# Numerical Solutions of Certain New Models of the TimeFractional Gray-Scott 

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#### Abstract

A reaction-diffusion system can be represented by the Gray-Scott model. In this study, we discuss a one-dimensional timefractional Gray-Scott model with Liouville-Caputo, Caputo-Fabrizio-Caputo, and Atangana-Baleanu-Caputo fractional derivatives. We utilize the fractional homotopy analysis transformation method to obtain approximate solutions for the timefractional Gray-Scott model. This method gives a more realistic series of solutions that converge rapidly to the exact solution. We can ensure convergence by solving the series resultant. We study the convergence analysis of fractional homotopy analysis transformation method by determining the interval of convergence employing the $\hbar_{\mu, \nu}$-curves and the average residual error. We also test the accuracy and the efficiency of this method by comparing our results numerically with the exact solution. Moreover, the effect of the fractionally obtained derivatives on the reaction-diffusion is analyzed. The fractional homotopy analysis transformation method algorithm can be easily applied for singular and nonsingular fractional derivative with partial differential equations, where a few terms of series solution are good enough to give an accurate solution.


## 1. Introduction

Differential equations play a significant role within the field of finance, engineering, physics, and biology. Therefore, these applications can be modelled through differential equations [1, 2].

Reaction-diffusion system (RDS) is known as a set of partial differential equations, which correspond to many physical phenomena. RDS can be applied in physics, biology, chemistry, epidemiology, etc. (see, for example, $[3,4]$ ).

An RDS can be represented by the Gray-Scott model (GSM). The classical (integer derivative) GSM has been studied by several numerical techniques [5, 6]. Moreover, the existence and stability of the solution to this model in one dimension are discussed in [7]. In recent years, solutions to the fractional (noninteger) GSM have been spread at the same rate with the classical (integer derivative) GSM [8, 9].

Fractional calculus (FC) deals with integrals and derivatives of noninteger order. Scholars have shown an increasing interest in FC since it can study all phenomena accurately than what has been modelled through integer differential equations [10-21].

Three are many definitions of FC, such as the RiemannLiouville and the Liouville-Caputo [22]. Recently, Caputo and Fabrizio (CF) proposed a new concept of fractional differentiation using the exponential decay as the kernel instead of the power law [23, 24]. Thereafter, Atangana and Baleanu (AB) developed a new concept of differentiation with nonsingular [25, 26], based on the general Mittag-Leffler function. These two concepts with fractional order in RiemannLiouville and Liouville-Caputo sense have a nonlocal kernel.

Despite the difficulty of finding exact solutions in FC's case, the numerical and approximate technique to obtain approximate solutions is needed. Several methods have been
applied for solving fractional differential equations, such as the fractional natural decomposition method [27, 28], q -homotopy analysis transform method [28-30], and Adams Bashforth and the Fourier spectral methods [31]. Khan et al. [32] and Kumar et al. [33, 34] coupled the homotopy analysis method (HAM) [35-37] with the Laplace transform to solve a nonlinear differential equation. This method is called the fractional homotopy analysis transform method (FHATM). The main advantage of this method is its ability to combine two powerful methods to obtain a rapidly convergent series for fractional differential equations. The FHATM provides us with a convenient way to control the convergence of the series solution.

In this paper, RDS can be represented by GSM. In order to find an approximate solution to the proposed model, the FHATM is applied. To the best of our knowledge, this paper is the first one that introduced the approximate analytic solution for the time-fractional Gray-Scott system using a nonsingular fractional derivative.

## 2. Preliminaries and Notations

2.1. The Model. We consider the reaction-diffusion system for the cubic autocatalysis. This system contains two chemical species $\mathscr{U}$ and $\mathscr{V}$, whose concentration is referred by variables $u$ and $v$, respectively. Cubic autocatalysis is given by two reactions, which occur at a different rate:

$$
\begin{gather*}
\mathscr{U}+\mathscr{V}->3 \mathscr{V}  \tag{1}\\
\mathscr{V}->\mathscr{P}
\end{gather*}
$$

where $\mathscr{P}$ is some inert product of reaction.
Following [38], when quantity depends on one spatial coordinate $(\xi)$, the GSM in one space dimension is equivalent to the following two equations:

$$
\begin{align*}
& \frac{\partial u}{\partial \rho}=\Delta u-u v^{2}+A(1-u)  \tag{2}\\
& \frac{\partial v}{\partial \rho}=\Delta v+u v^{2}-B v \tag{3}
\end{align*}
$$

The left-hand side of the above equations represents the change in concentration of $\mathscr{U}$ (upper equation) and the concentration of $\mathscr{V}$ (lower equation) over time. Moreover, $\Delta u$ and $\Delta v$ represent the Laplacian operator on 1-D. The second term in both equations (the concentration of $\mathscr{U}$ times the square of the concentration of $\mathscr{V}$ ) represents the reaction term. As shown by the minus $u v^{2}$ in $u$ (upper equation) and the positive $u v^{2}$ in $v$ (lower equation), the decrease in $u$ equals the increase in $v$. This term shows that $\mathscr{U}$ is converted to $\mathscr{V}$. As a result, this amount $u v^{2}$ is subtracted from the first equation and added to the second equation. The third term in the upper equation represents the replenishment term, while the third term in the lower equation represents the diminished term. The chemical $\mathscr{U}$ is added to a given rate $(+A$, scaled by $(1-u)$, so $u$ does not exceed 1$)$. On the other hand, the chemical $\mathscr{V}$ is removed to a given removal rate $(-B)$,

Table 1: List of parameters and functions.

| Parameters <br> and functions | Description |
| :--- | :---: |
| $\mathscr{U}$ | Chemical species |
| $\mathscr{V}$ | Chemical species |
| $u$ | The concentration of $\mathscr{U}$ |
| $v$ | The concentration of $\mathscr{V}$ |
| $A$ | The feed rate of $\mathscr{U}$ |

scaled by the concentration of $\mathscr{V}$, so $v$ does not go below zero. As a result, 1 would be the maximum value for $u$, and 0 would be the minimum value of $v$. In the context of this model, $A$ $\leq B$. The Gray-Scot model's parameters and functions (2) and (3) are given in Table 1.

