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Research Article

Invariant Analysis, Analytical Solutions, and Conservation Laws for Two-Dimensional Time Fractional Fokker-Planck Equation

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The main purpose of this paper is to apply the Lie symmetry analysis method for the two-dimensional time fractional Fokker-Planck (FP) equation in the sense of Riemann–Liouville fractional derivative. The Lie point symmetries are derived to obtain the similarity reductions and explicit solutions of the governing equation. By using the new conservation theorem, the new conserved vectors for the two-dimensional time fractional Fokker-Planck equation have been constructed with a detailed derivation. Finally, we obtain its explicit analytic solutions with the aid of the power series expansion method.

1. Introduction

Fractional calculus has attracted more attention of many researches in various scientific areas including biology, physics, financial theory, gas dynamics, engineering, fluid mechanics, and other areas of science, see for example [1–8]. The theory of fractional calculus is considered as a generalization of classical differential and integral calculus; it is an excellent tool for describing the memory effect and hereditary properties of various processes and viscoelastic materials. Due to its realistic senses, many researchers have tried to look for exact, analytical, and numerical solutions of fractional partial differential equations using different powerful methods such us (G'/G)-expansion method [9], the Variational iteration method [10, 11], functional variable method [12], subequation method [13], Finite difference method [14], Exp function method [15], Homotopy analysis method [16], Adomian decomposition method [17], the First integral method [18], Laplace transform method [19], Sumudu transform method [20], and so many other approaches.

The Lie symmetry method was firstly advocated by the Norwegian mathematician Sophus Lie [21, 22], who has made great achievements in the theories of continuous groups and differential equations. It is an efficient approach and widely employed for solving ordinary differential equations (ODEs), partial differential equations (PDEs), and fractional partial differential equations (FPDEs). This popularity is due to its utility in determining the explicit solutions of both ODEs and PDEs, linearization of some nonlinear equations, reducing the order of independent variables, and so on. Many papers focused on constructing symmetries of different fractional differential equations [23, 24–27]. Furthermore, the concept of conservation laws is fundamental and widely used in the study of the resolution of PDEs. Moreover, they convey a large deal of information about the studied physical system. The new conservation laws were introduced by Ibragimov [28], based on the notion of Lie symmetry generators without Lagrangian for solving FPDEs. Therefore, this new conservation law plays an increasingly important role in solving the conservation laws of FPDEs. More details about conservation laws can be found in [29–32].

In this paper, we consider the following two-dimensional time fractional Fokker-Planck equation

$$\frac{\partial^\alpha u}{\partial t^\alpha} = -\frac{a^2x^2}{2}\frac{\partial u^2}{\partial x^2} - \frac{b^2y^2}{2}\frac{\partial u^2}{\partial y^2} - kabxy\frac{\partial u^2}{\partial x \partial y} - rx\frac{\partial u}{\partial x} - ry\frac{\partial u}{\partial y} + ru. \tag{1}$$

It is well known that the Markovian diffusion process can be described with the Fokker-Planck equation. The Fokker-Planck equation is a partial differential equation for the probability density and the transition probability of these stochastic processes. It plays an important role in control theory, fluid mechanics, astrophysics, and quantum [33]. Moreover, it has been applied in various natural science fields such as quantum optics, solid-state physics, chemical physics, theoretical biology, and circuit theory. It is firstly proposed by Fokker and Planck to characterize the Brownian motion of particles [34]. Many researchers have solved the Fokker-Planck equation using various powerful methods, for more details see [35–37].

The fractional derivatives described here are in the Riemann-Liouville sense of order $\alpha(\alpha > 0)$, see [38, 39], which is defined by

$$D^\alpha u(t,x) = \frac{\partial^\alpha u}{\partial t^\alpha} = \begin{cases} \frac{\partial^m u}{\partial t^m}, \alpha = m \in \mathbb{N}, \\ \frac{1}{\Gamma(m-\alpha)}\frac{\partial^m}{\partial t^m} \end{cases}, \cdot \int_0^t (t-\tau)^{m-\alpha-1} u(\tau,x)d\tau, m-1 < \alpha < m, m \in \mathbb{N}^* \tag{2}$$

where $\Gamma(z)$ is the Gamma function defined by

$$\Gamma(z) = \int_0^{+\infty} e^{-t}t^{z-1}. \tag{3}$$

The main motivation behind this article is to make use of the Lie symmetry method to get the infinitesimal generators, group invariant solutions for the time fractional two-dimensional Fokker-Planck equation (FP), and to construct conservation laws given by Ibragimov [28]. Therefore, the new conserved vectors have been obtained using the new conservation theorem. Based on the power series method [32], the explicit power series solutions of the two-dimensional time fractional Fokker-Planck equation are derived.

The rest of this paper is organized as follows: in Section 2, we review some basic definitions of the Lie Symmetry method for fractional partial differential equations (FPDEs) and its properties. By employing the proposed method, Lie point symmetries of the Eq. (1) are obtained; by using similarity variables, the reduced equations are obtained; solving some of them, then the similarity solutions of Eq. (1) are deduced in Section 3. In Section 4, the conservation laws of Eq. (1) are obtained. Section 5 is devoted to constructing the explicit analytical power series solutions. Some conclusions and discussions are given in Section 6.

