Hindawi
Journal of Function Spaces
Volume 2021, Article ID 2500421, 8 pages
https://doi.org/10.1155/2021/2500421

Research Article

On the JK Iterative Process in Banach Spaces

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Received 10 May 2021; Accepted 9 July 2021; Published 27 July 2021

Academic Editor: Fanglei Wang

In the recent progress, different iterative procedures have been constructed in order to find the fixed point for a given self-map in an effective way. Among the other things, an effective iterative procedure called the JK iterative scheme was recently constructed and its strong and weak convergence was established for the class of Suzuki mappings in the setting of Banach spaces. The first purpose of this research is to obtain the strong and weak convergence of this scheme in the wider setting of generalized α-nonexpansive mappings. Secondly, by constructing an example of generalized α-nonexpansive maps which is not a Suzuki map, we show that the JK iterative scheme converges faster as compared the other iterative schemes. The presented results of this paper properly extend and improve the corresponding results of the literature.

1. Introduction

A mapping $\mathcal{S}$ on a subset U of a Banach space is called contraction provided that for all $z, z' \in U$ follows that

$$\left\| \mathcal{S}z - \mathcal{S}z' \right\| \le \delta \left\| z - z' \right\|, \tag{1}$$

where $\delta \in [0, 1)$ is fixed. A point v_0 is called a fixed point for $\mathcal{S}$ if $v_0 = \mathcal{S}v_0$. Normally, we denote the set of all fixed points of $\mathcal{S}$ by $F_{\mathcal{S}}$, that is, $F_{\mathcal{S}} = \{v_0 \in U : \mathcal{S}v_0 = v_0\}$. The Banach–Caccioppoli fixed point theorem (BCFPT) [1, 2] provides the existence of a unique fixed point for every self-contraction of a complete metric space.

We say that a self-map $\mathcal{S} : U \longrightarrow U$ is nonexpansive on the set U provided that

$$\left\| \mathcal{S}z - \mathcal{S}z' \right\| \le \left\| z - z' \right\|, \quad \text{for all } z, z' \in U. \tag{2}$$

We may observe that every contraction of a subset U of a Banach space is nonexpansive but the converse may not hold in general. Unlike contractions, every self-nonexpansive mapping of a complete metric space does not admit a fixed point. After many years of BCFPT, Browder [3], Gohde [4] and Kirk [5] independently obtained that a self-nonexpansive mapping of a closed bounded convex subset of a uniformly convex Banach space (UCBS, for short) always has a fixed point.

In 2008, Suzuki [6] provided a new type of generalization of nonexpansive mappings and proved some related fixed point results for this class of mappings in Banach spaces. Notice that a self-map $\mathcal{S} : U \longrightarrow U$ is mapping with the (C) property (also called Suzuki mapping) if any $z, z' \in U$ follows that

$$\frac{1}{2}\|z - \mathcal{S}z\| \le \left\| z - z' \right\| \Rightarrow \left\| \mathcal{S}z - \mathcal{S}z' \right\| \le \left\| z - z' \right\|. \tag{3}$$

In 2011, Aoyama and Kohsaka [7] provided the idea of α-nonexpansive mappings. A self-map $\mathcal{S} : U \longrightarrow U$ is called α-nonexpansive if any $z, z' \in U$ follows that

$$\left\|\mathcal{S}z - \mathcal{S}z'\right\|^2 \leq \alpha\left\|z - \mathcal{S}z'\right\|^2 + \alpha\left\|z' - \mathcal{S}z\right\|^2 + (1-2\alpha)\left\|z - z'\right\|^2, \tag{4}$$

where $\alpha \in [0, 1)$.

In 2017, Pant and Shukla [8] defined a very general class of nonexpansive mappings which properly contains the class of Suzuki mappings and partially extends the class of α-nonexpansive mappings. A self-map $\mathcal{S} : U \longrightarrow U$ is called generalized α-nonexpansive if any $z, z' \in U$ follows that

$$\frac{1}{2}\|z - \mathcal{S}z\| \leq \left\|z - z'\right\| \Rightarrow \left\|\mathcal{S}z - \mathcal{S}z'\right\| \leq \alpha\left\|z - \mathcal{S}z'\right\| + \alpha\left\|z' - \mathcal{S}z\right\| + (1-2\alpha)\left\|z - z'\right\|, \tag{5}$$

where $\alpha \in [0, 1)$.

Fixed point approximation for nonexpansive mappings under a suitable iterative method is a very active field of research and provides many interesting and important applications in applied sciences (cf. [9–12] and others). Finding the fixed points for nonexpansive and generalized nonexpansive under Picard iteration is not possible in general. A simple situation of such a case which is the rotation of the unit disk about the origin in a plane is a best example of a nonexpansive mapping which has a unique fixed point but Picard iteration does not converge to this point. In order to find fixed points of nonexpansive and hence generalized nonexpansive mappings and secondly to obtain relatively high accuracy, some authors introduced different types of iterative procedures (cf. Mann [13], Ishikawa [14], Noor [15], Agarwal et al. [16], Abbas and Nazir [17], Thakur et al. [18] and references therein). Suppose that U is a closed nonempty convex subset of a given Banach space, and assume further that ξ_k, $\eta_k, \mu_k \in (0, 1)$, $k \in \mathbb{N}$, and $\mathcal{S}$ is a self-map of U.