In this study, we extend the classical GS model to the following time-fractional Gray-Scott model (TFGSM) of the orders $\delta$ and $\eta$. Let $u(\xi, \rho)=u$ and $v(\xi, \rho)=v$; then

$$
\begin{equation*}
{ }_{0}^{(.)} D_{\rho}^{\delta} u=\Delta u-u v^{2}+A(1-u), 0<\delta \leq 1, \tag{4}
\end{equation*}
$$

$$
\begin{equation*}
{ }_{0}^{(.)} D_{\rho}^{\eta} v=\Delta v+u v^{2}-B v, 0<\eta \leq 1 \tag{5}
\end{equation*}
$$

with initial conditions

$$
\begin{align*}
& u(\xi, 0)=u_{0}(\xi, 0)  \tag{6}\\
& v(\xi, 0)=v_{0}(\xi, 0)
\end{align*}
$$

and homogeneous Neumann boundary conditions, where $(\xi, \rho) \in[0, \rho] \times[0, L], \rho \geq 0, L \geq 0$, and the operators ${ }_{0}^{(.)} D_{\rho}^{\delta, \eta}$ can be of type Liouville-Caputo ${ }_{0}^{L C} D_{\rho}^{\delta, \eta}$, Caputo-FabrizioCaputo ${ }_{0}^{C F C} D_{\rho}^{\delta, \eta}$, and Atangana-Baleanu-Caputo ${ }_{0}^{A B C} D_{\rho}^{\delta \eta}$ time-fractional derivatives with orders $\delta$ and $\eta$.
2.2. Fractional Calculus. The Liouville-Caputo fractional derivative [22], the Caputo-Fabrizio fractional derivative [23], and the Atangana-Baleanu fractional derivative [26] in the Caputo sense are defined, respectively, as

$$
\begin{aligned}
\left({ }_{0}^{L C} D_{\tau}^{\gamma} u\right)(\xi, \tau)= & \frac{1}{\Gamma(1-\gamma)} \int_{0}^{\tau}(\tau-\rho)^{-\gamma} \frac{\partial u}{\partial \rho} d \rho, 0<\gamma<1, \\
\left({ }_{0}^{C F C} D_{\tau}^{\gamma} u\right)(\xi, \tau)= & \frac{F(\gamma)}{\Gamma(1-\gamma)} \int_{0}^{\tau} \operatorname{Exp}\left(\frac{-\gamma}{1-\gamma}(\tau-\rho)\right) \frac{\partial u}{\partial \rho} d \rho, \\
& 0<\gamma<1, \\
\left({ }_{0}^{A B C} D_{\tau}^{\gamma} u\right)(\xi, \tau)= & \frac{F(\gamma)}{\Gamma(1-\gamma)} \int_{0}^{\tau} E_{\gamma}\left(\frac{-\gamma}{1-\gamma}(\tau-\rho)^{\gamma}\right) \frac{\partial u}{\partial \rho} d \rho, \\
& 0<\gamma<1,
\end{aligned}
$$

where $\rho>0$ and $F(\gamma)>0$ is a normalization function satisfying

$$
\begin{equation*}
F(\gamma)=(1-\gamma)+\frac{\gamma}{\Gamma(\gamma)} \tag{8}
\end{equation*}
$$

where $F(0)=F(1)=1$ and $E_{\gamma}($.$) denotes the Mittag-Leffler$ function, defined by

$$
\begin{equation*}
E_{\gamma}(z)=\sum_{k=0}^{\infty} \frac{z^{k}}{\Gamma(\gamma k+1)} \tag{9}
\end{equation*}
$$

The Liouville-Caputo fractional integral [22], the Caputo-Fabrizio fractional integral [39], and the Atangana-Baleanu fractional integral [40] in the Caputo sense are defined, respectively, as follows:

$$
\begin{align*}
\left({ }^{L C} I^{\gamma} u\right)(\xi, \tau)= & \frac{1}{\Gamma(\gamma)} \int_{0}^{\tau}(\tau-\rho)^{\gamma-1} u(\xi, \rho) d \rho, \quad 0<\gamma \\
\left({ }^{C F C} I^{\gamma} u\right)(\xi, \tau)= & \frac{2(1-\gamma)}{(2-\gamma) F(\gamma)} u(\xi, \tau) \\
& +\frac{2 \gamma}{(2-\gamma) F(\gamma)} \int_{0}^{\tau} u(\xi, \rho) d \rho, \quad 0<\gamma<1, \\
\left({ }^{A B C} I^{\gamma} u\right)(\xi, \tau)= & \frac{1-\gamma}{F(\gamma)} u(\xi, \tau)+\frac{\gamma}{F(\gamma) \Gamma(\gamma)} \int_{0}^{\tau} u(\xi, \rho) \\
& \cdot(\tau-\rho)^{\gamma-1} d \rho, \quad 0<\gamma<1 \tag{10}
\end{align*}
$$

Here, when $\gamma$ equals zero, the initial function is recovered, and when $\gamma$ equals unity, the classical ordinary integral is obtained.

