all k. It implies

$$n_k + 1 < m_k < m_k + 1 \text{ for all } k. \tag{60}$$

Using triangle inequality and by (58) and (59), we have

$$\delta < d(\varpi_{m_k}, \varpi_{n_k}) \le d(\varpi_{m_k}, \varpi_{m_k-1}) + d(\varpi_{m_k-1}, \varpi_{n_k}) \le \lambda_{m_k} + \delta. \tag{61}$$

Letting limit $k \longrightarrow \infty$ in (61) and using (53), we get

$$\lim_{k\longrightarrow\infty} d(\varpi_{m_k}, \varpi_{n_k}) = \delta. \tag{62}$$

Now, by using (53) and (62), we obtain

$$\lim_{k\longrightarrow\infty} d(\varpi_{m_k+1}, \varpi_{n_k+1}) = \delta. \tag{63}$$

Then, from $(N1)$, $(N2)$, and monotonicity of F_1, we get

$$\begin{aligned}
&\chi(d(\varpi_{m_k}, \varpi_{n_k})) + F_1(d(\varpi_{m_{k+1}}, \varpi_{n_{k+1}})) \\
&\quad = \chi(d(\varpi_{m_k}, \varpi_{n_k})) + F_1(D(\varpi_{m_{k+1}}, \Omega\varpi_{n_k})) \\
&\quad \le \chi(d(\varpi_{m_k}, \varpi_{n_k})) + F_1(H(\Omega\varpi_{m_k}, \Omega\varpi_{n_k})) \\
&\quad \le \chi(d(\varpi_{m_k}, \varpi_{n_k})) + F_2(H(\Omega\varpi_{m_k}, \Omega\varpi_{n_k})) \\
&\quad \le F_1(\rho(d(\varpi_{m_k}, \varpi_{n_k}), D(\varpi_{m_k}, \Omega\varpi_{m_k}), D(\varpi_{n_k}, \Omega\varpi_{n_k}), \\
&\qquad D(\varpi_{m_k}, \Omega\varpi_{n_k}), D(\varpi_{n_k}, \Omega\varpi_{m_k}))) \\
&\quad \le F_1(\rho(d(\varpi_{m_k}, \varpi_{n_k}), d(\varpi_{m_k}, \varpi_{m_{k+1}}), d(\varpi_{n_k}, \varpi_{n_{k+1}}), \\
&\qquad d(\varpi_{n_{k+1}}, \varpi_{n_k}) + d(\varpi_{n_k}, \varpi_{m_k}), d(\varpi_{n_k}, \varpi_{n_{k+1}}) + d(\varpi_{n_{k+1}}, \varpi_{m_{k+1}}))).
\end{aligned} \tag{64}$$

Since F_1 is continuous, so by passing the limit $k \longrightarrow \infty$, using equations (62) and (63), we have

$$\lim_{k\longrightarrow\infty} \chi(d(\varpi_{m_k}, \varpi_{n_k})) + F_1(\delta) \le F_1(\rho(\delta, 0, 0, \delta, \delta)) \le F_1(\delta\rho(1, 0, 0, 1, 1)). \tag{65}$$

Now, since $\rho \in \mathbb{P}$, we have $\rho(1, 0, 0, 1, 1) \in (0, 1]$; so, (65) implies

$$\lim_{s\longrightarrow\delta^+} \inf \phi(s) \le 0, \tag{66}$$

which is a contradiction to (17). Hence, (55) holds, which implies that $\{\varpi_n\}$ is a Cauchy sequence. Completeness of $\mathfrak{Z}$ ensures the existence of $\varpi^* \in \mathfrak{Z}$ such that

$$\lim_{n\longrightarrow\infty} \varpi_n = \varpi^*. \tag{67}$$

By following the same steps as in the proof of Theorem 12, we get $\varpi^* \in \Omega\varpi^*$. This completes the proof. □

Corollary 18. *Let $(\mathfrak{Z}, d)$ be a complete MS and $\Omega : \mathfrak{Z} \longrightarrow K(\mathfrak{Z})$ be a multivalued mapping. Assume that there exists $\chi \in \Phi$, a continuous, nondecreasing real-valued function F_1 on $(0, \infty)$ and a real valued function F_2 on $(0, \infty)$ satisfying condition $(F2')$ such that (N1) and the following condition holds:*

$$\begin{aligned}
H(\Omega\varpi, \Omega\omega)) > 0 \text{ implies } &\chi(d(\varpi, \omega)) + F_2(H(\Omega\varpi, \Omega\omega)) \\
&\le F_1(\wp_1(d(\varpi, \omega) + \wp_2 D(\varpi, \Omega\varpi) + \wp_3 D(\omega, \Omega\omega) \\
&\quad + \wp_4 D(\varpi, \Omega\omega) + \wp_5 D(\omega, \Omega\varpi) \text{ for all } \varpi, \omega \in \mathfrak{Z},
\end{aligned} \tag{68}$$

where $\wp_i \ge 0$, $\wp_1 + \wp_2 + \wp_3 + 2\wp_4 = 1$ and $\wp_1 + \wp_3 + \wp_4 \le 1$. Then, fix Ω is nonempty.

Proof. Define $\rho : [0,\infty)^5 \longrightarrow [0,\infty)$ by

$$\rho(\varpi_1, \varpi_2, \varpi_3, \varpi_4, \varpi_5) = \wp_1\varpi_1 + \wp_2\varpi_2 + \wp_3\varpi_3 + \wp_4\varpi_4 + \wp_5\varpi_5, \tag{69}$$

where $\wp_i \ge 0$, $\wp_1 + \wp_2 + \wp_3 + 2\wp_4 = 1$ and $\wp_1 + \wp_3 + \wp_4 \le 1$. Then, $\rho \in \mathbb{P}$ and result follow from Theorem 17. □

Remark 19. Corollary 18 improves Theorem 1 of [35]. In fact, by taking $F_1 = F_2$ and by defining $\Omega\varpi = \{\varpi\}$ for all ϖ, $\omega \in \mathfrak{Z}$ in Corollary 18, then we are back to Theorem 1 of [34]. In Corollary 18, condition $(F2)$ is weakened to the condition $(F2')$.

Next, we consider $\Omega\varpi$ that are closed subsets of $\mathfrak{Z}$ instead of compact subsets for all $\mathfrak{Z}$ and obtain the following theorems.