2. Method of Lie Symmetry Analysis for FPDEs

In this section, we briefly review the main points about Lie symmetry analysis of FPDEs [39–41] of the following form

$$D_t^\alpha u = F\left(x, y, t, u, u_t, u_x, u_y, u_{xx}, u_{yy}, \cdots\right), \alpha > 0. \tag{4}$$

We assume that the Eq. (4) is invariant under a one-parameter ε Lie group of infinitesimal transformations which are given as

$$\begin{aligned}
\widehat{x} &= x + \varepsilon\xi(x,y,t,u) + O\left(\varepsilon^2\right),\\
\widehat{y} &= y + \varepsilon\zeta(x,y,t,u) + O\left(\varepsilon^2\right),\\
\widehat{t} &= t + \varepsilon\tau(x,y,t,u) + O\left(\varepsilon^2\right),\\
\widehat{u} &= u + \varepsilon\eta(x,y,t,u) + O\left(\varepsilon^2\right),\\
\frac{\partial^\alpha \widehat{u}}{\partial t^\alpha} &= \frac{\partial^\alpha u}{\partial t^\alpha} + \varepsilon\eta_\alpha^0(x,y,t,u) + O\left(\varepsilon^2\right),\\
\frac{\partial \widehat{u}}{\partial \widehat{x}} &= \frac{\partial u}{\partial x} + \varepsilon\eta^x(x,y,t,u) + O\left(\varepsilon^2\right),\\
\frac{\partial \widehat{u}}{\partial \widehat{y}} &= \frac{\partial u}{\partial y} + \varepsilon\eta^y(x,y,t,u) + O\left(\varepsilon^2\right),\\
\frac{\partial^2 \widehat{u}}{\partial x\wedge^2} &= \frac{\partial^2 u}{\partial x^2} + \varepsilon\eta^{xx}(x,y,t,u) + O\left(\varepsilon^2\right),\\
\frac{\partial^2 \widehat{u}}{\partial y\wedge^2} &= \frac{\partial^2 u}{\partial y^2} + \varepsilon\eta^{yy}(x,y,t,u) + O\left(\varepsilon^2\right),\\
&\vdots
\end{aligned} \tag{5}$$

where $\varepsilon < <1$ is a group parameter and ξ, ζ, η, and τ are infinitesimals and η_α^0 is extended infinitesimal. The explicit expressions of $\eta^x, \eta^y, \eta^{xx}, \eta^{yy}$ are given by

$$\begin{aligned}
\eta^x &= D_x(\eta) - u_x D_x(\xi) - u_y D_x(\zeta) - u_t D_x(\tau),\\
\eta^y &= D_y(\eta) - u_x D_y(\xi) - u_y D_y(\zeta) - u_t D_y(\tau),\\
\eta^{xx} &= D_x(\eta^{xx}) - u_{xxx} D_x(\xi) - u_{xxy} D_x(\zeta) - u_{xxt} D_x(\tau),\\
&\vdots
\end{aligned} \tag{6}$$

D_x, D_y, and D_t are the total derivatives with respect to x, y , and t, respectively, which are defined as

$$D_{x^i} = \frac{\partial}{\partial x^i} + u_i\frac{\partial}{\partial u} + u_{ij}\frac{\partial}{\partial u_j} + \cdots, i, j = 1, 2, 3, \cdots, \tag{7}$$

where $u_i = \partial u/\partial x^i$, $u_{ij} = \partial^2 u/\partial x^i \partial x^j$, and so on.

The infinitesimal generator X is given by the following expression

$$X=\xi(x,y,t,u)\frac{\partial}{\partial x}+\zeta(x,y,t,u)\frac{\partial}{\partial x}+\tau(x,y,t,u)\frac{\partial}{\partial t}+\eta(x,y,t,u)\frac{\partial}{\partial u}. \tag{8}$$

The infinitesimal generator X satisfy the following invariance condition of Eq. (4):

$$Pr^{(n)}X(\Delta)\Big|_{\Delta=0}=0, n=1,2,\cdots, \tag{9}$$

where

$$\Delta\coloneqq D_t^\alpha u-F\left(x,t,u,u_t,u_x,u_y,u_{xx},u_{yy}\cdots\right). \tag{10}$$

The structure of the Riemann-Liouville derivative must be invariant under transformations (5), because the lower limit of the integral (2) is fixed. The invariance condition yields

$$\tau(x,y,t,u)|_{t=0}=0. \tag{11}$$

The explicit form of the α^{th} extended infinitesimal can be obtained as follows:

$$\begin{aligned}\eta_\alpha^0=&\frac{\partial^\alpha\eta}{\partial t^\alpha}+(\eta_u-\alpha D_t(\tau))\frac{\partial^\alpha u}{\partial t^\alpha}-u\frac{\partial^\alpha\eta_u}{\partial t^\alpha}\\&+\mu+\sum_{n=1}^{\infty}\left[\begin{pmatrix}\alpha\\n\end{pmatrix}\frac{\partial^\alpha\eta_u}{\partial t^\alpha}-\begin{pmatrix}\alpha\\n+1\end{pmatrix}D_t^{n+1}(\tau)\right]D_t^{\alpha-n}(u)\\&-\sum_{n=1}^{\infty}\begin{pmatrix}\alpha\\n\end{pmatrix}D_t^n(\xi)D_t^{\alpha-n}(u_x)-\sum_{n=1}^{\infty}\begin{pmatrix}\alpha\\n\end{pmatrix}D_t^n(\zeta)D_t^{\alpha-n}(u_y),\end{aligned} \tag{12}$$