The Laplace transformation of the Liouville-Caputo fractional derivative [22], the Caputo-Fabrizio fractional derivative [23], and the Atangana-Baleanu fractional derivative [26] in the Caputo sense are given, respectively, as follows:

$$
\begin{align*}
& \mathscr{L}\left\{{ }_{0}^{L C} D_{\rho}^{\gamma} u\right\}(\xi, s)=s^{\gamma} \mathscr{L}\{u\}(\xi, s)-\sum_{k=0}^{m-1} u^{(k)}(\xi, 0+) s^{\gamma-k-1}, \\
& \mathscr{L}\left\{{ }_{0}^{C F C} D_{\rho}^{\gamma} u\right\}(\xi, s)=\frac{F(\gamma)}{1-\gamma} \frac{s \mathscr{L}\{u\}(\xi, s)-u(\xi, 0)}{s+(\gamma / 1-\gamma)}, \\
& \mathscr{L}\left\{{ }_{0}^{A B C} D_{\rho}^{\gamma} u\right\}(\xi, s)=\frac{F(\gamma)}{1-\gamma} \frac{s^{\gamma} \mathscr{L}\{u\}(\xi, s)-u(\xi, 0) s^{\gamma-1}}{s^{\gamma}+(\gamma / 1-\gamma)} . \tag{11}
\end{align*}
$$

2.3. Homotopy Series. The following properties can be found in [41]. Let $\varphi_{1}$ and $\varphi_{2}$ be a homotopy series of a homotopy parameter $q$ given by

$$
\begin{align*}
\varphi_{1} & =\sum_{i=0}^{+\infty} u_{i} q^{i}  \tag{12}\\
\varphi_{2} & =\sum_{i=0}^{+\infty} v_{i} q^{i}
\end{align*}
$$

Then, the $n$ th-order homotopy derivative is given as

$$
\begin{equation*}
D_{n}\left(\varphi_{1}\right)=\left.\frac{1}{n!} \frac{\partial^{n} \varphi_{1}}{\partial q^{n}}\right|_{q=0} \tag{13}
\end{equation*}
$$

which holds the following:

$$
\begin{align*}
D_{n}\left(\varphi_{1}\right) & =u_{n} \\
D_{n}\left(q^{k} \varphi_{1}\right) & =D_{n-k}\left(\varphi_{1}\right)=u_{n-k} \tag{14}
\end{align*}
$$

(a) $D_{n}\left(\varphi_{1}^{m} \varphi_{2}^{l}\right)=\sum_{i=0}^{n} D_{i}\left(\varphi_{1}^{m}\right) D_{n-i}\left(\varphi_{2}^{l}\right)=\sum_{i=0}^{n} D_{i}\left(\varphi_{2}^{l}\right) D_{n-i}($ $\left.\varphi_{1}^{m}\right)$, where $n \geq 0, m \geq 0, l \geq 0$, and $0 \leq k \leq n$ are integers
(b) If $\mathscr{L}$ is a linear operator independent of the auxiliary parameter $q$, then for homotopy series, (12) holds $D_{n}\left(L \varphi_{1}\right)=L D_{n}\left(\varphi_{1}\right)$
(c) If $\mathscr{G}$ and $\mathscr{F}$ are functions independent of the auxiliary parameter $q$, then for homotopy series, (12) holds $D_{n}\left(\mathscr{G} \varphi_{1} \pm \mathscr{F} \varphi_{2}\right)=\mathscr{G} D_{n}\left(\varphi_{1}\right) \pm \mathscr{F} D_{n}\left(\varphi_{2}\right)$.

## 3. Homotopy and Laplace Transform for FHATM

Applying the Laplace transformation on Equations (4) and (5), using the Laplace transformation formula of LC, CFC, and ABC , and then simplifying these equations, we obtain

$$
\begin{aligned}
\mathscr{L}\{u(\xi, \rho)\}(s)= & \frac{u(\xi, 0)}{s}+\frac{A}{s} Y_{1, \delta}(.)-Y_{1, \delta}(.) \mathscr{L}\left((u(\xi, \rho))_{\xi \xi}\right. \\
& \left.+A(1-u(\xi, \rho))-u(\xi, \rho) v^{2}(\xi, \rho)\right)(s)
\end{aligned}
$$

$$
\begin{align*}
\mathscr{L}\{v(\xi, \rho)\}(s)= & \frac{v(\xi, 0)}{s}-Y_{1, \eta}(.) \mathscr{L}\left((v(\xi, \rho))_{\xi \xi}-B v(\xi, \rho)\right.  \tag{15}\\
& \left.+u(\xi, \rho) v^{2}(\xi, \rho)\right)(s)
\end{align*}
$$

where $Y_{1, \delta}($.$) and Y_{1, \eta}($.$) are defined in Table 2. It is difficult$ to evaluate the Laplace transformation of unknown solutions $u$ and $v$ specifically when combined in a nonlinear form.