Theorem 20. *Let $(\mathfrak{Z}, d)$ be a complete MS and $\Omega : \mathfrak{Z} \longrightarrow C(X)$ be a multivalued mapping. Assume that there exists $\chi \in \Phi$, $F \in \Delta(F*)$ and a real-valued function L on $(0, \infty)$ such that following holds:*

(G1) $F(c) \le L(c)$ for all $c > 0$

(G2) $H(\Omega\varpi, \Omega\omega)) > 0$ implies

$$\begin{aligned}
&\chi(d(\varpi, \omega)) + L(H(\Omega\varpi, \Omega\omega)) \\
&\quad \le F(\rho(d(\varpi, \omega), D(\varpi, \Omega\varpi), D(\omega, \Omega\omega), D(\varpi, \Omega\omega), D(\omega, \Omega\varpi))),
\end{aligned} \tag{70}$$

for all $\varpi, \omega \in \mathfrak{Z}$ and $\rho \in \mathscr{P}$.

Then, fix (Ω) is nonempty.

Proof. Let $x_0 \in \mathfrak{Z}$ be an arbitrary point and $\varpi_1 \in \Omega\varpi_0$. Assume that $\varpi_1 \in \Omega\varpi_1$; otherwise, ϖ_1 is a fixed point of Ω, and the proof is complete. Then, since $\Omega\varpi_1$ is closed, $D(\varpi_1, \Omega\varpi_1) > 0$ and consequently, $H(\Omega\varpi_0, \Omega\varpi_1) > 0$. Due to $(F4)$, we obtain

$$F(D(\varpi_1, \Omega\varpi_1)) = \inf_{z\in\Omega\varpi_1} F(d(\varpi_1, z)). \tag{71}$$

□

Then, (71) with (G1) and (G2) gives

$$\begin{aligned}\inf_{z\in\Omega\varpi_1} F(d(\varpi_1,z)) &= F(D(\varpi_1,\Omega\varpi_1)) \le F(H(\Omega\varpi_0,\Omega\varpi_1))\\ &\le L(H(\Omega\varpi_0,\Omega\varpi_1))\\ &\le F(\rho(d(\varpi_0,\varpi_1),D(\varpi_0,\Omega\varpi_0),D(\varpi_1,\Omega\varpi_1),\\ &\quad D(\varpi_0,\Omega\varpi_1),D(\varpi_1,\Omega\varpi_0))) - \chi(d(\varpi_0,\varpi_1))\\ &< F(\rho(d(\varpi_0,\varpi_1),d(\varpi_0,\varpi_1),d(\varpi_1,\varpi_2),d(\varpi_0,\varpi_2),0)).\end{aligned} \tag{72}$$

Thus, there exists $x_2 \in \Omega\varpi_1$ such that

$$F(d(\varpi_1,\varpi_2)) < F(\rho(d(\varpi_0,\varpi_1),d(\varpi_0,\varpi_1),d(\varpi_1,\varpi_2),d(\varpi_0,\varpi_1),0)). \tag{73}$$

Since F is an nondecreasing function, (73) with (ρ_3) implies that

$$\begin{aligned}d(\varpi_1,\varpi_2) &< \rho(d(\varpi_0,\varpi_1),d(\varpi_0,\varpi_1),d(\varpi_1,\varpi_2),d(\varpi_0,\varpi_1),0)\\ &\le \rho(d(\varpi_0,\varpi_1),d(\varpi_0,\varpi_1),d(\varpi_1,\varpi_2),d(\varpi_0,\varpi_1)+d(\varpi_1,\varpi_2),0).\end{aligned} \tag{74}$$

By using Lemma 11, (74) implies

$$d(\varpi_1,\varpi_2) < d(\varpi_0,\varpi_1). \tag{75}$$

Next, arguing as previous, we get $\varpi_3 \in \Omega\varpi_2$ with $D(\varpi_2, \Omega\varpi_2) > 0$. Also, by using Lemma 11, from (G1) and (G2), we obtain

$$d(\varpi_2,\varpi_3) < d(\varpi_1,\varpi_2) \tag{76}$$

Continuing in the same manner, we get a sequence $\{\varpi_n\} \subset \mathfrak{Z}$ such that $\varpi_{n+1} \in \Omega\varpi_n$ with $D(\varpi_n,\Omega\varpi_{n+1}) > 0$ and

$$d(\varpi_n,\varpi_{n+1}) < d(\varpi_{n-1},\varpi_n), \tag{77}$$

for all $n \in \mathbb{N}$. (77) implies that $\{d(\varpi_n,\varpi_{n+1})\}_{n\in\mathbb{N}}$ is a decreasing sequence of positive real numbers. Hence, from $(F4)$, (G1), and (G2), we get

$$\begin{aligned}\inf_{z\in\Omega\varpi_n} F(d(\varpi_n,z)) &= F(D(\varpi_n,\Omega\varpi_n)) \le F(H(\Omega\varpi_{n-1},\Omega\varpi_n))\\ &\le L(H(\Omega\varpi_{n-1},\Omega\varpi_n)) \le F(\rho(d(\varpi_{n-1},\varpi_n),D(\varpi_{n-1},\Omega\varpi_{n-1}),\\ &\quad D(\varpi_n,\Omega\varpi_n),D(\varpi_{n-1},\Omega\varpi_n),D(\varpi_n,\Omega\varpi_{n-1})))\\ &\quad - \chi(d(\varpi_{n-1},\varpi_n))\\ &\le F(\rho(d(\varpi_{n-1},\varpi_n),d(\varpi_{n-1},\varpi_n),d(\varpi_n,\varpi_{n+1}),\\ &\quad d(\varpi_{n-1},\varpi_n)+d(\varpi_n,\varpi_{n+1}),0)) - \chi(d(\varpi_{n-1},\varpi_n))\\ &\le F(\rho(d(\varpi_{n-1},\varpi_n),d(\varpi_{n-1},\varpi_n),d(\varpi_{n-1},\varpi_n),\\ &\quad 2d(\varpi_{n-1},\varpi_n),0)) - \chi(d(\varpi_{n-1},\varpi_n))\\ &\le F(d(\varpi_{n-1},\varpi_n)\rho(1,1,1,2,0)) - \chi(d(\varpi_{n-1},\varpi_n))\\ &\le F(d(\varpi_{n-1},\varpi_n)) - \chi(d(\varpi_{n-1},\varpi_n)).\end{aligned} \tag{78}$$