The Mann [13] iteration process is stated as follows:

$$\begin{aligned} p_1 &= p \in U, \\ p_{k+1} &= (1-\xi_k)p_k + \xi_k \mathcal{S}p_k. \end{aligned} \tag{6}$$

The Ishikawa [14] iterative process may be viewed as a two-step Mann iteration, which is given by

$$\begin{aligned} p_1 &= p \in U, \\ q_k &= (1-\eta_k)p_k + \eta_k \mathcal{S}p_k, \\ p_{k+1} &= (1-\xi_k)p_k + \xi_k \mathcal{S}q_k. \end{aligned} \tag{7}$$

In 2000, Noor [15] suggested a three-step iterative process which is more general than the Mann and Ishikawa iteration processes as follows:

$$\begin{aligned} p_1 &= p \in U, \\ r_k &= (1-\mu_k)p_k + \mu_k \mathcal{S}p_k, \\ q_k &= (1-\eta_k)p_k + \eta_k \mathcal{S}r_k, \\ p_{k+1} &= (1-\xi_k)p_k + \xi_k \mathcal{S}q_k. \end{aligned} \tag{8}$$

In 2007, Agarwal et al. [16] suggested a new iteration process, which converges faster than the Mann iteration for contraction mappings in Banach spaces:

$$\begin{aligned} p_1 &= p \in U, \\ q_k &= (1-\eta_k)p_k + \eta_k \mathcal{S}p_k, \\ p_{k+1} &= (1-\xi_k)\mathcal{S}p_k + \xi_k \mathcal{S}q_k. \end{aligned} \tag{9}$$

In 2014, Abbas and Nazir [17] proposed a new three-step iteration which converges faster than all of the Picard, Mann, Ishikawa, and Agarwal iterative processes for nonexpansive mappings, as follows:

$$\begin{aligned} p_1 &= p \in U, \\ r_k &= (1-\mu_k)p_k + \mu_k \mathcal{S}p_k, \\ q_k &= (1-\eta_k)\mathcal{S}p_k + \eta_k \mathcal{S}r_k, \\ p_{k+1} &= (1-\xi_k)\mathcal{S}q_k + \xi_k \mathcal{S}r_k. \end{aligned} \tag{10}$$

In 2016, Thakur et al. [18] suggested the following iteration process, which converges faster than all of the above iterative processes for Suzuki mappings:

$$\begin{aligned} p_1 &= p \in U, \\ r_k &= (1-\eta_k)p_k + \eta_k \mathcal{S}p_k, \\ q_k &= \mathcal{S}((1-\xi_k)p_k + \xi_k r_k), \\ p_{k+1} &= \mathcal{S}q_k. \end{aligned} \tag{11}$$

Very recently, Ahmad et al. [19] introduced a new iterative process named JK iteration, as follows:

$$\begin{aligned} p_1 &= p \in U, \\ r_k &= (1-\eta_k)p_k + \eta_k \mathcal{S}p_k, \\ q_k &= \mathcal{S}r_k, \\ p_{k+1} &= \mathcal{S}((1-\xi_k)\mathcal{S}r_k + \xi_k \mathcal{S}q_k). \end{aligned} \tag{12}$$

They observed that JK iteration (12) can be used for fixed points of Suzuki mappings. Moreover, they proved by providing a novel example of Suzuki mappings that the JK iteration process converges faster than all of the above iterative processes including the leading Thakur iteration (11). In this paper, firstly, we improve and extend the main results of Ahmad et al. [19] from the context of Suzuki mappings to the more general framework of generalized α-nonexpansive mappings. We then provide a novel example of generalized α-nonexpansive mappings and show that its JK iterative process is better than the mentioned iterative processes. Our results can be used for finding the solutions of split feasibility problems, solutions of differential and integral equations provided that the operator is generalized α-nonexpansive.

2. Preliminaries

We now provide some definitions.

Definition 1 [20]. A Banach space W is said to be endowed with Opial's property if every weakly convergence sequence $\{p_k\} \subseteq W$ having a weak limit $v_0 \in W$ follows that

$$\limsup_{k \longrightarrow \infty} \|p_k - v_0\| < \limsup_{k \longrightarrow \infty} \|p_k - u_0\|, \quad \text{for every choice of } u_0 \neq v_0. \tag{13}$$

Definition 2 [21]. A self-mapping $\mathcal{S}$ of a subset U of a Banach space is said to be endowed with condition I if one has a nondecreasing function $\eta : [0, \infty) \longrightarrow [0, \infty)$ having $\eta(0) = 0$ and $\eta(v) > 0$ for every $v \in (0, \infty)$ and $\|z - \mathcal{S}z\| \geq \eta(d(z, F_{\mathcal{S}}))$ for every $z \in U$; here, $d(z, F_{\mathcal{S}})$ stands for the distance of z from $F_{\mathcal{S}}$.

Definition 3. Suppose that W is any given Banach space and $\{p_k\} \subseteq W$ is bounded. Assume that $\varnothing \neq U \subseteq W$ is closed and convex. Then, the asymptotic radius of the sequence $\{p_k\}$ relative to the set U is given by $r(U, \{p_k\}) = \inf \{\limsup_{k \longrightarrow \infty} \|p_k - z\| : z \in U\}$. Moreover, the asymptotic center of $\{p_k\}$ with respect to U is given by $A(U, p_k\}) = \{z \in U : \limsup_{k \longrightarrow \infty} \|p_k - z\| = r(U, p_k)\}$.

Remark 4. The most well-known fact about the set $A(U, \{p_k\})$ is that it is always singleton whenever W is UCBS [22]. The fact that the set $A(U, \{p_k\})$ is convex and nonempty also known in the case when U is weakly compact and convex [23, 24].

Now, we combine some elementary properties of generalized α-nonexpansive mappings, which can be found in [8].