We define the homotopy maps as follows:

$$
\begin{align*}
\mathscr{H}_{u}(\widehat{u}(\xi, \rho ; q), \widehat{v}(\xi, \rho ; q))= & (1-q) \mathscr{L}\left[\widehat{u}(\xi, \rho ; q)-u_{0}(\xi, \rho)\right](s) \\
& -q \hbar_{u} N_{u}[\widehat{u}(\xi, \rho ; q), \widehat{v}(\xi, \rho ; q)], \tag{16}
\end{align*}
$$

Table 2: Values of $Y_{s, \delta, \eta}(),. s=1, \cdots, 4$.

|  | LC | CFC |
| :--- | :---: | :---: |
| $Y_{1, \delta}()$. | $\frac{1}{s^{\delta}}$ | $\frac{\delta+(1-\delta) s}{s F(\delta)}$ |
| $Y_{1, \eta}()$. | $\frac{\eta+(1-\eta) s}{s F(\eta)}$ | $\frac{\delta+(1-\delta) s^{\delta}}{s^{\delta} F(\delta)}$ |
| $Y_{2, \delta}()$. | $\frac{1}{s^{\eta}}$ | $\frac{1}{F(\delta)}((1-\delta)+\delta \rho)$ |
| $Y_{2, \eta}()$. | $\frac{\rho^{\delta}}{\Gamma(\delta+1)}$ | $\frac{\eta+(1-\eta) s^{\eta}}{s^{\eta}}$ |
| $Y_{3, \delta}()$. | $\frac{\rho^{\eta}}{\Gamma(\eta+1)}((1-\eta)+\eta \rho)$ | $\frac{1}{F(\delta)}\left((1-\delta)+\frac{\delta \rho^{\delta}}{\Gamma(\delta+1)}\right)$ |
| $Y_{3, \eta}()$. | $\frac{\rho^{2 \delta}}{\Gamma(2 \delta+1)}$ | $\left(\frac{1}{F(\delta)}\right)^{2}\left((1-\delta)^{2}+2(1-\delta) \delta \rho+\frac{(\delta \rho)^{2}}{2}\right)$ |
| $Y_{4}()$. | $\frac{\rho^{2 \eta}}{\Gamma(2 \eta+1)}$ | $\frac{\rho^{\delta \eta}}{\Gamma(\delta \eta+1)}$ |

$$
\begin{align*}
\mathscr{H}_{v}(\widehat{u}(\xi, \rho ; q), \widehat{v}(\xi, \rho ; q))= & (1-q) \mathscr{L}\left[\hat{v}(\xi, \rho ; q)-v_{0}(\xi, \rho)\right](s) \\
& -q \hbar_{v} N_{v}[\widehat{u}(\xi, \rho ; q), \widehat{v}(\xi, \rho ; q)], \tag{17}
\end{align*}
$$

where

$$
\begin{align*}
& N_{u}[\widehat{u}(\xi, \rho ; q), \widehat{v}(\xi, \rho ; q)] \\
& =  \tag{18}\\
& =\mathscr{L}\{\widehat{u}(\xi, \rho ; q)\}(s)-\frac{1}{s}\left(\widehat{u}(\xi, 0)+A Y_{1, \delta}(\cdot)\right) \\
& \quad+Y_{1, \delta}(\cdot) \mathscr{L}(\widehat{u}(\xi, \rho))_{\xi \xi}+A(1-\widehat{u}(\xi, \rho)) \\
& \left.\quad-\widehat{u}(\xi, \rho) \widehat{v}^{2}(\xi, \rho)\right)(s) \\
& N_{v}[\widehat{u}(\xi, \rho ; q), \widehat{v}(\xi, \rho ; q)]  \tag{19}\\
& =\mathscr{L}\{\widehat{v}(\xi, \rho ; q)\}(s)-\frac{\widehat{v}(\xi, 0)}{s}+Y_{1, \eta}(.) \mathscr{L} \\
& \quad \cdot\left((\widehat{v}(\xi, \rho))_{\xi \xi}-B \widehat{v}(\xi, \rho)+\widehat{u}(\xi, \rho) \widehat{v}^{2}(\xi, \rho)\right)(s)
\end{align*}
$$

The rest of the parameters and functions are defined in Table 3.

By requiring the left-hand side of Equations (16) and (17) to be zero, we construct the so-called zeroth-order deformation equation

$$
\begin{align*}
& (1-q) \mathscr{L}\left[\widehat{u}(\xi, \rho ; q)-u_{0}(\xi, \rho)\right](s)  \tag{20}\\
& \quad=q \hbar_{u} N_{u}[\widehat{u}(\xi, \rho ; q), \widehat{v}(\xi, \rho ; q)], \\
& (1-q) \mathscr{L}\left[\widehat{v}(\xi, \rho ; q)-v_{0}(\xi, \rho)\right](s)  \tag{21}\\
& \quad=q \hbar_{v} N_{v}[\widehat{u}(\xi, \rho ; q), \widehat{v}(\xi, \rho ; q)],
\end{align*}
$$

subject to the initial conditions

$$
\begin{align*}
& \widehat{u}(\xi, \rho ; q)=u_{0}(\xi, \rho)=u_{0}  \tag{22}\\
& \widehat{v}(\xi, \rho ; q)=v_{0}(\xi, \rho)=v_{0}
\end{align*}
$$

There are three cases of solutions depending on the parameter $q \in[0,1]$ :
(a) If $q=0$ (we are on the linear operator), where

$$
\begin{align*}
& \widehat{u}(\xi, \rho ; 0)=u_{0}(\xi, \rho),  \tag{23}\\
& \widehat{v}(\xi, \rho ; 0)=v_{0}(\xi, \rho)
\end{align*}
$$

(b) If $q=1$ (we are on the nonlinear operator), where

$$
\begin{align*}
& \widehat{u}(\xi, \rho ; 1)=u(\xi, \rho),  \tag{24}\\
& \widehat{v}(\xi, \rho ; 1)=v(\xi, \rho)
\end{align*}
$$

(c) If $q$ varies from zero to one, the solution of the Equations (4) and (5) vary from the initial guesses $u_{0}(\xi, \rho)$ and $v_{0}(\xi, \rho)$ to the exact solutions $u(\xi, \rho)$ and $v(\xi, \rho)$.