where

$$\begin{aligned}\mu=&\sum_{n=2}^{\infty}\sum_{m=2}^{n}\sum_{k=2}^{m}\sum_{r=0}^{k-1}\begin{pmatrix}\alpha\\n\end{pmatrix}\begin{pmatrix}n\\m\end{pmatrix}\begin{pmatrix}k\\r\end{pmatrix}\frac{1}{k!}\frac{t^{n-\alpha}}{\Gamma(n+1-\alpha)}(-u)^r\frac{\partial^m}{\partial t^m}\\&\cdot\left(u^{k-r}\right)\frac{\partial^{n-m+k}\eta}{\partial x^{n-m}\partial u^k}.\end{aligned} \tag{13}$$

It is worth noting that $\mu=0$ if the infinitesimal η is linear in u, due to the presence of $\partial^k\eta/\partial u^k$.

Definition 1 (see [42]). The function $u=\theta(x,y,t)$ is an invariant solution of Eq. (4) if and only if

(i) $u=\theta(x,y,t)$ is an invariant surface, that is to say

$$\begin{aligned}X\theta=0\Longleftrightarrow\Big(&\xi(x,y,t,u)\frac{\partial}{\partial x}+\zeta(x,y,t,u)\frac{\partial}{\partial y}\\&+\tau(x,y,t,u)\frac{\partial}{\partial t}+\eta(x,y,t,u)\frac{\partial}{\partial u}\Big)\theta=0.\end{aligned} \tag{14}$$

(ii) $u=\theta(x,y,t)$ satisfies Eq. (4)

3. Symmetry Analysis and Similarity Reductions of the Two-Dimensional Time Fractional Fokker-Planck Equation

In the present section, the Lie symmetry analysis method has been applied for deriving the infinitesimal generators of the two-dimensional time fractional Fokker-Planck (1). By using the third prolongation [43, 44], the symmetry determining equation for Eq. (1) has been obtained as

$$\begin{aligned}\eta_0^\alpha+&\frac{a^2x^2}{2}\eta^{xx}+a^2x\xi u_{xx}+\frac{b^2}{2}y^2\eta^{yy}+b^2\varphi yu_{yy}+kabxy\eta^{xy}\\&+kab(x\varphi+y\xi)u_{xy}+rx\eta^x+r\xi u_x+ry\eta^y+ryu_y-ru=0.\end{aligned} \tag{15}$$

By substituting the expressions η_α^0 given in Eq. (6) and Eq. (12) into Eq. (15), and equating various powers of derivatives of u to zero, we obtain an overdetermined system of linear equations; by solving this system, we obtain the following infinitesimals

$$\begin{aligned}\eta(x,y,u,t)&=(C_1\ln(x)+C_2\ln(y)+C_3)u+f(x,y,t),\\\tau(x,y,u,t)&=-\frac{abk(C_1+C_3)}{\alpha r}t+C_4,\\\xi(x,y,u,t)&=\left(-\frac{C_1abkln(x)}{r}+C_6\right)x,\\\varphi(x,y,u,t)&=\left(-\frac{C_3abkln(y)}{r}+C_5\right)y,\end{aligned} \tag{16}$$

where C_i, $i=1,..,6$ are arbitrary constants. So, the associated vector fields of Eq. (1) are given by

$$\begin{aligned}X_1&=y\frac{\partial}{\partial y},\\X_2&=x\frac{\partial}{\partial x},\\X_3&=\ln(x)\frac{\partial}{\partial u}+\frac{4a^2t}{\alpha(a^2-2r)}\frac{\partial}{\partial t}+\frac{2a^2\ln(x)}{a^2-2r}x\frac{\partial}{\partial x},\\X_\infty&=f(x,y,t)\frac{\partial}{\partial u}.\end{aligned} \tag{17}$$

Case 2. For $X_1 = \partial/\partial y$, the characteristic equation is

$$\frac{dy}{y} = \frac{dx}{0} = \frac{du}{0} = \frac{dt}{0}. \tag{18}$$

By solving the above characteristic equation, we obtain the solution $u = f(x, t)$. Substituting it into Eq. (1), we derive the following reduced fractional ordinary Fokker-Planck equation:

$$D_t^\alpha f(x,t) = -\frac{b^2 x^2}{2}\frac{\partial^2 f}{\partial x^2}(x,t) - rx\frac{\partial f}{\partial x}(x,t) + rf(x,t). \tag{19}$$