Thus, for all $n \in \mathbb{N}$,

$$\inf_{z\in\Omega\varpi_n} F(d(\varpi_n,z)) \le F(d(\varpi_{n-1},\varpi_n)) - \chi(d(\varpi_{n-1},\varpi_n)). \tag{79}$$

Thus, from (79), there exists $\varpi_{n+1} \in \Omega\varpi_n$ such that

$$F(d(\varpi_n,\varpi_{n+1})) \le F(d(\varpi_{n-1},\varpi_n)) - \chi(d(\varpi_{n-1},\varpi_n)). \tag{80}$$

Since $\chi \in \Phi$, there exists $h > 0$ and $n_0 \in \mathbb{N}$ such that $\chi(d(\varpi_n,\varpi_{n+1})) < h$, for all $n \ge n_0$. From (80), we obtain

$$\begin{aligned}F(d(\varpi_n,\varpi_{n+1})) &\le F(d(\varpi_{n-1},\varpi_n)) - \chi(d(\varpi_{n-1},\varpi_n))\\ &\le F((\varpi_{n-2},\varpi_{n-1})) - \chi(d(\varpi_{n-2},\varpi_{n-1})) - \chi(d(\varpi_{n-1},\varpi_n)) \vdots\\ &\le F(d(\varpi_0,\varpi_1)) - \sum_{i=1}^{n-1}\chi(d(\varpi_{i-1},\varpi_i))\\ &= F(d(\varpi_0,\varpi_1)) - \sum_{i=1}^{n_0-1}\chi(d(\varpi_{i-1},\varpi_i)) - \sum_{i=n_0}^{n-1}\chi(d(\varpi_{i-1},\varpi_i))\\ &= F(d(\varpi_0,\varpi_1)) - (n-n_0)h, n \ge n_0.\end{aligned} \tag{81}$$

Taking $n \longrightarrow \infty$ in (81), we get $F(d(\varpi_{n-1},\varpi_n)) \longrightarrow -\infty$ and by $(F2')$, we have

$$\lim_{n\longrightarrow\infty} d(\varpi_{n-1},\varpi_n) = 0, \tag{82}$$

Now, from $(F3)$, there exists $k \in (0,1)$ such that

$$\lim_{n\longrightarrow\infty} (d(\varpi_{n-1},\varpi_n))^k F(d(\varpi_{n-1},\varpi_n)) = 0. \tag{83}$$

Then, from (81), for all $n \in \mathbb{N}$, we have

$$\begin{aligned}&(d(\varpi_{n-1},\varpi_n))^k F(d(\varpi_{n-1},\varpi_n)) - (d(\varpi_{n-1},\varpi_n))^k F(d(\varpi_0,\varpi_1))\\ &\le (d(\varpi_{n-1},\varpi_n))^k (F(d(\varpi_{n-1},\varpi_n)) - (n-n_0)h)\\ &\quad - (d(\varpi_{n-1},\varpi_n))^k F(d(\varpi_0,\varpi_1))\\ &= -(d(\varpi_{n-1},\varpi_n))^k (n-n_0)h \le 0.\end{aligned} \tag{84}$$

Taking limit $n \longrightarrow \infty$, in (84) and using (82) and (83), we have

$$\lim_{n\longrightarrow\infty} n(d(\varpi_{n-1},\varpi_n))^k = 0. \tag{85}$$

Observe that from (85), there exists $n_1 \in \mathbb{N}$ such that $n(d(\varpi_{n-1},\varpi_n))^k \le 1$ for all $n \ge n_1$. Thus, for all $n \ge n_1$, we have

$$d(\varpi_{n-1},\varpi_n) \le \frac{1}{n^{1/k}} \text{ for } all\ n \ge n_1, \tag{86}$$

Now, in order to show that $\{x_n\}_{n\in\mathbb{N}}$ is Cauchy sequence,

consider $m, n \in \mathbb{N}$ such that $m > n > n_1$. From (86), we get

$$d(\varpi_m, \varpi_n) \leq \sum_{i=n}^{m-1} d(\varpi_i, \varpi_{i+1}) \leq \sum_{i=n}^{m-1} \frac{1}{i^{1/k}} \leq \sum_{i=n}^{\infty} \frac{1}{i^{1/k}}. \quad (87)$$

As a result of the above and the series' convergence, $\sum_{i=n}^{\infty} (1/i^{1/k})$, we receive that $\{\varpi_n\}_{n\in\mathbb{N}}$ is Cauchy sequence. Since $\mathfrak{Z}$ is a complete space, so there exists $\varpi^* \in \mathfrak{Z}$ such that

$$\lim_{n\longrightarrow\infty} x_n = \varpi^*. \quad (88)$$

Now,

$$\begin{aligned} F(\mathrm{H}(\Omega\varpi, \Omega\omega)) &\leq L(\mathrm{H}(\Omega\varpi, \Omega\omega)) < \phi(d(\varpi, \omega)) + L(H(\Omega\varpi, \Omega\omega) \\ &\leq F(\rho(d(\varpi, \omega), D(\varpi, \Omega\varpi), D(\omega, \Omega\omega), D(\varpi, \Omega\omega), D(\omega, \Omega\varpi))). \end{aligned} \quad (89)$$

Since F is nondecreasing function, we obtain

$$H(\Omega\varpi, \Omega\omega) \leq \rho(d(\varpi, \omega), D(\varpi, \Omega\varpi), D(\omega, \Omega\omega), D(\varpi, \Omega\omega), D(\omega, \Omega\varpi)), \text{for all } \varpi, \omega \in \mathfrak{Z}. \quad (90)$$

We claim that ϖ^* is a fixed point of $\mathfrak{Z}$. On contrary, suppose that $D(\varpi^*, \Omega\varpi^*) > 0$ and by equation (90), we have

$$\begin{aligned} D(\varpi^*, \Omega\varpi^*) &\leq d(\varpi^*, \varpi_{n+1}) + D(\varpi_{n+1}, \Omega\varpi^*) \\ &< d(\varpi^*, \varpi_{n+1}) + H(\Omega\varpi_n, \Omega\varpi^*) \\ &< d(\varpi^*, \varpi_{n+1}) + \rho(d(\varpi_n, \varpi^*), D(\varpi_n, \Omega\varpi_n), \\ &\quad D(\varpi^*, \Omega\varpi^*), D(\varpi_n, \Omega\varpi^*), D(\varpi^*, \Omega\varpi_n)) \\ &\leq d(\varpi^*, \varpi_{n+1}) + \rho(d(\varpi_n, \varpi^*), d(\varpi_n, \varpi_{n+1}), \\ &\quad D(\varpi^*, \Omega\varpi^*), D(\varpi_n, \varpi^*) + D(\varpi^*, \Omega\varpi^*), \\ &\quad d(\varpi^*, \varpi_{n+1})). \end{aligned} \quad (91)$$