Proposition 5. *Suppose that U is any nonempty subset of a Banach space W and $\mathcal{S} : U \longrightarrow U$.*

(a) If $\mathcal{S}$ is Suzuki mapping, then, $\mathcal{S}$ is generalized α-nonexpansive

(b) If $\mathcal{S}$ is generalized α-nonexpansive having nonempty fixed point set, then, for any $v_0 \in F_{\mathcal{S}}$, $\|\mathcal{S}z - \mathcal{S}v_0\| \leq \|z - v_0\|$ for all $z \in U$

(c) If $\mathcal{S}$ is generalized α-nonexpansive, then, the set $F_{\mathcal{S}}$ is closed in U. Also, $F_{\mathcal{S}}$ is convex in the case when W is strictly convex and U is convex

(d) If $\mathcal{S}$ is a generalized α-nonexpansive mapping, then, for every choice of $z, z' \in U$, the following holds:

$$\|z - \mathcal{S}z'\| \leq \frac{(3 + \alpha)}{(1 - \alpha)} \|z - \mathcal{S}z\| + \|z - z'\| \tag{14}$$

(e) Suppose that $\mathcal{S}$ is generalized α-nonexpansive and W is endowed with the Opial property. If $\{p_k\}$ is weakly convergent to l_0 and $\lim_{k \longrightarrow \infty} \|p_k - \mathcal{S}p_k\| = 0$, it follows that $l_0 \in F_{\mathcal{S}}$

The following useful lemma can be found in [25].

Lemma 6. *Suppose that $0 < i \leq y_k \leq j < 1$ for each $k \in \mathbb{N}$ and $\lambda \geq 0$. If $\{p_k\}$ and $\{q_k\}$are any sequences in a UCBS W endowed with $\limsup_{k \longrightarrow \infty} \|p_k\| \leq \lambda$, $\limsup_{k \longrightarrow \infty} \|q_k\| \leq \lambda$, and $\lim_{k \longrightarrow \infty} \|y_k p_k + (1 - y_k) q_k\| = \lambda$, then $\lim_{k \longrightarrow \infty} \|p_k - q_k\| = 0$.*

3. Main Results

The aim of this section is at giving some important weak and strong convergence of JK (12) for the class of generalized α-nonexpansive mappings. We start the section with a key lemma.

Lemma 7. *Suppose that W is UCBS, $\varnothing \neq U \subseteq W$ is closed convex, and $\mathcal{S} : U \longrightarrow U$ is a generalized α-nonexpansive having $F_{\mathcal{S}} \neq \varnothing$. If $\{p_k\}$ is a JK iteration sequence as provided in (12). Then, $\lim_{k \longrightarrow \infty} \|p_k - v_0\|$ exists for each $v_0 \in F_{\mathcal{S}}$.*

Proof. If we choose $v_0 \in F_{\mathcal{S}}$, then, using (12) along with Proposition 5 (b), we have

$$\begin{aligned} \|r_k - v_0\| &= \|(1 - \eta_k) p_k + \eta_k \mathcal{S}p_k - v_0\| \\ &\leq (1 - \eta_k)\|p_k - v_0\| + \eta_k \|\mathcal{S}p_k - v_0\| \\ &\leq (1 - \eta_k)\|p_k - v_0\| + \eta_k \|p_k - v_0\| \\ &\leq \|p_k - v_0\|. \end{aligned} \tag{15}$$

Hence,

$$\begin{aligned} \|p_{k+1} - v_0\| &= \|\mathcal{S}((1 - \xi_k)\mathcal{S}r_k + \xi_k \mathcal{S}q_k) - v_0\| \\ &\leq \|(1 - \xi_k)\mathcal{S}r_k + \xi_k \mathcal{S}q_k - v_0\| \\ &\leq (1 - \xi_k)\|\mathcal{S}r_k - v_0\| + \xi_k \|\mathcal{S}q_k - v_0\| \\ &\leq (1 - \xi_k)\|r_k - v_0\| + \xi_k \|q_k - v_0\| \\ &= (1 - \xi_k)\|r_k - v_0\| + \xi_k \|\mathcal{S}r_k - v_0\| \\ &\leq (1 - \xi_k)\|r_k - v_0\| + \xi_k \|r_k - v_0\| \\ &= \|r_k - v_0\| \leq \|p_k - v_0\|. \end{aligned} \tag{16}$$

Consequently, we conclude that $\{\|p_k - v_0\|\}$ is nonincreasing and bounded; accordingly, we must have that $\lim_{k \longrightarrow \infty} \|p_k - v_0\|$ exists for every element v_0 of $F_{\mathcal{S}}$. ◻

Theorem 8. *Suppose that W is UCBS, $\varnothing \neq U \subseteq W$ is closed convex, and $\mathcal{S} : U \longrightarrow U$ is a generalized α-nonexpansive.*

If $\{p_k\}$ is a JK iteration sequence as provided in (12). Then, $F_{\mathcal{S}} \neq \emptyset$ if and only if $\{p_k\}$ is bounded and $\lim_{k\longrightarrow\infty}\|p_k - \mathcal{S}p_k\| = 0$.