Table 3: Parameters and variables in equation homotopy maps given in Equations (16) and (17).

|  | Meaning | Condition |
| :--- | :---: | :---: |
| $q$ | The embedding parameter | $q \in[0,1]$ |
| $\hbar_{u, v}$ | The nonzero auxiliary | $\hbar_{u, v} \neq 0$ |
| $\mathscr{L}_{u, v}$ | parameter | The auxiliary linear |
| $N_{u, v}$ | The nonlinear operator | $\mathscr{L}(c)=0$, where $c$ is a |
| Tee Equations (18) and (19) |  |  |

Expanding $\widehat{u}(\xi, \rho ; q)$ and $\widehat{v}(\xi, \rho ; q)$ by the Taylor series with respect to the embedding parameter $q$, we obtain

$$
\begin{align*}
& \widehat{u}(\xi, \rho ; q)=u_{0}(\xi, \rho)+\sum_{m=1}^{\infty} u_{m}(\rho) q^{m},  \tag{25}\\
& \widehat{v}(\xi, \rho ; q)=v_{0}(\xi, \rho)+\sum_{m=1}^{\infty} v_{m}(\rho) q^{m}, \tag{26}
\end{align*}
$$

where

$$
\begin{align*}
& u_{m}(\rho)=\left.\frac{1}{m!} \frac{\partial^{m} \widehat{\mathcal{u}}(\xi, \rho ; q)}{\partial q^{m}}\right|_{q=0},  \tag{27}\\
& v_{m}(\rho)=\left.\frac{1}{m!} \frac{\partial^{m} \widehat{v}(\xi, \rho ; q)}{\partial q^{m}}\right|_{q=0} .
\end{align*}
$$

If $u_{0}(\xi, \rho), v_{0}(\xi, \rho)$, the auxiliary parameter $\hbar_{u, v}$, and the auxiliary linear operator $L$ are properly chosen, then according to [36], the series (25) and (26) converges at $q=1$, and we have

$$
\begin{align*}
\widehat{u}(\xi, \rho ; 1) & =u_{0}(\xi, \rho)+\sum_{m=1}^{\infty} u_{m}(\rho) \text { i.e } u(\xi, \rho) \\
& =u_{0}(\xi, \rho)+\sum_{m=1}^{\infty} u_{m}(\rho), \tag{28}
\end{align*}
$$

$$
\begin{align*}
\widehat{v}(\xi, \rho ; 1) & =v_{0}(\xi, \rho)+\sum_{m=1}^{\infty} v_{m}(\rho) i . e v(\xi, \rho) \\
& =v_{0}(\xi, \rho)+\sum_{m=1}^{\infty} v_{m}(\rho), \tag{29}
\end{align*}
$$

which must be one of the solutions of Equations (4) and (5).
Let us define the vectors that deduce the $m$ th-order deformation equations from the zeroth-deformation Equations (20) and (21), given as follows:

$$
\begin{array}{ll}
\vec{u}_{m}(\xi, \rho)=\left\{u_{0}(\xi, \rho), u_{1}(\xi, \rho), \cdots, u_{m}(\xi, \rho)\right\}, & m=1,2, \cdots, n \\
\vec{v}_{m}(\xi, \rho)=\left\{v_{0}(\xi, \rho), v_{1}(\xi, \rho), \cdots, v_{m}(\xi, \rho)\right\}, & m=1,2, \cdots, n \tag{30}
\end{array}
$$

Upon differentiating the zeroth-deformation in Equations (20) and (21) $m$ times with respect to the embedding parameter $q$, setting $q=0$, and finally dividing them by $m!$, we have the so-called $m$ th-order deformation equations as follows:

$$
\begin{align*}
& \mathscr{L}\left[u_{m}(\xi, \rho)-\chi_{m} u_{m-1}(\xi, \rho)\right] \\
& \quad=\hbar_{u} R_{m, u}\left(\vec{u}_{m-1}, \vec{v}_{m-1}\right), \quad m=1,2, \cdots, n  \tag{31}\\
& \mathscr{L}\left[v_{m}(\xi, \rho)-\chi_{m} v_{m-1}(\xi, \rho)\right] \\
& \quad=\hbar_{v} R_{m, v}\left(\vec{u}_{m-1}, \vec{v}_{m-1}\right), \quad m=1,2, \cdots, n \tag{32}
\end{align*}
$$

Applying the inverse Laplace transform to Equations (31) and (32), we obtain

$$
\begin{align*}
u_{m}(\xi, \rho)= & \chi_{m} u_{m-1}(\xi, \rho)+\hbar_{u} \mathscr{L}^{-1} \\
& \cdot\left(R_{m, u}\left(\vec{u}_{m-1}, \vec{v}_{m-1}\right)\right), \quad m=1,2, \cdots, n \\
v_{m}(\xi, \rho)= & \chi_{m} v_{m-1}(\xi, \rho)+\hbar_{v} \mathscr{L}^{-1} \\
& \cdot\left(R_{m, v}\left(\vec{u}_{m-1}, \vec{v}_{m-1}\right)\right), \quad m=1,2, \cdots, n \tag{33}
\end{align*}
$$