Using the symmetry method, we obtain the following infinitesimals:

$$\begin{aligned} \eta_1 &= \ln(x) M_2 u + M_1 u + f(x,t), \\ \tau_1 &= \frac{4M_2 a^2 t}{\alpha(a^2 - 2r)} + M_3, \\ \xi_1 &= \left(\frac{4M_2 a^2 t}{\alpha(a^2 - 2r)} + M_4\right)x, \end{aligned} \tag{20}$$

where M_i, $i = 1, .., 4$ are arbitrary constants. Then, the Lie algebra of infinitesimal symmetries of Eq. (19) is given by

$$\begin{aligned} X_{11} &= x\frac{\partial}{\partial x}, \\ X_{12} &= u\frac{\partial}{\partial u}, \\ X_{13} &= \ln(x)\frac{\partial}{\partial u} + \frac{2a^2 \ln(x)}{a^2 - 2r}x\frac{\partial}{\partial x} + \frac{4a^2 t}{\alpha(a^2 - 2r)}t\frac{\partial}{\partial t}, \\ X_\infty &= f(x,t)\frac{\partial}{\partial u}. \end{aligned} \tag{21}$$

Case 3. The characteristic equation for the infinitesimal generator X_{11} can be expressed symbolically as follows:

$$\frac{dx}{1} = \frac{dt}{0} = \frac{du}{0}. \tag{22}$$

By solving the above characteristic equation, we obtain the solution $u = f(t)$. Substituting it into Eq. (19), we derive the reduced fractional ordinary equation:

$$D_t^\alpha f = rf. \tag{23}$$

The above can be solved through the Laplace transform method

$$\mathscr{L}(D_t^\alpha f) = r\mathscr{L}(f). \tag{24}$$

Since the Laplace transform of the Riemann–Liouville derivative is defined by the following form

$$\mathscr{L}\{D_x^\alpha f(x), s\} = s^\alpha F(s) - \sum_{k=0}^{n-1} s^k D^{\alpha-k-1} f(0_+), \tag{25}$$

then,

$$\mathscr{L}(D_t^\alpha f) = s^\alpha \mathscr{L}(f) - s^{\alpha-1}. \tag{26}$$

According to Eq. (25), we have

$$\mathscr{L}(f) = \frac{s^{\alpha-1}}{s^\alpha - r}. \tag{27}$$

By using the inverse Laplace transform, it gives

$$f(t) = E_{\alpha,1}(rt^\alpha), \tag{28}$$

where

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \alpha, \beta \in \mathbb{C}, \tag{29}$$

is the Mittag-Leffler function.

Case 4. For $X_{11} + \gamma X_{12}$, the similarity transformation corresponding to this generator can be derived by solving the associated characteristic equation

$$\frac{dx}{x} = \frac{du}{u} = \frac{dt}{t}, \tag{30}$$

which take the form

$$u = x^\gamma g(t), \tag{31}$$

replacing it in Eq. (19) yields the following reduced FODE:

$$D_t^\alpha(g(t)) = \left(-\frac{b^2}{2}\alpha(\alpha-1) - r(\alpha-1)\right)g(t). \tag{32}$$

By using the Laplace transform, we obtain the following solution

$$g(t) = E_{\alpha,1}\left(\left(\frac{b^2}{2}\alpha(\alpha-1) + r(\alpha-1)\right)t^\alpha\right). \tag{33}$$

4. Conservation Laws of the Two-Dimensional Time Fractional Fokker-Planck Equation

In this section, the conservation laws of the two-dimensional time fractional Fokker-Planck equation have been investigated by using a new conservation theorem [28]. The conserved vectors C_t, C_x, C_y have been obtained, and it satisfies the following conservation equation:

$$D_t\left(C^t\right) + D_x(C^x) + D_y(C^y) = 0. \tag{34}$$

The formal Lagrangian for Eq. (1) can be written as follows:

$$L = \omega(x, y, t)\left[\frac{\partial^\alpha u}{\partial t^\alpha} + \frac{a^2x^2}{2}\frac{\partial u^2}{\partial x^2} + \frac{b^2y^2}{2}\frac{\partial u^2}{\partial y^2} + kabxy\frac{\partial u^2}{\partial x\partial y} + rx\frac{\partial u}{\partial x} + ry\frac{\partial u}{\partial y} - ru\right], \tag{35}$$

where $\omega(x, y, t)$ is the new dependent variable. Based on the definition of the Lagrangian, the action integral of Eq. (35) is given by

$$\int_0^T\int_{\Omega_x}\int_{\Omega_y} L\left(x, y, t, \omega, u, D_t^\alpha u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}\right)dxdydt. \tag{36}$$

The Euler–Lagrange operator is defined as

$$\frac{\delta}{\delta u} = \frac{\partial}{\partial u} + (D_t^\alpha u)^*\frac{\partial}{\partial D_t^\alpha} - D_x\frac{\partial}{\partial u_x} - D_y\frac{\partial}{\partial u_y} + D_x^2\frac{\partial}{\partial u_{xx}} + D_y^2\frac{\partial}{\partial u_{yy}} + D_xD_y\frac{\partial}{\partial u_{xy}}. \tag{37}$$

The adjoint operator $(D_t^\alpha)^*$ of is defined by

$$(D_t^\alpha)^* = (-1)^n R_T^{n-\alpha}(D_t^n) = {}_t^C D_T^\alpha, \tag{38}$$

where $R_T^{n-\alpha}$ is the right-sided operator of fractional integration of order $(n-\alpha)$ that is defined by

$$R_T^{n-\alpha}f(x, t) = \frac{1}{\Gamma(n-\alpha)}\int_t^T(\tau - t)^{n-\alpha-1}f(x, \tau)d\tau. \tag{39}$$