Proof. Firstly, we may take $F_{\mathcal{S}} \neq \emptyset$. According to Lemma 7, one concludes that $\{p_k\}$ is bounded and $\lim_{k\longrightarrow\infty}\|p_k - v_0\|$ exists for every element v_0 of $F_{\mathcal{S}}$. We now suppose

$$\lim_{k\longrightarrow\infty}\|p_k - v_0\| = \lambda. \tag{17}$$

We need to obtain that $\lim_{k\longrightarrow\infty}\|p_k - \mathcal{S}p_k\| = 0$. Then, using Lemma 7, we have

$$\begin{aligned}\|r_k - v_0\| \leq \|p_k - v_0\| \Rightarrow \limsup_{k\longrightarrow\infty}\|r_k - v_0\| \\ \leq \limsup_{k\longrightarrow\infty}\|p_k - v_0\| = \lambda.\end{aligned} \tag{18}$$

Since $v_0 \in F_{\mathcal{S}}$, so by Proposition 5 (b), we infer

$$\begin{aligned}\|\mathcal{S}p_k - v_0\| \leq \|p_k - v_0\| \Rightarrow \limsup_{k\longrightarrow\infty}\|\mathcal{S}p_k - v_0\| \\ \leq \limsup_{k\longrightarrow\infty}\|p_k - v_0\| = \lambda.\end{aligned} \tag{19}$$

Now from (16), we get

$$\|p_{k+1} - v_0\| \leq \|r_k - v_0\|. \tag{20}$$

Using this together with (17), we obtain

$$\lambda \leq \liminf_{k\longrightarrow\infty}\|r_k - v_0\|. \tag{21}$$

From (18) and (21), we deduce

$$\lambda = \lim_{k\longrightarrow\infty}\|r_k - v_0\|. \tag{22}$$

Using (22), we get

$$\begin{aligned}\lambda &= \lim_{k\longrightarrow\infty}\|r_k - v_0\| = \lim_{k\longrightarrow\infty}\|(1-\eta_k)p_k + \eta_k\mathcal{S}p_k - v_0\| \\ &= \lim_{k\longrightarrow\infty}\|(1-\eta_k)(p_k - v_0) + \eta_k(\mathcal{S}p_k - v_0)\|.\end{aligned} \tag{23}$$

Thus,

$$\lambda = \lim_{k\longrightarrow\infty}\|(1-\eta_k)(p_k - v_0) + \eta_k(\mathcal{S}p_k - v_0)\|. \tag{24}$$

Using (17), (19), and (24) and keeping Lemma 6 in mind, one concludes that

$$\lim_{k\longrightarrow\infty}\|p_k - \mathcal{S}p_k\| = 0. \tag{25}$$

Conversely, we may suppose that $\{p_k\}$ is bounded in U such that $\lim_{k\longrightarrow\infty}\|p_k - \mathcal{S}p_k\| = 0$. The aim is to prove that $F_{\mathcal{S}} \neq \emptyset$. If we take any $v_0 \in A(U, \{p_k\})$, then, using Proposition 5 (d), it follows that

$$\begin{aligned}A(\mathcal{S}v_0, \{p_k\}) &= \limsup_{k\longrightarrow\infty}\|p_k - \mathcal{S}v_0\| \leq \frac{(3+\alpha)}{(1-\alpha)}\limsup_{k\longrightarrow\infty}\|p_k - \mathcal{S}p_k\| \\ &\quad + \limsup_{k\longrightarrow\infty}\|p_k - v_0\| = \limsup_{k\longrightarrow\infty}\|p_k - v_0\| \\ &= A(v_0, \{p_k\}).\end{aligned} \tag{26}$$

It follows that $\mathcal{S}v_0 \in A(U, \{p_k\}$. Since W is UCBS, $A(U, \{p_k\}$ contains only one element, that is, we must have $\mathcal{S}v_0 = v_0$. Hence, $v_0 \in F_{\mathcal{S}}$, that is, the fixed point $F_{\mathcal{S}}$ is nonempty. □

Now, we are in the position to prove our weak convergence result.

Theorem 9. *Suppose that W is UCBS, $\emptyset \neq U \subseteq W$ is closed convex, and $\mathcal{S}: U \longrightarrow U$ is a generalized α-nonexpansive having $F_{\mathcal{S}} \neq \emptyset$. If $\{p_k\}$ is a JK iteration sequence as provided in (12) and W has Opial's property, then, $\{p_k\}$ converges weakly to a point of $F_{\mathcal{S}}$.*

Proof. By Theorem 8, $\{p_k\}$ is bounded. The uniform convexity of W follows reflexivity of W, that is, $\{p_k\}$ has a weakly convergent subsequence $\{p_{k_t}\}$ with a weak limit, namely, l_0. According to Theorem 8, $\lim_{m\longrightarrow\infty}\|p_{k_m} - \mathcal{S}p_{k_m}\| = 0$. Hence, using Proposition 5 (e), we get $l_0 \in F_{\mathcal{S}}$. We claim that l_0 is the weak limit of $\{p_k\}$. We may suppose on the contrary that l_0 is not the weak limit of $\{p_k\}$, that is, $\{p_k\}$ has another weakly convergent subsequence $\{p_{k_s}\}$ with a weak limit, namely, $l_0' \neq l_0$. According to Theorem 8, $\lim_{s\longrightarrow\infty}\|p_{k_s} - \mathcal{S}p_{k_s}\| = 0$. Hence, using Proposition 5 (e), we get $l_0' \in F_{\mathcal{S}}$. Now using Lemma 7 and Opial's property, we have

$$\begin{aligned}\limsup_{k\longrightarrow\infty}\|p_k - l_0\| &= \limsup_{m\longrightarrow\infty}\left\|p_{k_m} - l_0\right\| < \limsup_{m\longrightarrow\infty}\left\|p_{k_m} - l_0'\right\| \\ &= \limsup_{k\longrightarrow\infty}\|p_k - l_0'\| = \limsup_{s\longrightarrow\infty}\left\|p_{k_s} - l_0'\right\| \\ &< \limsup_{s\longrightarrow\infty}\left\|p_{k_s} - l_0\right\| = \limsup_{k\longrightarrow\infty}\|p_k - l_0\|.\end{aligned} \tag{27}$$

Consequently, we obtained $\limsup_{k\longrightarrow\infty}\|p_k - l_0\| < \limsup_{k\longrightarrow\infty}\|p_k - l_0\|$, which suggests a contradiction. Therefore, we conclude that l_0 is the weak limit of the sequence $\{p_k\}$. □

Now, we prove the following strong convergence result.