Here,

$$
\begin{aligned}
R_{m}\left(\vec{u}_{m-1}, \xi, \rho\right)= & \mathscr{L}\left[u_{m-1}(\xi, \rho)\right]-\left(1-\chi_{m}\right) \frac{1}{s}\left(u_{0}+A Y_{1, \delta}(.)\right) \\
& +Y_{1, \delta}(\cdot) \mathscr{L}\left(\left(u_{m-1}\right)_{\xi \xi}+A u_{m-1}\right. \\
& \left.-\sum_{i=0}^{m-1} u_{m-1-i}(\xi, \rho) \sum_{j=0}^{i} v_{j}(\xi, \rho) v_{i-j}(\xi, \rho)\right)
\end{aligned}
$$

- (s) red,

$$
\begin{aligned}
R_{m}\left(\vec{v}_{m-1}, \xi, \rho\right)= & \mathscr{L}\left[v_{m-1}(\xi, \rho)\right](s)-\left(1-\chi_{m}\right)\left(\frac{v_{0}}{s}\right) \\
& +Y_{1, \eta}(\cdot) \mathscr{L}\left(\left(v_{m-1}\right)_{\xi \xi}-B v_{m-1}\right. \\
& \left.+\sum_{i=0}^{m-1} u_{m-1-i}(\xi, \rho) \sum_{j=0}^{i} v_{j}(\xi, \rho) v_{i-j}(\xi, \rho)\right)
\end{aligned}
$$

- (s)red,

$$
\chi_{m}= \begin{cases}0 & \mathrm{~m} \leq 1  \tag{34}\\ 1 & \mathrm{~m}>1\end{cases}
$$

Consequently, the solutions of the $m$ th-order deformation equation are given as

$$
\begin{align*}
u_{m}(\xi, \rho)= & \left(\chi_{m}+\hbar_{u}\right) u_{m-1}(\xi, \rho)-\hbar_{u}\left(1-\chi_{m}\right) \\
& \cdot\left(u_{0}+A \mathscr{L}^{-1}\left(Y_{1, \delta}(\cdot)\right)\right)-\hbar_{u} \mathscr{L}^{-1} \\
& \cdot\left(Y _ { 1 , \delta } ( \cdot ) \mathscr { L } \left(\left(u_{m-1}\right)_{\xi \xi}+A u_{m-1}\right.\right.  \tag{35}\\
& \left.\left.-\sum_{i=0}^{m-1} u_{m-1-i}(\xi, \rho) \sum_{j=0}^{i} v_{j}(\xi, \rho) v_{i-j}(\xi, \rho)\right)\right)
\end{align*}
$$

Consider the initial guesses $u_{0}^{(.)}(\xi, \rho)=u(\xi, 0)$ and $u_{0}^{(.)}$ $(\xi, \rho)=v(\xi, 0)$; then using Equations (35) and (36), the first two terms are given as

$$
\begin{align*}
u_{1}^{(\cdot)}(\xi, \rho)= & -\hbar_{u} Y_{2, \delta}(\cdot)\left(\left(u_{0}\right)_{\xi \xi}-u_{0} v_{0}^{2}+A\left(1-u_{0}\right)\right)  \tag{37}\\
v_{1}^{(.)}(\xi, \rho)= & -\hbar_{v} Y_{2, \eta}(\cdot)\left(\left(v_{0}\right)_{\xi \xi}+u_{0} v_{0}^{2}-B v_{0}\right)  \tag{38}\\
u_{2}^{(.)}(\xi, \rho)= & \left(1+\hbar_{u}\right) u_{1}^{(\cdot)}(\xi, \rho)-\hbar_{u}^{2} Y_{3, \delta}(\cdot) \\
& \cdot\left(\left(\left(u_{0}\right)_{\xi \xi}-u_{0} v_{0}^{2}+A\left(1-u_{0}\right)\right)\left(v_{0}^{2}+A\right)\right.  \tag{39}\\
& \left.+\left(\left(u_{0}\right)_{\xi \xi \xi \xi}-\left(u_{0} v_{0}^{2}\right)_{\xi \xi}-A\left(u_{0}\right)_{\xi \xi}\right)\right) \\
& -2 \hbar_{u}^{2} Y_{4}(\cdot)\left(u_{0} v_{0}^{2}-B v_{0}\right)\left(u_{0} v_{0}\right) \\
v_{2}^{(\cdot)}(\xi, \rho)= & \left(1+\hbar_{v}\right) v_{1}^{(\cdot)}(\xi, \rho)+\hbar_{v}^{2} Y_{3, \eta}(.) \\
& \cdot\left(\left(\left(v_{0}\right)_{\xi \xi}+u_{0} v_{0}^{2}-B v_{0}\right)\left(2 u_{0} v_{0}-B\right)\right. \\
& \left.+\left(\left(v_{0}\right)_{\xi \xi \xi \xi}+\left(u_{0} v_{0}^{2}\right)_{\xi \xi}-B\left(v_{0}\right)_{\xi \xi}\right)\right)  \tag{40}\\
& +\hbar_{v}^{2} Y_{4}(.) v_{0}^{2}\left(A\left(1-u_{0}\right)-u_{0} v_{0}^{2}\right)
\end{align*}
$$

where $Y_{s}(),. s=2,3,4$ is defined in Table 2. In a similar way, $u_{m}^{(.)}(\xi, \rho)$ and $v_{m}^{(.)}(\xi, \rho)$ for $m \geq 3$ can be obtained.