So, the adjoint equation of Eq. (1) as the Euler–Lagrange equation, given by

$$\frac{\delta L}{\delta u} = 0. \tag{40}$$

For the case of three independent variables x, y, t and one dependent variable $u(x, y, t)$, we get

$$\bar{X} + D_t(\tau)I + D_x(\xi)I + D_y(\zeta)I = W\frac{\delta}{\delta u} + D_t\left(C^t\right)I + D_x(C^x)I + D_y(C^y)I, \tag{41}$$

where I is the identity operator and $\delta/\delta u$ is denoted as the Euler-Lagrange operator. So $\bar{X}$ is presented as

$$\bar{X} = \xi\frac{\partial}{\partial x} + \tau\frac{\partial}{\partial t} + \eta\frac{\partial}{\partial u} + \zeta\frac{\partial}{\partial y} + \eta_\alpha^0\frac{\partial}{\partial D_t^\alpha u} + \eta^x\frac{\partial}{\partial u_x} + \eta^{xx}\frac{\partial}{\partial u_{xx}} + \eta^y\frac{\partial}{\partial u_y} + \eta^{yy}\frac{\partial}{\partial u_{yy}} + \eta^{xy}\frac{\partial}{\partial u_{xy}}, \tag{42}$$

and the Lie characteristic function W is given by

$$W = \eta - \tau u_t - \xi u_x - \zeta u_y. \tag{43}$$

Using the Lie symmetries V_1, V_2, V_3, we have

$$W_1 = -xu_x,\ W_2 = -u_t,\ W_3 = -yu_y,\ W_4 = u\ln(x)\frac{\partial}{\partial u} + \frac{abk}{\alpha r}t\frac{\partial}{\partial t} + \frac{abk}{r}x\ln(x)\frac{\partial}{\partial x},$$

$$W_5 = u\ln(y)\frac{\partial}{\partial u} + \frac{abk}{\alpha r}t\frac{\partial}{\partial t} + \frac{abk}{r}y\ln(y)\frac{\partial}{\partial y},\ W_6 = u,\ W_\infty = f(x, y, t). \tag{44}$$

Based on the fractional generalizations of the Noether operators, the components of conserved vectors can be presented as follows:

$$C^t = \tau L + \sum_{k=0}^{n-1}(-1)^k{}_0D_t^{\alpha-1-k}(W)D_t^k\frac{\partial L}{\partial_0 D_t^\alpha u} - (-1)^n J\left(W, D_t^k\frac{\partial L}{\partial_0 D_t^\alpha u}\right), \tag{45}$$

where $J(.)$ is defined by

$$J(f, g) = \frac{1}{\Gamma(n-\alpha)}\int_0^t\int_t^T\frac{f(\tau, x, y)g(\theta, x, y)}{(\theta-\tau)^{\alpha+1-n}}d\theta d\tau. \tag{46}$$

And the other components C^i are defined as

$$C^i = \xi^i L + W_\theta\left[\frac{\partial L}{\partial u_i^\theta} - D_j\left(\frac{\partial L}{\partial u_{ij}^\theta}\right) + D_jD_k\left(\frac{\partial L}{\partial u_{ijk}^\theta}\right) - \cdots\right] + D_j(W_\theta)\left[\frac{\partial L}{\partial u_{ij}^\theta} - D_k\left(\frac{\partial L}{\partial u_{ijk}^\theta}\right) + \cdots\right] + D_jD_k(W_\theta)\left(\frac{\partial L}{\partial u_{ijk}^\theta} - \cdots\right) + \cdots, \tag{47}$$

where $\xi^1 = \xi,\ \xi^2 = \zeta,\ \theta = 1, 2$. Using Eqs. (45) and (47), we obtain the following components of conserved vectors.

Case 5. For $W_1 = -xu_x$, we have

$$C^t = \tau L + {}_0D_t^{\alpha-1}(W_2)D_t^k \frac{\partial L}{\partial_0 D_t^\alpha u} + J\left(W_2, D_t \frac{\partial L}{\partial_0 D_t^\alpha u}\right) = w(x,y,t)D_t^{\alpha-1}(-u_t) + J(-u_t, w_t),$$

$$C^x = \xi L + W_2\left[\frac{\partial L}{\partial u_x} - D_x\frac{\partial L}{\partial u_{xx}} - D_y\frac{\partial L}{\partial u_{xy}}\right] + D_x(W_2)\left(\frac{\partial L}{\partial u_{xx}}\right) + D_y(W_2)\left(\frac{\partial L}{\partial u_{xy}}\right) = -u_t\left(rxw - \frac{a^2x^2}{2}w_x - a^2xw - kabxyw_y - kabxw\right) - u_{xt}w\frac{a^2x^2}{2} - u_{yt}kabxyw,$$

$$C^y = \xi L + W_2\left[\frac{\partial L}{\partial u_y} - D_x\frac{\partial L}{\partial u_{xy}} - D_y\frac{\partial L}{\partial u_{yy}}\right] + D_x(W_2)\left[\frac{\partial L}{\partial u_{xy}}\right] + D_y(W_2)\left(\frac{\partial L}{\partial u_{yy}}\right) = -u_t\left[ryw - kabxyw_x - kabyw - \frac{b^2y^2}{2}w_y - b^2yw\right] - u_{xt}kabxyw - u_{yt}\frac{b^2y^2}{2}w. \tag{49}$$