Passing to limit as $n \longrightarrow \infty$ in the above inequality, we obtain

$$D(\varpi^*, \Omega\varpi^*) < \rho(0, 0, D(\varpi^*, \Omega\varpi^*), 0 + D(\varpi^*, \Omega\varpi^*), 0), \quad (92)$$

which implies by Lemma 1 that

$$0 < D(x^*, \Omega\varpi^*) < 0, \quad (93)$$

which is a contradiction. Hence, $D(\varpi^*, \Omega\varpi^*) = 0$. Since $\Omega\varpi^*$ is closed, therefore, $\varpi^* \in \Omega\varpi^*$.

Corollary 21. *Let $(\mathfrak{Z}, d)$ be a complete MS and $\Omega : \mathfrak{Z} \longrightarrow C(\mathfrak{Z})$ be a multivalued mapping. Assume that there exists $\chi \in \Phi$, $F \in \Delta(F*)$ and a real-valued function L on $(0, \infty)$ such that (G1) and the following condition holds:*

$$\begin{aligned} H(\Omega\varpi, \Omega\omega)) > 0 \text{ implies } \chi(d(\varpi, \omega)) + F(H(\Omega\varpi, \Omega\omega)) \\ \leq L(\wp_1(d(\varpi, \omega) + \wp_2 D(\varpi, \Omega\varpi) + \wp_3 D(\omega, \Omega\omega) \\ + \wp_4 D(\varpi, \Omega\omega) + \wp_5 D(\omega, \Omega\varpi) \text{ for all } \varpi, \omega \in \mathfrak{Z}, \end{aligned} \quad (94)$$

where $\wp_i \geq 0$ and $\wp_1 + \wp_2 + \wp_3 + 2\wp_4 = 1$. Then, fix Ω is nonempty.

Proof. Define $\rho : [0,\infty)^5 \longrightarrow [0,\infty)$ by

$$\rho(\varpi_1, \varpi_2, \varpi_3, \varpi_4, \varpi_5) = \wp_1\varpi_1 + \wp_2\varpi_2 + \wp_3\varpi_3 + \wp_4\varpi_4 + \wp_5\varpi_5, \quad (95)$$

where $\wp_i \geq 0$ and $\wp_1 + \wp_2 + \wp_3 + 2\wp_4 = 1$. Then. $\rho \in \mathscr{P}$ and result follow from Theorem 20. □

Corollary 22. *Let $(\mathfrak{Z}, d)$ be a complete MS and $\Omega : \mathfrak{Z} \longrightarrow C(\mathfrak{Z})$ be a multivalued mapping. Assume that there exists $\chi \in \Phi$, $F \in \Delta(F*)$ and a real-valued function L on $(0, \infty)$ such that (G1) and the following condition holds:*

$$\begin{aligned} H(\Omega\varpi, \Omega\omega)) > 0 \text{ implies } \chi(d(\varpi, \omega)) + F(H(\Omega\varpi, \Omega\omega)) \\ \leq L(d(\varpi, \omega) + \lambda D(\omega, \Omega\varpi)) \text{ for all } \varpi, \omega \in \mathfrak{Z}, \end{aligned} \quad (96)$$

where $\lambda \geq 0$. Then, fix Ω is nonempty.

Proof. Define $\rho : [0,\infty)^5 \longrightarrow [0,\infty)$ by

$$\rho(\varpi_1, \varpi_2, \varpi_3, \varpi_4, \varpi_5) = \varpi_1 + \lambda\varpi_5, \quad (97)$$

where $\lambda \geq 0$. Then, $\rho \in \mathscr{P}$ and result follow from Theorem 20. □

Remark 23. Corollary 21 generalizes Theorem 24 of [33]. Indeed, by considering $L = F$ and by defining $\chi(t) = 2\tau$ for all $\varpi, \omega \in \mathfrak{Z}$, where $\tau > 0$ in Corollary 21, we obtain Theorem 32 of [33]. Also, by considering $L = F$ and by defining $\chi(t) = \tau$ for all $\varpi, \omega \in \mathfrak{Z}$, where $\tau > 0$ in Corollary 22, we get back the Theorem 24 of [32].

Theorem 24. *Let $(\mathfrak{Z}, d)$ be a complete MS and $\Omega : \mathfrak{Z} \longrightarrow C(\mathfrak{Z})$ be a multivalued mapping. Assume that there exists $\chi \in \Phi$, a nondecreasing and continuous real-valued function $F : (0,\infty) \longrightarrow \mathbb{R}$ satisfying condition $(F2')$ and a real-valued function L on $(0, \infty)$ such that following holds:*

(G1) $F(c) \leq L(c)$ for all $c > 0$

(G2) $H(\Omega\varpi, \Omega\omega)) > 0$ implies

$$\chi(d(\varpi,\omega)) + L(H(\Omega\varpi,\Omega\omega)) \le F(\rho(d(\varpi,\omega), D(\varpi,\Omega\varpi), D(\omega,\Omega\omega), D(\varpi,\Omega\omega), D(\omega,\Omega\varpi))), \tag{98}$$

for all $\varpi, \omega \in \mathfrak{Z}$ and $\rho \in \mathbb{P}$.
Then, fix Ω is nonempty.