Theorem 10. *Suppose that W is UCBS, $\emptyset \neq U \subseteq W$ is compact convex, and $\mathcal{S}: U \longrightarrow U$ is a generalized α-nonexpansive* *having $F_{\mathcal{S}} \neq \varnothing$. If $\{p_k\}$ is a JK iteration sequence as provided in (12), then, $\{p_k\}$ converges strongly to a point of $F_{\mathcal{S}}$.*

Proof. Since $\{p_k\} \subseteq U$ and U is compact, so we can find a subsequence, namely, $\{p_{k_m}\}$ of $\{p_k\}$ such that $\lim_{m\longrightarrow\infty}\|p_{k_m} - u_0\| = 0$ for some element $u_0 \in U$. Moreover, since $F_{\mathcal{S}} \neq \varnothing$, so according to the Theorem 8, $\lim_{m\longrightarrow\infty}\|p_{k_m} - \mathcal{S}p_{k_m}\| = 0$. Applying Proposition 5 (d), we get

$$\left\|p_{k_m} - \mathcal{S}u_0\right\| \leq \frac{(3+\alpha)}{(1-\alpha)}\left\|p_{k_m} - \mathcal{S}p_{k_m}\right\| + \left\|p_{k_m} - u_0\right\|. \quad (28)$$

Consequently, $p_{k_m} \longrightarrow \mathcal{S}u_0$ provided that $m \longrightarrow \infty$. But W is a Banach space, and so, the limit of a convergent sequence is always unique. Thus, $\mathcal{S}u_0 = u_0$. Lemma 7 provides us that $\lim_{k\longrightarrow\infty}\|p_k - u_0\|$ exists. Hence, u_0 is the strong limit of $\{p_k\}$. □

We now state and then prove another strong convergence theorem as follows.

Theorem 11. *Suppose that W is UCBS, $\varnothing \neq U \subseteq W$ is closed convex, and $\mathcal{S} : U \longrightarrow U$ is a generalized α-nonexpansive having $F_{\mathcal{S}} \neq \varnothing$. If $\{p_k\}$ is a JK iteration sequence as provided in (12), then, $\{p_k\}$ converges strongly to a point $F_{\mathcal{S}}$ whenever $\liminf_{k\longrightarrow\infty} d(p_k, F_{\mathcal{S}}) = 0$*

Proof. According to Lemma 7, $\lim_{k\longrightarrow\infty}\|p_k - v_0\|$ exists, for every choice of fixed point v_0 of $\mathcal{S}$. It follows that $\lim_{k\longrightarrow\infty} d(p_k, F_{\mathcal{S}})$ exists. Accordingly, we have

$$\lim_{k\longrightarrow\infty} d(p_k, F_{\mathcal{S}}) = 0. \quad (29)$$

The above strong limit suggests the existence of two subsequences $\{p_{k_s}\}$, $\{v_s\}$ in $\{p_k\}$ and $F_{\mathcal{S}}$, respectively, with the property $\|p_{k_s} - v_s\| \leq (1/2^s)$ for every natural constant s. According to the proof of Lemma 7, the iterative sequence $\{p_k\}$ is nonincreasing. Accordingly, we have

$$\left\|p_{k_{s+1}} - v_s\right\| \leq \left\|p_{k_s} - v_s\right\| \leq \frac{1}{2^s}. \quad (30)$$

Using the above and triangle inequality, one has

$$\begin{aligned}\|v_{s+1} - v_s\| &\leq \left\|v_{s+1} - p_{k_{s+1}}\right\| + \left\|p_{k_{s+1}} - v_s\right\| \leq \frac{1}{2^{s+1}} + \frac{1}{2^s} \\ &\leq \frac{1}{2^{s-1}} \longrightarrow 0, \quad \text{provided that } s \longrightarrow \infty.\end{aligned} \quad (31)$$

Accordingly, we obtained $\lim_{s\longrightarrow\infty}\|v_{s+1} - v_s\| = 0$, that is, $\{v_s\}$ form the Cauchy sequence in the closed set $F_{\mathcal{S}} \subseteq U$. It follows that $\lim_{s\longrightarrow\infty} v_s = u_0$ for some $u_0 \in F_{\mathcal{S}}$. Cosequently, $u_0 \in F_{\mathcal{S}}$. By Lemma 7, $\lim_{k\longrightarrow\infty}\|p_k - v_0\|$ exists, that is, u_0 is also the strong limit of $\{p_k\}$. □

Table 1: Strong convergence comparison of JK (12), Thakur (11), Abbas (10), Agarwal (9), Noor (8), Ishikawa (7), and Mann (6) iterates for the self-map $\mathcal{S}$ in Example 13.