Finally, we approximate the solutions of $u(\xi, \rho)$ and $v(\xi, \rho)$ of Equations (4) and (5) by

$$
\begin{align*}
& u^{(\cdot)}(\xi, \rho)=\sum_{m=0}^{\infty} u_{m}^{(.)}(\xi, \rho),  \tag{41}\\
& v^{(\cdot)}(\xi, \rho)=\sum_{m=0}^{\infty} v_{m}^{(\cdot)}(\xi, \rho)
\end{align*}
$$

where the superscript (.) is replaced by (LC), (CFC), and (ABC).


Figure 1: The $\hbar_{u, v}$ curves obtained from the 3rd order of the FHATM solutions using the ABC, CFC, and LC when $\delta$ and $\eta$ tend to 1 and $\xi=10$.

Table 4: List of variables and parameters values.

|  | $\varrho$ | $\xi$ | $\delta, \eta$ | $\hbar_{u, v}$ | $A$ | $B$ | $L$ | $L_{1}$ | $M$ | $N$ |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Figure 1 | 0.0001 | 10 | 0.99 | $\hbar_{u, v} \in(-3.4,1.4)$ | 0.125 | 0.125 | 10 | 10 | - | - |
| Figure 2 | $\frac{k L_{1}}{N}$ | $\frac{s L}{M}$ | 0.99 | $\hbar_{u, v} \in(-0.8,-0.2)$ | 0.125 | 0.125 | 10 | 10 | 10 | 10 |
| Figure 3 | 5 | $\xi \in(-20,20)$ | Varies | $\hbar_{u, v}^{*}\left(\right.$ optimal value of $\left.\hbar_{u, v}\right)$ | 0.125 | 0.125 | - | - | - | - |
| Table 5 | $\frac{k L_{1}}{N}$ | $\frac{s L}{M}$ | 0.99 | Varies | 0.125 | 0.125 | 10 | 10 | 10 | 10 |
| Table 4 | $\rho \in(0,80)$ | $\xi \in(0,80)$ | 0.99 | -0.25 | 0.125 | 0.125 | 100 | - | - | - |

Table 5: Regions of convergence, optimal values of $\hbar_{u, v}$ and minimum values.

| Operator | $\hbar_{u}$ | $\hbar_{u}^{*}\left(\right.$ optimal value of $\left.\hbar_{u}\right)$ | Minimum value of $E_{u}\left(\hbar_{u}\right)$ |
| :--- | :---: | :---: | :---: |
| $u(\xi$, Q $)$ |  |  |  |
| ABC | $-1.9 \leq \hbar_{u} \leq-0.19$ | -0.597282 | 0.0000803049 |
| CFC | $-1.5 \leq \hbar_{u} \leq 0.15$ | -0.473649 | 0.000232817 |
| LC | $-1.9 \leq \hbar_{u} \leq-0.19$ | -0.596935 | 0.0000223021 |
| $v(\xi$, @ $)$ |  |  |  |
| ABC | $-1.9 \leq \hbar_{v} \leq-0.19$ | -0.597282 | 0.0000803049 |
| CFC | $-1.5 \leq \hbar_{v} \leq 0.15$ | -0.473649 | 0.000232817 |
| LC | $-1.9 \leq \hbar_{v} \leq-0.19$ | -0.596935 | 0.0000223021 |








Figure 2: The square residual function Equations (38) and (39) using the third-order approximation solution of the FHATM solutions using the $\mathrm{ABC}, \mathrm{CFC}$, and LC.


Figure 3: Different values of $\delta$ and $\eta$ using the ABC, CFC, and LC.

Table 6: The absolute error of $u(\xi, \rho)$ and $v(\xi, \rho)$.

| $u(\xi, \varrho)$ |  |  |  |  |  |
| :--- | :---: | :---: | :---: | :---: | :---: |
|  | $\varrho$ | $\xi$ | ABC | CFC | LC |
|  | 0 | 10 | $2.88537 \times 10^{-6}$ | $2.88537 \times 10^{-6}$ | 0 |
| Absolute error | 20 | 20 | $1.04222 \times 10^{-6}$ | $1.096 \times 10^{-6}$ | $1.04725 \times 10^{-6}$ |
|  | 40 | 40 | $5.01874 \times 10^{-12}$ | $5.86325 \times 10^{-12}$ | $5.14655 \times 10^{-12}$ |
|  | 60 | 60 | $2.9643 \times 10^{-14}$ | $3.60822 \times 10^{-14}$ | $1.29896 \times 10^{-14}$ |
| $v(\xi, \varrho)$ |  |  |  | CFC | LC |
|  | $\varrho$ | $\xi$ | ABC | $2.88537 \times 10^{-6}$ | 0 |
|  | 0 | 10 | $2.88537 \times 10^{-6}$ | $1.096 \times 10^{-6}$ | $1.04725 \times 10^{-6}$ |
|  | 20 | $1.04222 \times 10^{-6}$ | $5.86808 \times 10^{-12}$ | $5.14111 \times 10^{-12}$ |  |
|  | 20 | 60 | $5.01377 \times 10^{-12}$ | $1.77636 \times 10^{-14}$ | $2.4869 \times 10^{-14}$ |

## 4. Numerical Results

To demonstrate the efficiency of the FHATM for solving the time-fractional Gray-Scott equation, we present the solution in figures and tables for several values of fractional derivatives.