Proof. Let $\varpi_0 \in \mathfrak{Z}$ be an arbitrary point and $\varpi_1 \in \Omega\varpi_0$. Then, as in proof of Theorem 20, we get a sequence $\{\varpi_n\} \subset \mathfrak{Z}$ such that $\varpi_{n+1} \in \Omega\varpi_n$ with $D(\varpi_n, \Omega\varpi_{n+1}) > 0$,

$$d(\varpi_n, \varpi_{n+1}) < d(\varpi_{n-1}, \varpi_n), \text{ for } all\ n \in \mathbb{N}, \tag{99}$$

$$F(d(\varpi_{n-1}, \varpi_n)) \le F(d(\varpi_0, \varpi_1)) - (n - n_0)h, n \ge n_0. \tag{100}$$

Taking $n \longrightarrow \infty$ in (100), we get $F(d(\varpi_{n-1}, \varpi_n)) \longrightarrow -\infty$ and by $(F2')$, we have

$$\lim_{n\longrightarrow\infty} d(\varpi_{n-1}, \varpi_n) = 0. \tag{101}$$

Next, we claim that

$$\lim_{n,m\longrightarrow\infty} d(\varpi_n, \varpi_m) = 0. \tag{102}$$

If (102) is not true, then there exists $\delta > 0$ such that for all $r \ge 0$, there exists $m_k > n_k > r$

$$d(\varpi_n, \varpi_m) > \delta. \tag{103}$$

Also, there exists $r_0 \in \mathbb{N}$ such that

$$\lambda_{r_0} = d(\varpi_{n-1}, \varpi_n) < \delta \text{ forall } n \ge r_0. \tag{104}$$

Consider two subsequences $\{\varpi_{n_k}\}$ and $\{\varpi_{m_k}\}$ of $\{x_n\}$; then, as is proof of Theorem 17, we get

$$\begin{aligned} \lim_{k\longrightarrow\infty} d(\varpi_{m_k}, \varpi_{n_k}) &= \delta, \\ \lim_{k\longrightarrow\infty} d(\varpi_{m_k+1}, \varpi_{n_k+1}) &= \delta. \end{aligned} \tag{105}$$

Then, from (G1), (G2), and monotonicity of F, we get

$$\begin{aligned} &\chi(d(\varpi_{m_k}, \varpi_{n_k})) + F(d(\varpi_{m_{k+1}}, \varpi_{n_{k+1}})) \\ &= \chi(d(\varpi_{m_k}, \varpi_{n_k})) + F(D(\varpi_{m_{k+1}}, \Omega\varpi_{n_k})) \\ &\le \chi(d(\varpi_{m_k}, \varpi_{n_k})) + F(H(\Omega\varpi_{m_k}, \Omega\varpi_{n_k})) \\ &\le \chi(d(\varpi_{m_k}, \varpi_{n_k})) + L(H(\Omega\varpi_{m_k}, \Omega\varpi_{n_k})) \\ &\le F(\rho(d(\varpi_{m_k}, \varpi_{n_k}), D(\varpi_{m_k}, \Omega\varpi_{m_k}), D(\varpi_{n_k}, \Omega\varpi_{n_k}), \\ &\quad D(\varpi_{m_k}, \Omega\varpi_{n_k}), D(\varpi_{n_k}, \Omega\varpi_{m_k}))) \\ &\le F(\rho(d(\varpi_{m_k}, \varpi_{n_k}), d(\varpi_{m_k}, \varpi_{m_{k+1}}), d(\varpi_{n_k}, \varpi_{n_{k+1}}), d(\varpi_{m_{k+1}}, \varpi_{n_k}) \\ &\quad + d(\varpi_{n_k}, \varpi_{m_k}), d(\varpi_{n_k}, \varpi_{n_{k+1}}) + d(\varpi_{n_{k+1}}, \varpi_{m_{k+1}}))). \end{aligned} \tag{106}$$

Since F is continuous, so by passing the limit $k \longrightarrow \infty$, using equations (105) and (106), we have

$$\lim_{k\longrightarrow\infty} \chi(d(\varpi_{m_k}, \varpi_{n_k})) + F(\delta) \le F(\rho(\delta, 0, 0, \delta, \delta)) \le F(\delta\rho(1, 0, 0, 1, 1)). \tag{107}$$

Now, since $\rho \in \mathbb{P}$, we have $\rho(1, 0, 0, 1, 1) \in (0, 1]$; so, (107) implies

$$\lim_{s\longrightarrow\delta^+} \inf \phi(s) \le 0, \tag{108}$$

which is a contradiction to (17). Hence, (102) holds, which implies that $\{\varpi_n\}$ is Cauchy sequence. Completeness of $\mathfrak{Z}$ ensures the existence of $\varpi^* \in \mathfrak{Z}$ such that

$$\lim_{n\longrightarrow\infty} \varpi_n = \varpi^*. \tag{109}$$

By following the same steps as in the proof of Theorem 20, we get $\varpi^* \in \Omega\varpi^*$. This completes the proof. □

Corollary 25. *Let $(\mathfrak{Z}, d)$ be a complete MS and $\Omega : \mathfrak{Z} \longrightarrow C(\mathfrak{Z})$ be a multivalued mapping. Assume that there exists $\chi \in \Phi$, a nondecreasing and continuous real-valued function $F : (0,\infty) \longrightarrow \mathbb{R}$ satisfying condition $(F2')$ and a real-valued function L on $(0, \infty)$ such that (G1) and the following condition hold:*

$$\begin{aligned} H(\Omega\varpi, \Omega\omega)) > 0 \text{ implies } &\chi(d(\varpi,\omega)) + F(H(\Omega\varpi,\Omega\omega)) \\ &\le L(\wp_1(d(\varpi,\omega) + \wp_2 D(\varpi,\Omega\varpi) + \wp_3 D(\omega,\Omega\omega) \\ &\quad + \wp_4 D(\varpi,\Omega\omega) + \wp_5 D(\omega,\Omega\varpi) \text{ for all } \varpi, \omega \in \mathfrak{Z}, \end{aligned} \tag{110}$$

where $\wp_i \ge 0$, $\wp_1 + \wp_2 + \wp_3 + 2\wp_4 = 1$, and $\wp_1 + \wp_3 + \wp_4 \le 1$. Then, fix Ω is nonempty.