Case 7. For $W_2 = -yu^y$, we have

$$C^t = \tau L + {}_0D_t^{\alpha-1}(W_3)D_t^k \frac{\partial L}{\partial_0 D_t^\alpha u} + J\left(W_3, D_t \frac{\partial L}{\partial_0 D_t^\alpha u}\right) = -yw(x,y,t)D_t^{\alpha-1}(u_y) - J(yu_y, w_t),$$

$$C^x = \xi L + W_3\left[\frac{\partial L}{\partial u_x} - D_x\frac{\partial L}{\partial u_{xx}} - D_x\frac{\partial L}{\partial u_{xx}} - D_y\frac{\partial L}{\partial u_{xy}}\right] + D_x(W_3)\left(\frac{\partial L}{\partial u_{xx}}\right) + D_y(W_3)\left(\frac{\partial L}{\partial u_{xy}}\right) = -u_y\left(rxyw - \frac{a^2x^2}{2}yw_x - a^2xyw - kabxy^2w_y - kabxyw\right) - u_{xy}yw\frac{a^2x^2}{2} - u_{yy}kabxy^2w,$$

$$C^y = \xi L + W_3\left[\frac{\partial L}{\partial u_y} - D_x\frac{\partial L}{\partial u_{xy}} - D_y\frac{\partial L}{\partial u_{yy}}\right] + D_x(W_3)\left[\frac{\partial L}{\partial u_{xy}}\right] + D_y(W_3)\left(\frac{\partial L}{\partial u_{yy}}\right) = -u_y\left[ry^2w - kabxy^2w - kabxy^2w_x - \frac{b^2y^3}{2}w_y - b^2y^2w\right] - u_{xt}kaby^2w - u_{yy}\frac{b^2y^3}{2}w. \tag{50}$$

Case 8. For $W_4 = uln(x) + (kab/\alpha r)tu_t + (kab/r)xln(x)u_x$, we have

$$C^t = \tau L + {}_0D_t^{\alpha-1}(W_4)D_t^k \frac{\partial L}{\partial_0 D_t^\alpha u} + J\left(W_4, D_t \frac{\partial L}{\partial_0 D_t^\alpha u}\right) = w(x,y,t)D_t^{\alpha-1}\left(-uln(x) + \frac{kab}{\alpha r}tu_t + \frac{kab}{r}xln(x)u_x\right) - J\left(-uln(x) + \frac{kab}{\alpha r}tu_t + \frac{kab}{r}xln(x), w_t\right),$$

$$C^x = \xi L + W_4\left[\frac{\partial L}{\partial u_x} - D_x\frac{\partial L}{\partial u_{xx}} - D_x\frac{\partial L}{\partial u_{xx}} - D_y\frac{\partial L}{\partial u_{xy}}\right] + D_x(W_4)\left(\frac{\partial L}{\partial u_{xx}}\right) + D_y(W_4)\left(\frac{\partial L}{\partial u_{xy}}\right) = \left(-u\ln(x) + \frac{kab}{\alpha r}tu_t + \frac{kab}{r}x\ln(x)\right)\left(rxw - \frac{a^2x^2}{2}w_x - a^2xw - kabxyw_y - kabxw\right) + \left(\frac{u}{x} + (\ln(x)+1)\frac{kab}{r}u_x + \frac{kab}{\alpha r}tu_{xt} + \frac{kab}{r}x\ln(x)u_{xx}\right)\frac{a^2x^2}{2}w + \frac{ka^2b^2}{\alpha r}k^2xytwu_{ty} + \frac{ka^2b^2}{r}kx^2y\ln(x)u_{xy},$$

$$C^y = \xi L + W_4\left[\frac{\partial L}{\partial u_y} - D_x\frac{\partial L}{\partial u_{xy}} - D_y\frac{\partial L}{\partial u_{yy}}\right] + D_x(W_4)\left[\frac{\partial L}{\partial u_{xy}}\right] + D_y(W_4)\left(\frac{\partial L}{\partial u_{yy}}\right) = \left(u\ln(x) + \frac{kab}{\alpha r}tu_t + \frac{kab}{r}x\ln(x)u_x\right) \cdot \left[ryw - kabyw - kabxyw_x - \frac{b^2y^2}{2}w_y - b^2yw\right] + \left(\frac{u}{x} + (\ln(x)+1)\frac{kab}{r}u_x + \frac{kab}{\alpha r}tu_{xt} + \frac{kab}{r}x\ln(x)u_{xx}\right)kabxyw + \frac{ka^2b^3}{2\alpha r}kty^2wu_{ty} + \frac{kab^3}{2r}kwxy^2\ln(x)u_{xy}. \tag{51}$$