k	JK	Thakur	Abbas	Agarwal	Noor	Ishikawa	Mann
1	6.5000	6.5000	6.5000	6.5000	6.5000	6.5000	6.5000
2	5.1645	5.2897	5.3983	5.5794	5.7361	5.8044	5.9750
3	5.0180	5.0559	5.1057	5.2238	5.3613	5.4313	5.6338
4	5.0020	5.0108	5.0281	5.0864	5.1773	5.2313	5.4119
5	5.0002	5.0021	5.0075	5.0334	5.0870	5.1240	5.2678
6	*5.0000*	5.0004	5.0020	5.0129	5.0427	5.0665	5.1740
7	5.0000	5.0001	5.0005	5.0050	5.0210	5.0357	5.1131
8	5.0000	*5.0000*	5.0001	5.0019	5.0103	5.0191	5.0735
9	5.0000	5.0000	*5.0000*	5.0007	5.0050	5.0103	5.0478
10	5.0000	5.0000	5.0000	5.0003	5.0025	5.0055	5.0311
11	5.0000	5.0000	5.0000	5.0001	5.0012	5.0029	5.0202
12	5.0000	5.0000	5.0000	*5.0000*	5.0006	5.0016	5.0131
13	5.0000	5.0000	5.0000	5.0000	5.0003	5.0008	5.0085
14	5.0000	5.0000	5.0000	5.0000	5.0001	5.0005	5.0055
15	5.0000	5.0000	5.0000	5.0000	5.0001	5.0002	5.0036
16	5.0000	5.0000	5.0000	5.0000	*5.0000*	5.0001	5.0023
17	5.0000	5.0000	5.0000	5.0000	5.0000	5.0001	5.0015
18	5.0000	5.0000	5.0000	5.0000	5.0000	*5.0000*	5.0010
19	5.0000	5.0000	5.0000	5.0000	5.0000	5.0000	5.0006
20	5.0000	5.0000	5.0000	5.0000	5.0000	5.0000	5.0004
21	5.0000	5.0000	5.0000	5.0000	5.0000	5.0000	5.0003
22	5.0000	5.0000	5.0000	5.0000	5.0000	5.0000	5.0002
23	5.0000	5.0000	5.0000	5.0000	5.0000	5.0000	5.0001
24	5.0000	5.0000	5.0000	5.0000	5.0000	5.0000	5.0001
25	5.0000	5.0000	5.0000	5.0000	5.0000	5.0000	*5.0000*

We finish this section with a strong convergence theorem under the condition I.

Theorem 12. *Suppose that W is UCBS, $\varnothing \neq U \subseteq W$ is closed convex, and $\mathcal{S} : U \longrightarrow U$ is a generalized α-nonexpansive having $F_{\mathcal{S}} \neq \varnothing$. If $\{p_k\}$ is a JK iteration sequence as provided in (12). Then, $\{p_k\}$ converges strongly to a point of $F_{\mathcal{S}}$ whenever $\mathcal{S}$ is endowed with condition I.*

Proof. According to Theorem 8, one can conclude that $\liminf_{k\longrightarrow\infty}\|p_k - \mathcal{S}p_k\| = 0$. Applying the condition I of $\mathcal{S}$, one obtain $\liminf_{k\longrightarrow\infty} d(p_k, F_{\mathcal{S}}) = 0$. It now follows from Theorem 11 that $\{p_k\}$ is strongly convergent in the set $F_{\mathcal{S}}$. □

4. Numerical Example

The aim of this section is to provide a new example of generalized α-nonexpansive mappings that exceeds the class of Suzuki mappings. We connect the mentioned iterative schemes with this example to show the effectiveness of our obtained results.

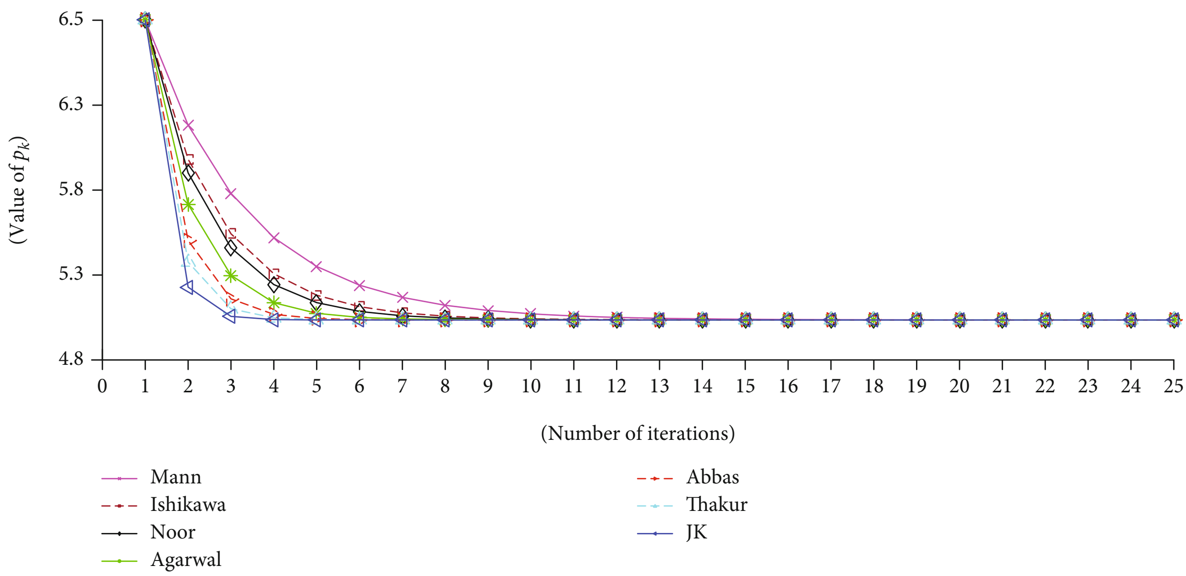

Figure 1: Convergence behavior of the sequences developed by remarkable iterative processes.