Proof. Define $\rho : [0,\infty)^5 \longrightarrow [0,\infty)$ by

$$\rho(\varpi_1, \varpi_2, \varpi_3, \varpi_4, \varpi_5) = \wp_1\varpi_1 + \wp_2\varpi_2 + \wp_3\varpi_3 + \wp_4\varpi_4 + \wp_5\varpi_5, \tag{111}$$

where $\wp_i \ge 0$, $\wp_1 + \wp_2 + \wp_3 + 2\wp_4 = 1$, and $\wp_1 + \wp_3 + \wp_4 \le 1$. Then, $\rho \in \mathbb{P}$ and result follow from Theorem 24. □

If we restrict $\lambda = 0$ in Corollary 22, then ρ defined in the proof of Corollary 22 also satisfies $\rho(1, 0, 0, 1, 1) = 1$ and hence $\rho \in \mathbb{P}$. Consequently, from Theorem 24, we get

Corollary 26. *Let $(\mathfrak{Z}, d)$ be a complete MS and $\Omega : \mathfrak{Z} \longrightarrow C(\mathfrak{Z})$ be a multivalued mapping. Assume that there exists $\chi \in \Phi$, a nondecreasing and continuous real-valued function $F : (0,\infty) \longrightarrow \mathbb{R}$ satisfying condition $(F2')$ and a real-valued function L on $(0, \infty)$ such that (G1) and the following*

condition hold:

$$H(\Omega\varpi, \Omega\omega)) > 0 \text{ implies } \chi(d(\varpi, \omega)) + F(H(\Omega\varpi, \Omega\omega)) \leq L(d(\varpi, \omega) + \lambda D(\omega, \Omega\varpi)) \text{ for all } \varpi, \omega \in \mathfrak{Z}, \tag{112}$$

where $\lambda \geq 0$. Then, fix Ω is nonempty.

2. Data Dependence

Let $\mathfrak{Z}, \mathfrak{B}$ be two nonempty sets and $\Omega : \mathfrak{Z} \longrightarrow P(\mathfrak{B})$. Denote by $G(\Omega)$, the graph of the multivalued operator is Ω. A multivalued operator $\Omega : \mathfrak{Z} \longrightarrow P(\mathfrak{B})$ is said to be closed if $G(\Omega)$ is a closed set in $\mathfrak{Z} \times \mathfrak{B}$. A selection for Ω is a singlevalued operator $\Omega : \mathfrak{Z} \longrightarrow \mathfrak{B}$ such that $t(\varpi) \in \Omega(\varpi)$, for each $\varpi \in \mathfrak{Z}$. Mo t and Petrusel in [36] discussed some basic problems including data dependence of the fixed point theory for a new type contractive multivalued operator. In [37], Rus et al. gave an important abstract notion as follows:

Definition 27. Let $(\mathfrak{Z}, d)$ be a MS and $\Omega : \mathfrak{Z} \longrightarrow CL(\mathfrak{Z})$ a multivalued operator. Then, Ω is a multivalued weakly Picard operator (briefly MWP operator) if for all $\varpi \in \mathfrak{Z}$ and $\omega \in \Omega\varpi$, there exists a sequence $\{\varpi_n\}_{n\in\mathbb{N}}$ such that

(i) $\varpi_0 = \varpi$, $\varpi_1 = \omega$

(ii) $\varpi_{n+1} \in \Omega\varpi_n$, for all $n \in \mathbb{N}$

(iii) The sequence $\{\varpi_n\}_{n\in\mathbb{N}}$ is convergent, and its limit is a fixed point of Ω

A sequence $\{\varpi_n\}_{n\in\mathbb{N}}$ satisfying the conditions (i) and (ii) in Definition 27 is also called a sequence of successive approximations of Ω starting from ϖ_0. Now, we present the main result of this section.

Theorem 28. *Let $(\mathfrak{Z}, d)$ be a MS and $\Omega_1, \Omega_2 : \mathfrak{Z} \longrightarrow K(\mathfrak{Z})$ be two multivalued operators. Assume that there exists $\chi \in \Phi$, a nondecreasing real-valued function F_1 on $(0, \infty)$ and a real valued function F_2 on $(0, \infty)$ satisfying condition $(F2')$ and $(F3)$ such that Ω_i satisfies (N1) and (N2) for all $i \in \{1, 2\}$:*

(i) There exists $\lambda > 0$ such that $H(\Omega_1(\varpi), \Omega_2(\varpi)) \leq \lambda$, for all $\varpi \in \mathfrak{Z}$

(ii) Then

(iii) Fix $(\Omega_i) \in CL(\mathfrak{Z})$ and $i \in \{1, 2\}$

(iv) Ω_1 and Ω_2 are MWP operators and

$$H(Fix(\Omega_1), Fix(\Omega_2)) \leq \frac{\lambda}{1 - \max\{\rho_1(1, 1, 1, 2, 0), \rho_2(1, 1, 1, 2, 0)\}}. \tag{113}$$

Proof. (a) By Theorem 12, we have that fix $(\Omega_i) \neq \varnothing$, for $i \in \{1, 2\}$. Next, we prove that the fixed point set of multivalued operators Ω_i is closed for $i \in \{1, 2\}$. For this, let $\{w_n\}$ be a sequence in fix (Ω_i) such that $w_n \longrightarrow w$ as $n \longrightarrow \infty$. Then,

$$\begin{aligned} F_1(H(\Omega\varpi, \Omega\omega)) &\leq F_2(H(\Omega\varpi, \Omega\omega)) < \phi(d(\varpi, \omega)) + F_2(H(\Omega\varpi, \Omega\omega) \\ &\leq F_1(\rho(d(\varpi, \omega), D(\varpi, \Omega\varpi), D(\omega, \Omega\omega), D(\varpi, \Omega\omega), D(\omega, \Omega\varpi))). \end{aligned} \tag{114}$$

Since F_1 is nondecreasing function, we obtain for all $\varpi, \omega \in \mathfrak{Z}$

$$H(\Omega\varpi, \Omega\omega) \leq \rho(d(\varpi, \omega), D(\varpi, \Omega\varpi), D(\omega, \Omega\omega), D(\varpi, \Omega\omega), D(\omega, \Omega\varpi)). \tag{115}$$

Suppose that $D(w, \Omega w) > 0$, then we have

$$\begin{aligned} D(w, \Omega w) &\leq d(w, w_{n+1}) + D(w_{n+1}, \Omega w) < d(w, w_{n+1}) + H(\Omega w_n, \Omega w) \\ &< d(w, w_{n+1}) + \rho(d(w_n, w), D(w_n, \Omega w_n), D(w, \Omega w), \\ &\quad D(w_n, \Omega w), D(w, \Omega w_n)) \\ &\leq d(w, w_{n+1}) + \rho(d(w_n, w), d(w_n, w_{n+1}), D(w, Tw), \\ &\quad d(w_n, w) + D(w, Tw), d(w, w_{n+1})). \end{aligned} \tag{116}$$

Passing to limit as $n \longrightarrow \infty$ in the above inequality, we obtain

$$D(w, \Omega w) < \rho(0, 0, D(w, \Omega w), 0 + D(w, \Omega w), 0), \tag{117}$$

which implies by Lemma 1 that

$$0 < D(w, Tw) < 0, \tag{118}$$

which is a contradiction. Hence, $D(w, \Omega w) = 0$. Since Ωw is closed, so $w \in \Omega w$.