Case 9. For $W_5 = u\ln(y) + (kab/\alpha r)tu_t + (kab/r)y\ln(y)u_y$, we have

$$C^t = \tau L + {}_0D_t^{\alpha-1}(W_5)D_t^k \frac{\partial L}{\partial_0 D_t^\alpha u} + J\left(W_5, D_t \frac{\partial L}{\partial_0 D_t^\alpha u}\right) = w(x,y,t)D_t^{\alpha-1}\left(u\ln(y) + \frac{kab}{\alpha r}tu_t + \frac{kab}{r}y\ln(y)u_y\right) - J\left(u\ln(y) + \frac{kab}{\alpha r}tu_t + \frac{kab}{r}y\ln(y)u_y, w_t\right),$$

$$C^x = \xi L + W_4 \left[\frac{\partial L}{\partial u_x} - D_x \frac{\partial L}{\partial u_{xx}} - D_x \frac{\partial L}{\partial u_{xx}} - D_y \frac{\partial L}{\partial u_{xy}} \right] + D_x(W_5)\left(\frac{\partial L}{\partial u_{xx}}\right) + D_y(W_5)\left(\frac{\partial L}{\partial u_{xy}}\right) = \left(u \ln(y) + \frac{kab}{\alpha r} t u_t + \frac{kab}{r} y \ln(y) u_y\right)\left(rxw - \frac{a^2x^2}{2} w_x - a^2xw - kabxyw_y - kabxw\right) + \left(\frac{u}{y} + (\ln(y)+1)\frac{kab}{r} u_y + \frac{kab}{\alpha r} t u_{yt} + \frac{kab}{r} y \ln(y) u_{yy}\right) kxyabw + \frac{kab}{\alpha r} t u_{xt} + \frac{ka^3b}{2r} kx^2 y \ln(y) w u_{xy},$$

$$C^y = \xi L + W_5 \left[\frac{\partial L}{\partial u_y} - D_x \frac{\partial L}{\partial u_{xy}} - D_y \frac{\partial L}{\partial u_{yy}} \right] + D_x(W_5)\left[\frac{\partial L}{\partial u_{xy}}\right] + D_y(W_5)\left(\frac{\partial L}{\partial u_{yy}}\right) = \left(u \ln(y) + \frac{kab}{\alpha r} t u_t + \frac{kab}{r} y \ln(y) u_y\right) \cdot \left[ryw - kabyw - kabxyw_x - \frac{b^2y^2}{2} w_y - b^2yw \right] + \left(\frac{u}{y} + (\ln(y)+1)\frac{kab}{r} u_y + \frac{kab}{\alpha r} t u_{yt} + \frac{kab}{r} y \ln(y) u_{yy}\right) \frac{b^2y^2}{2} w + \frac{k^2a^2b^2}{\alpha r} ktxywu_{xt}. \tag{52}$$

Case 10. For $W_6 = u$, we have

$$C^t = \tau L + {}_0D_t^{\alpha-1}(W_6) D_t^k \frac{\partial L}{\partial_0 D_t^\alpha u} + J\left(W_6, D_t \frac{\partial L}{\partial_0 D_t^\alpha u}\right) = w(x,y,t) D_t^{\alpha-1}(u) - J(u, w_t),$$

$$C^x = \xi L + W_6 \left[\frac{\partial L}{\partial u_x} - D_x \frac{\partial L}{\partial u_{xx}} - D_x \frac{\partial L}{\partial u_{xx}} - D_y \frac{\partial L}{\partial u_{xy}} \right] + D_x(W_6)\left(\frac{\partial L}{\partial u_{xx}}\right) + D_y(W_6)\left(\frac{\partial L}{\partial u_{xy}}\right) = u\left(rxw - \frac{a^2x^2}{2} w_x - a^2xw - kabxyw_y - kabxw\right) + \frac{a^2x^2}{2} wu_x + kabxywu_y,$$

$$C^y = \xi L + W_6 \left[\frac{\partial L}{\partial u_y} - D_x \frac{\partial L}{\partial u_{xy}} - D_y \frac{\partial L}{\partial u_{yy}} \right] + D_x(W_6)\left[\frac{\partial L}{\partial u_{xy}}\right] + D_y(W_6)\left(\frac{\partial L}{\partial u_{yy}}\right) = u\left[ryw - kabyw - kabxyw_x - \frac{b^2y^2}{2} w_y - b^2yw \right] + \frac{b^2y^2}{\alpha r} wu_y + kabxywu_x. \tag{53}$$

Case 11. For $W_\infty = f(x,y,t)$, we have

$$C^t_\infty = \tau L + {}_0D_t^{\alpha-1}(f(x,y,t)) D_t^k \frac{\partial L}{\partial_0 D_t^\alpha u} + J\left(f(x,y,t), D_t \frac{\partial L}{\partial_0 D_t^\alpha u}\right) = w(x,y,t) D_t^{\alpha-1}(f(x,y,t)) - J(f(x,y,t), w_t),$$

$$C^x_\infty = \xi L + W_\infty \left[\frac{\partial L}{\partial u_x} - D_x \frac{\partial L}{\partial u_{xx}} - D_x \frac{\partial L}{\partial u_{xx}} - D_y \frac{\partial L}{\partial u_{xy}} \right] + D_x(W_\infty)\left(\frac{\partial L}{\partial u_{xx}}\right) + D_y(W_\infty)\left(\frac{\partial L}{\partial u_{xy}}\right) = f(x,y,t)\left(rxw - \frac{a^2x^2}{2} w_x - a^2xw - kabxyw_y - kabxw\right) + \frac{a^2x^2}{2} wf_x(x,y,t) + kabxywf_y(x,y,t),$$

$$C^y_\infty = \xi L + W_\infty \left[\frac{\partial L}{\partial u_y} - D_x \frac{\partial L}{\partial u_{xy}} - D_y \frac{\partial L}{\partial u_{yy}} \right] + D_x(W_\infty)\left[\frac{\partial L}{\partial u_{xy}}\right] + D_y(W_\infty)\left(\frac{\partial L}{\partial u_{yy}}\right) = f(x,y,t)\left[ryw - kabyw - kabxyw_x - \frac{b^2y^2}{2} w_y - b^2yw \right] + \frac{b^2y^2}{\alpha r} wf_y(x,y,t) + f_x(x,y,t) kabxyw. \tag{54}$$