Example 13. We take a set $U = [5, 10]$ and set a self-map on U by the following rule:

$$\mathcal{S}z = \begin{cases} \dfrac{z+5}{2}, & \text{if } z < 10, \\ 5, & \text{if } z = 10. \end{cases} \tag{32}$$

We show that $\mathcal{S}$ is generalized α-nonexpansive having $\alpha = (1/2)$, but not Suzuki mapping. This example thus exceeds the class of Suzuki mappings.

Case 1. When $z, z' \in \{10\}$, we have

$$\frac{1}{2}\left|z - \mathcal{S}z'\right| + \frac{1}{2}\left|z' - \mathcal{S}z\right| + \left(1 - 2\left(\frac{1}{2}\right)\right)\left|z - z'\right| \geq 0 = \left|\mathcal{S}z - \mathcal{S}z'\right|. \tag{33}$$

Case 2. When $z, z' \in [5, 10)$, we have

$$\begin{aligned} &\frac{1}{2}\left|z - \mathcal{S}z'\right| + \frac{1}{2}\left|z' - \mathcal{S}z\right| + \left(1 - 2\left(\frac{1}{2}\right)\right)\left|z - z'\right| \\ &= \frac{1}{2}\left|z' - \left(\frac{z+5}{2}\right)\right| + \frac{1}{2}\left|z - \left(\frac{z'+5}{2}\right)\right| \\ &\geq \frac{1}{2}\left|\left(z' - \left(\frac{z+5}{2}\right)\right) - \left(z - \left(\frac{z'+5}{2}\right)\right)\right| \\ &= \frac{1}{2}\left|\frac{2z' - z - 5 - 2z + z' + 5}{2}\right| = \frac{1}{2}\left|\frac{3z' - 3z}{2}\right| \\ &= \frac{3}{4}\left|z - z'\right| \geq \frac{1}{2}\left|z - z'\right| = \left|\mathcal{S}z - \mathcal{S}z'\right|. \end{aligned} \tag{34}$$

Case 3. When $z' \in \{10\}$ and $z \in [5, 10)$, we have

$$\begin{aligned} &\frac{1}{2}\left|z - \mathcal{S}z'\right| + \frac{1}{2}\left|z' - \mathcal{S}z\right| + \left(1 - 2\left(\frac{1}{2}\right)\right)\left|z - z'\right| \\ &= \frac{1}{2}|z - 5| + \frac{1}{2}\left|z' - \left(\frac{z+5}{2}\right)\right| \geq \frac{1}{2}|z - 5| \\ &= \left|\frac{z-5}{2}\right| = \left|\mathcal{S}z - \mathcal{S}z'\right|. \end{aligned} \tag{35}$$

The above cases clearly suggest that $\mathcal{S}$ is generalized 1/2 nonexpansive mapping having $F_{\mathcal{S}} = \{5\}$. Choose $z = 8.8$ and $z' = 10$; then, $|z - z'| = 1.2$, $|\mathcal{S}z - \mathcal{S}z'| = 1.9$, and $(1/2)|z - \mathcal{S}z| = 0.95$. Thus, it is seen that, $(1/2)|z - \mathcal{S}z| < |z - z'|$ but $|\mathcal{S}z - \mathcal{S}z'| > |z - z'|$. Thus, $\mathcal{S}$ exceeds the class of Suzuki mappings.

Now, we by choosing $\xi_k = 0.70$, $\eta_k = 0.65$, and $\mu_k = 0.80$, we may observe in Table 1 and Figure 1 that JK (12) iterative process converges faster to $5 \in F_{\mathcal{S}}$ as compared the other processes.

Now we show the further effectiveness of the JK iteration (12) in the class of generalized α-nonexpansive mappings. Using $\mathcal{S}$ defined in Example 13, we suggest some different values for the parameters and p_1. To do this, we set the stopping criteria $\|p_k - 5\| < 10^{-15}$. The obtained results are provided in Table 2. The bold numbers show that JK iteration (12) requires less iteration numbers as compared to the leading three-step Thakur (11) and leading two-step Agarwal (9).

Remark 14. The main outcome of this paper extended the corresponding results of Ahmad et al. [19] from the class of Suzuki maps to the setting of generalized α-nonexpansive maps. We have observed in Tables 1 and 2 as well as in Figure 1 that the JK iterative scheme (12) is still more effective than the other iterative schemes even in the general setting of generalized α-nonexpansive maps.

Table 2: Influence of parameters: comparison of various iterative schemes.

Iterations	Initial points 5.1	6.0	6.9	7.8	8.9	9.9
For $\xi_k = (1/k+5)$, $\eta_k = (k/k+3)$						
Agarwal	29	31	31	32	32	32
Thakur	18	20	20	20	20	21
JK	*14*	*15*	*15*	*15*	*16*	*16*
For $\xi_k = 1/k$, $\eta_k = \left(1/(k+23)^{1/4}\right)$						
Agarwal	43	47	48	48	49	49
Thakur	22	24	24	24	25	25
JK	*18*	*20*	*20*	*20*	*20*	*21*
For $\xi_k = 1-(1/(4k+6))^{1/5}$, $\eta_k = 1/k^4$						
Agarwal	44	48	49	49	50	50
Thakur	22	24	25	25	25	25
JK	*18*	*20*	*20*	*20*	*20*	*21*
For $\xi_k = ((k+1)/(2k+1))^{1/2}$, $\eta_k = 1/(3k+7)^{5/30}$						
Agarwal	45	48	49	50	50	50
Thakur	24	25	25	25	25	25
JK	*17*	*18*	*19*	*19*	*19*	*19*
For $\xi_k = 1-(1/(4k+3))^{1/4}$, $\eta_k = 1/k^7$						
Agarwal	44	48	49	49	50	50
Thakur	22	24	25	25	25	25
JK	*18*	*19*	*20*	*20*	*20*	*20*

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

No competing interests are associated with this paper.