(b) From the proof of Theorem 12, we immediately get that Ω_i operators are MWP operators for $i \in \{1, 2\}$. Now, we will show that $H(\text{Fix}(\Omega_1), \text{Fix}(\Omega_2)) \leq \lambda/1 - \max\{\rho_1(1, 1, 1, 2, 0), \rho_2(1, 1, 1, 2, 0)\}$. For this purpose, Let $q > 1$, $\varpi_0 \in Fix(\Omega_1)$, be arbitrary. Then, there exists $\varpi_1 \in \Omega_2\varpi_0$ such that $d(\varpi_0, \varpi_1) = D(\varpi_0, \Omega_2\varpi_0)$ and $d(\varpi_0, \varpi_1) \leq qH(\Omega_1\varpi_0, \Omega_2\varpi_0)$. Next, for $\varpi_1 \in \Omega_2\varpi_0$, there exists $\varpi_2 \in \Omega_2\varpi_1$ such that $d(\varpi_0, \varpi_1) = D(\varpi_0, \Omega_2\varpi_0)$ and $d(\varpi_1, \varpi_2) \leq qH(\Omega_2\varpi_0, \Omega_2\varpi_1)$. Then, by using (3.1), we get $d(\varpi_1, \varpi_2) \leq d(\varpi_0, \varpi_1)$ and

$$\begin{aligned} d(\varpi_1, \varpi_2) &\leq qH(\Omega_2\varpi_0, \Omega_2\varpi_1) \\ &\leq q\rho(d(\varpi_0, \varpi_1), D(\varpi_0, \Omega_2\varpi_0), D(\varpi_1, \Omega_2\varpi_1), \\ &\quad D(\varpi_0, \Omega_2\varpi_1), D(\varpi_1, \Omega_2\varpi_0)) \\ &\leq q\rho(d(\varpi_0, \varpi_1), d(\varpi_0, \varpi_1), d(\varpi_1, x_2), d(\varpi_0, \varpi_1) \\ &\quad + d(\varpi_1, x_2), 0)) \\ &\leq q\rho(d(\varpi_0, \varpi_1), d(\varpi_0, \varpi_1), d(\varpi_0, \varpi_1), 2d(\varpi_0, \varpi_1), 0)) \\ &\leq q\rho(1, 1, 1, 2, 0)d(\varpi_0, \varpi_1). \end{aligned} \tag{119}$$

Inductively, we will obtain a sequence of successive approximations for Ω_2 starting from ϖ_0, satisfying the

following:

$$d(\varpi_n, \varpi_{n+1}) \leq (q\rho_1(1,1,1,2,0))^n d(\varpi_0, \varpi_1), n \in \mathbb{N}, \quad (120)$$

which further implies for each $n \in \mathbb{N}$,

$$d(\varpi_n, \varpi_{n+m}) \leq \frac{(q\rho_1(1,1,1,2,0))^n}{1 - q\rho_1(1,1,1,2,0)} d(\varpi_0, \varpi_1). \quad (121)$$

Letting $n \longrightarrow \infty$, we get that $\{\varpi_n\}_{n\in\mathbb{N}}$ is Cauchy sequence in $(\mathfrak{Z}, d)$, and so it converges to an element $u \in \mathfrak{Z}$. As in the proof of Theorem 12, we get that $u \in \text{Fix}(\Omega_2)$. From (121), letting $m \longrightarrow \infty$ to get

$$d(\varpi_n, u) \leq \frac{(q\rho_1(1,1,1,2,0))^n}{1 - q\rho_1(1,1,1,2,0)} d(\varpi_0, \varpi_1), \text{for each } n \in \mathbb{N}. \quad (122)$$

Putting $n = 0$, we get that

$$d(\varpi_0, u) \leq \frac{1}{1 - q\rho_1 o(1,1,1,2,0)} d(\varpi_0, \varpi_1) \leq \frac{q\lambda}{1 - q\rho_1(1,1,1,2,0)}. \quad (123)$$

By interchanging the roles of Ω_1 and Ω_2, we obtain that for each $u_0 \in Fix(\Omega_2)$, there exists $x \in Fix(\Omega_1)$ such that

$$d(u_0, \varpi) \leq \frac{1}{1 - q\rho_2(1,1,1,0,2)} d(u_0, u_1) \leq \frac{q\lambda}{1 - q\rho_2(1,1,1,0,2)}. \quad (124)$$

Hence, $H(\text{Fix}(\Omega_1), \text{Fix}(\Omega_2)) \leq q\lambda/1 - \max\{q\rho_1(1,1,1,2,0), q\rho_2(1,1,1,2,0)$, and letting $q \longrightarrow 1$, we get the conclusion. □

3. Strict Fixed Points and Well Posedness

Firstly, we define the notions of well posedness of a fixed point problem.

Definition 29 [38, 39]. Let $(\mathfrak{Z}, d)$ be a MS, $\mathfrak{B} \in P(\mathfrak{Z})$, and $\Omega : \mathfrak{B} \longrightarrow C(\mathfrak{Z})$ be a multivalued operator. Then, the fixed point problem is well posed for Ω with respect to D if

(a_1) $Fix\,\Omega = \{\varpi^*;$

(b_1) If $\varpi_n \in \mathfrak{B}$, $n \in \mathbb{N}$, and $D(\varpi_n, \Omega\varpi_n) \longrightarrow 0$ as $n \longrightarrow \infty$, then $\varpi_n \xrightarrow{d} \varpi^* \in Fix\,\Omega$ as $n \longrightarrow \infty$

Definition 30 [38, 39]. Let $(\mathfrak{Z}, d)$ be a MS, $\mathfrak{B} \in P(\mathfrak{Z})$, and $\Omega : \mathfrak{B} \longrightarrow C(\mathfrak{Z})$ be a multivalued operator. Then, the fixed point problem is well posed for Ω with respect to H if

(a_2) $S\text{Fix}\,\Omega = \{\varpi^*\}$

(b_2) If $\varpi_{dn} \in \mathfrak{B}$, $n \in \mathbb{N}$, and $H(\varpi_n, \Omega\varpi_n) \longrightarrow 0$ as $n \longrightarrow \infty$, then $\varpi_n \xrightarrow{d} \varpi^* \in \text{SFix}\,\Omega$ as $n \longrightarrow \infty$

Remark 31. Note that if the fixed point problem is well posed for Ω with respect to D and $\text{Fix}\,\Omega = \text{SFix}\,\Omega$, then the fixed point problem is well posed for Ω with respect to H.