5. Power Series and Analytical Solutions for Eq. (1)

In this section, based on the power series method [45], the exact analytic solutions are a kind of exact power series solutions for Eq. (1), constructed with a detailed derivation.

$$u(x,y,t) = u(w), w = my + \rho x - \frac{\epsilon t^\alpha}{\Gamma(1+\alpha)}, \tag{55}$$

where m, k, α, and $\epsilon \neq 0$ are arbitrary. The time fractional Fokker-Planck Eq. (1) is reduced to the following ODE

$$(-\epsilon + rx\rho + rym)u' + \left(\frac{a^2x^2}{2}\rho^2 + \frac{by^2}{2}m^2 + \rho abkmxy\right)u'' - ru = 0. \tag{56}$$

We assume that the solution of Eq. (1) has the following form:

$$u(w) = \sum_{n=0}^{\infty} \sigma_n \xi^n, \tag{57}$$

where σ_n are constants to be determined later. According to

Eq. (57), we get

$$u'(w) = \sum_{n=0}^{\infty}(n+1)\sigma_{n+1}\xi^n,$$
$$u''(w) = \sum_{n=0}^{\infty}(n+1)(n+2)\sigma_{n+2}\xi^n. \tag{58}$$

Substituting (57) and (58) into (56), we obtain

$$(-\epsilon + rx\rho + rym)\sum_{n=0}^{\infty}(n+1)\sigma_{n+1}\xi^n + \left(\frac{a^2x^2}{2}\rho^2 + \frac{by^2}{2}m^2 + \rho abkmxy\right)\sum_{n=0}^{\infty}(n+1)(n+2)\sigma_{n+2}\xi^n - r\sum_{n=0}^{\infty}\sigma_n\xi^n = 0. \tag{59}$$

Observing coefficients in Eq. (59), when $n = 0$, we have

$$(-\epsilon + rx\rho + rym)\sigma_1 + 2\left(\frac{a^2x^2}{2}\rho^2 + \frac{by^2}{2}m^2 + \rho abkmxy\right)\sigma_2 - r\sigma_0 = 0. \tag{60}$$

By comparing coefficients of σ, we get

$$\sigma_2 = \frac{(\epsilon - rx\rho - rym)\sigma_1 + r\sigma_0}{ax^2\rho^2 + bm^2y^2 + 2\rho abkmxy}. \tag{61}$$

When $n \geq 1$, we have

$$\sigma_{n+2} = \frac{2}{(n+1)(n+2)}\frac{(\epsilon - rx\rho - rym)(n+1)\sigma_{n+1} + r\sigma_n}{ax^2\rho^2 + bm^2y^2 + 2\rho abkmxy}. \tag{62}$$

The power series solution for Eq. (5) can be rewritten as follows:

$$u(x, y, t) = a_0 + a_1\left(kx + my - \frac{\epsilon t^\alpha}{\Gamma(1+\alpha)}\right) + \frac{(\epsilon - rxk - rym)\sigma_1 + r\sigma_0}{ax^2k^2 + by^2m^2 + 2\rho xyabkm}\left(kx + my - \frac{\epsilon t^\alpha}{\Gamma(1+\alpha)}\right)^2 + \sum_{n=1}^{\infty}\frac{1}{(n+1)(n+2)}\frac{2(\epsilon - rxk - rym)(n+1)\sigma_{n+1} + r\sigma_n}{ax^2k^2 + by^2m^2 + 2kabxykm} \cdot \left(kx + my - \frac{\epsilon t^\alpha}{\Gamma(1+\alpha)}\right)^{n+2}. \tag{63}$$

6. Conclusion

In this paper, the invariance properties of the two-dimensional time fractional Fokker-Planck equation with the Riemann-Liouville fractional derivative have been investigated in the sense of Lie point symmetries. Then, the power series method has been applied to get an explicit solution for the two-dimensional time fractional Fokker-Planck equation. For obtaining new components of conserved vectors, a new theorem of conservation law has been employed along with the formal Lagrangian, which allows us to construct conservation laws for the two-dimensional time fractional Fokker-Planck equation. Our results show that the extended Lie group analysis approach and the power series method provide powerful mathematical tools to investigate other FDEs in different fields of applied mathematics. In addition, it shows that the proposed analysis is very efficient to construct conservation laws of the two-dimensional time fractional Fokker-Planck equation. Moreover, we can employ symmetry analysis to the time-space fractional Fokker-Planck equation; it will be valuable as future subject works.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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