Authors' Contributions

This research work is completed by the equal contribution of each of the author listed in this paper.

References

[1] S. Banach, "Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales," *Fundamenta Mathematicae*, vol. 3, pp. 133–181, 1922.

[2] R. Caccioppoli, "Un teorema generale sull' esistenza di elementi uniti in una transformazione funzionale," *Rendiconti Accademy Lincei*, vol. 11, pp. 794–799, 1930.

[3] F. E. Browder, "Nonexpansive nonlinear operators in a Banach space," *Proc. Natl. Acad. Sci. USA*, vol. 54, no. 4, pp. 1041–1044, 1965.

[4] D. Gohde, "Zum Prinzip der Kontraktiven Abbildung," *Mathematische Nachrichten*, vol. 30, no. 3-4, pp. 251–258, 1965.

[5] W. A. Kirk, "A fixed point theorem for mappings which do not increase distances," *The American Mathematical Monthly*, vol. 72, no. 9, pp. 1004–1006, 1965.

[6] T. Suzuki, "Fixed point theorems and convergence theorems for some generalized nonexpansive mappings," *Journal of Mathematical Analysis and Applications*, vol. 340, no. 2, pp. 1088–1095, 2008.

[7] K. Aoyama and F. Kohsaka, "Fixed point theorem for α-nonexpansive mappings in Banach spaces," *Nonlinear Analysis*, vol. 74, no. 13, pp. 4387–4391, 2011.

[8] R. Pant and R. Shukla, "Approximating fixed points of generalizedα-nonexpansive mappings in Banach spaces," *Numerical Functional Analysis and Optimization*, vol. 38, no. 2, pp. 248–266, 2017.

[9] J. Ahmad, K. Ullah, and M. Arshad, "Approximation of fixed points for a class of mappings satisfying property (CSC) in Banach spaces," *Mathematical Sciences*, vol. 15, no. 2, pp. 207–213, 2021.

[10] D. R. Sahu, F. Babu, and S. Sharma, "The S-iterative techniques on Hadamard manifolds and applications," *Journal of Applied and Numerical Optimization*, vol. 2, pp. 353–371, 2020.

[11] B. Tan and S. Li, "Strong convergence of inertial Mann algorithms for solving hierarchical fixed point problems," *Journal of Nonlinear and Variational Analysis*, vol. 4, pp. 337–355, 2020.

[12] J. Zhao, D. Hou, and H. Zong, "Several iterative algorithms for solving the multiple-set split common fixed-point problem of averaged operators," *Journal of Nonlinear Functional Analysis*, vol. 2019, no. 1, 2019.

[13] W. R. Mann, "Mean value methods in iteration," *Proceedings of the American Mathematical Society*, vol. 4, no. 3, pp. 506–510, 1953.

[14] S. Ishikawa, "Fixed points by a new iteration method," *Proceedings of American Mathematical Society*, vol. 44, no. 1, pp. 147–150, 1974.

[15] M. A. Noor, "New approximation schemes for general variational inequalities," *Journal of Mathematical Analysis and Applications*, vol. 251, no. 1, pp. 217–229, 2000.

[16] R. P. Agarwal, D. O'Regon, and D. R. Sahu, "Iterative construction of fixed points of nearly asymtotically non-expansive mappings," *Journal of Nonlinear and convex Analysis*, vol. 8, no. 1, pp. 61–79, 2007.

[17] M. Abbas and T. Nazir, "A new faster iteration process applied to constrained minimization and feasibility problems," *Matematički Vesnik*, vol. 66, pp. 223–234, 2014.

[18] B. S. Thakur, D. Thakur, and M. Postolache, "A new iterative scheme for numerical reckoning fixed points of Suzuki's generalized nonexpansive mappings," *Applied Mathematics and Computation*, vol. 275, pp. 147–155, 2016.

[19] J. Ahmad, K. Ullah, M. Arshad, and Z. Ma, "A new iterative method for Suzuki mappings in Banach spaces," *Journal of Mathematics*, vol. 2021, Article ID 6622931, 7 pages, 2021.

[20] Z. Opial, "Weak convergence of the sequence of successive approximations for nonexpansive mappings," *Bulletin of the American Mathematical Society*, vol. 73, no. 4, pp. 591–598, 1967.

[21] H. F. Senter and W. G. Dotson, "Approximating fixed points of nonexpansive mappings," *Proceedings of American Mathematical Society*, vol. 44, no. 2, pp. 375–380, 1974.

[22] J. A. Clarkson, "Uniformly convex spaces," *Transactions of the American Mathematical Society*, vol. 40, no. 3, pp. 396–414, 1936.

[23] R. P. Agarwal, D. O'Regan, and D. R. Sahu, *Fixed Point Theory for Lipschitzian-Type Mappings with Applications Series; Topological Fixed Point Theory and Its Applications Vol. 6*, Springer, New York, NY, USA, 2009.

[24] W. Takahashi, *Nonlinear Functional Analysis*, Yokohoma Publishers, Yokohoma, 2000.

[25] J. Schu, "Weak and strong convergence to fixed points of asymptotically nonexpansive mappings," *Bulletin of the Australian Mathematical Society*, vol. 43, no. 1, pp. 153–159, 1991.