Theorem 32. *Let $(\mathfrak{Z}, d)$ be a MS and $\Omega : \mathfrak{Z} \longrightarrow K(\mathfrak{Z})$ be a multivalued operators. Assume that*

(1) There exist $\chi \in \Phi$, a continuous, nondecreasing real-valued function F_1 on $(0, \infty)$ and a real-valued function F_2 on $(0, \infty)$ satisfying condition $(F2')$ such that (N1) and (N2) hold for $\rho \in \mathscr{P}$ with $\rho(1,0,0,1,1) \in (0,1)$,

$$SFix\,\Omega \neq \varnothing. \quad (125)$$

Then,

(a) $Fix\,\Omega = SFix\,\Omega = \{\varpi^*\}$;

(b) The fixed point problem is well posed for Ω with respect to H

Proof. (a) By Theorem 17, we have that fix $(\Omega) \neq \varnothing$. Next, We will prove that $\text{Fix}\,\Omega = \{\varpi^*\}$. From (N1) and (N2), we get that

$$\begin{aligned} F_1(H(\Omega\varpi, \Omega\omega)) &\leq F_2(H(\Omega\varpi, \Omega\omega)) < \phi(d(\varpi, \omega)) + F_2(H(\Omega\varpi, \Omega\omega) \\ &\leq F_1(\rho(d(\varpi, \omega), D(\varpi, \Omega\varpi), D(\omega, \Omega\omega), D(\varpi, \Omega\omega), D(\omega, \Omega\varpi))). \end{aligned} \quad (126)$$

Since F_1 is nondecreasing function, we obtain for all $\varpi, \omega \in \mathfrak{Z}$,

$$H(\Omega\varpi, \Omega\omega) \leq \rho(d(\varpi, \omega), D(\varpi, \Omega\varpi), D(\omega, \Omega\omega), D(\varpi, \Omega\omega), D(\omega, \Omega\varpi)). \quad (127)$$

Let $u \in Fix\,\Omega$, with $u \neq \varpi^*$; then $D(\varpi^*, \Omega u) > 0$, and we have

$$\begin{aligned} D(\varpi^*, \Omega u) &= H(\Omega\varpi^*, \Omega u) \\ &\leq \rho(d(\varpi^*, u), D(\varpi^*, \Omega\varpi^*), D(u, \Omega u), D(\varpi^*, \Omega u), D(u, \Omega\varpi^*)) \\ &\leq \rho(d(\varpi^*, u), 0, 0, d(\varpi^*, u), d(u, \varpi^*)) \leq d(\varpi^*, u)\rho(1,0,0,1,1). \end{aligned} \quad (128)$$

Since $\rho(1,0,0,1,1) \in (0,1)$, above inequality implies that

$$d(\varpi^*, u) = D(\varpi^*, \Omega u) < d(\varpi^*, u), \quad (129)$$

which is a contradiction. Hence, $d(\varpi^*, u) = 0$; so, $\varpi^* = u$.

(b) Let $\varpi_n \in \mathfrak{B}$, $n \in \mathbb{N}$, such that

$$\lim_{n\longrightarrow\infty} D(\varpi_n, \Omega\varpi_n) = 0. \quad (130)$$

We claim that

$$\lim_{n\longrightarrow\infty} d(\varpi_n, \varpi^*) = 0, \quad (131)$$

where $\varpi^* \in \text{Fix}\,\Omega$. If (131) is not true, there exists $\varepsilon > 0$ such

that for each $n \in \mathbb{N}$, we have that

$$d(\varpi_n, \varpi^*) > \varepsilon. \tag{132}$$

On the other hand, from (130), there exists $n_\varepsilon \in \mathbb{N} \setminus 0$ such that $D(\varpi_n, \Omega\varpi_n) < \varepsilon$ for each $n > n_\varepsilon$. Hence, for each $n > n_\varepsilon$, we get

$$\begin{aligned} d(\varpi_n, \varpi^*) &= D(\varpi_n, \Omega\varpi^*) = D(\varpi_n, \Omega\varpi_n) + H(\Omega\varpi_n, \Omega\varpi^*) \\ &\leq D(\varpi_n, \Omega\varpi_n) + \rho(d(\varpi_n, \varpi^*), D(\varpi_n, \Omega\varpi_n), \\ &\quad D(\varpi^*, \Omega\varpi^*), D(\varpi_n, \Omega\varpi^*), D(\varpi^*, \Omega\varpi_n)) \\ &\leq D(\varpi_n, \Omega\varpi_n) + \rho(d(\varpi_n, \varpi^*), D(\varpi_n, \Omega\varpi_n), \\ &\quad d(\varpi^*, \varpi^*), d(\varpi_n, \varpi^*), d(\varpi^*, \varpi_n) + D(\varpi_n, \Omega\varpi_n)). \end{aligned} \tag{133}$$

Since $\rho(1, 0, 0, 1, 1) \in (0, 1)$, so by passing the limit $n \longrightarrow \infty$, we obtain $d(x_n, \varpi^*) \longrightarrow 0$ as $n \longrightarrow \infty$, a contradiction. Consequently, proof is complete by Remark 31. □

4. Conclusion

In the theory of set-valued dynamic systems, fixed points and strict fixed points of multivalued operators are essential notions. A rest point of the dynamic system can be read as a fixed point for the multivalued map Ω, whereas a strict fixed point for Ω can be viewed as the system's endpoint. We have made a contribution in this approach by establishing some basic problems in multivalued fixed point and strict fixed point theory. We have proved several existence and data dependence results for multivalued nonlinear mappings satisfying a new class of contractive conditions via auxiliary functions. The obtained outcomes are backed up by a nontrivial example. The findings add to and expand on some of the most recent results in the literature.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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