According to [5], we take the initial conditions as

$$
\begin{align*}
& u(\xi, 0)=\frac{3-\sqrt{\mu}}{4}-\frac{\sqrt{2 \rho}}{4} \tanh \left(\frac{\sqrt{\rho}}{4} \xi\right),  \tag{42}\\
& v(\xi, 0)=\frac{1+\sqrt{\mu}}{4}+\frac{\sqrt{2 \rho}}{4} \tanh \left(\frac{\sqrt{\rho}}{4} \xi\right),
\end{align*}
$$

and the exact solution of Equations (2) and (3) is given by

$$
\begin{align*}
& u(\xi, \rho)=\frac{3-\sqrt{\mu}}{4}-\frac{\sqrt{2 \rho}}{4} \tanh \left(\frac{\sqrt{\rho}}{4}(\xi-c \rho)\right)  \tag{43}\\
& v(\xi, \rho)=\frac{1+\sqrt{\mu}}{4}+\frac{\sqrt{2 \rho}}{4} \tanh \left(\frac{\sqrt{\rho}}{4}(\xi-c \rho)\right) \tag{44}
\end{align*}
$$

We evaluated the intervals of convergence for the LC, CFC, and ABC by finding $\hbar_{u, v}$ curves, and the averaged residual error. Furthermore, we tested the accuracy of the results obtained by employing FHATM by comparing it with the exact solution. Figure 1 shows the $u^{\prime}(10,0)$ and $v^{\prime}(10,0)$ against $\hbar_{u, v}$ taking the values in Table 4 . We plot $\hbar_{u, v}$ curves of the third terms of the FHATM solution for the fractional-time LC, CFC, and ABC equations (4) and (5) with the aim to observe the intervals of convergence. From this figure, we note that the straight line parallel with the $\hbar_{u, v}$ -axis provides the region of convergence according to [36]. These valid regions are listed in Table 5. We notice that $\hbar_{u, v}$ curves do not give the optimal value of the auxiliary parameter $\hbar_{u, v}$ that can make Equations (28) and (29) converge fast. So, according to [42], we compute the optimal values of
parameter $\hbar_{u, v}$ from the minimum of the average residual errors. The square residual is defined as

$$
\begin{align*}
S E_{n, u}\left(\hbar_{u}\right)= & \frac{1}{(N+1)(M+1)} \sum_{k=0}^{N} \sum_{s=0}^{M} \\
& \cdot\left[E_{n, u}\left(\sum_{i=0}^{n} u_{i}\left(\frac{10 k}{N}, \frac{10 s}{M}\right)\right)\right]^{2},  \tag{45}\\
S E_{n, v}\left(\hbar_{v}\right)= & \frac{1}{(N+1)(M+1)} \sum_{k=0}^{N} \sum_{s=0}^{M} \\
& \cdot\left[E_{n, v}\left(\sum_{i=0}^{n} v_{i}\left(\frac{10 k}{N}, \frac{10 s}{M}\right)\right)\right]^{2} \tag{46}
\end{align*}
$$

corresponding to a nonlinear algebraic equations

$$
\begin{align*}
& \frac{d\left(S E_{n, u}\left(\hbar_{u}\right)\right)}{d \hbar_{u}}=0 \\
& \frac{d\left(S E_{n, v}\left(\hbar_{v}\right)\right)}{d \hbar_{v}}=0 \tag{47}
\end{align*}
$$

Figure 2 and Table 5 show the average residual error for the LC, CFC, and ABC operators. These show $S E_{n, u}\left(\hbar_{u}\right)$ and $S E_{n, v}\left(\hbar_{v}\right)$ for 3 terms obtained using FHATM. We set into Equations (45) and (46) the parameter values given in Table 4. Using the command "Minimize" in Mathematica, we plotted the residual error against $\hbar_{u, v}$ to get the optimal values $\hbar_{u, v}^{*}$. From Table 5, it is seen that the FHATM for LC, CFC, and ABC operators converges rapidly. Note that only three iterations are considered here. Therefore, the accuracy of the results can be improved by considering more terms, where the error converges to zero.

Figure 3 and Table 6 show the comparison of 3 terms in the FHATM solution for the LC, CFC, and ABC operators with the exact solution in Equations (43) and (44). Table 6 presents the absolute error of the FHATM solution using
parameter values given in Table 4. We noted from this table that the FHATM solution for LC, CFC, and ABC operators is in excellent agreement with the exact solutions. Moreover, Figure 3 shows the comparison between the exact solution and approximate solution obtained by 3 terms of FHATM for the $\mathrm{LC}, \mathrm{CFC}$, and ABC operators for parameter values listed in Table 4. We observe from Figure 3 that the solution obtained by FHATM increases rapidly to the exact solution following the increase in $\delta$ and $\eta$. Those tables and figures demonstrate the efficacy of the presented algorithm for solving the time-fractional Gray-Scott equation.

## 5. Conclusion

In this paper, the Gray-Scott equation was extended to the time-fractional Gray-Scott equation of Liouville-Caputo (LC), Caputo-Fabrizio-Caputo (CFC), and Atangana-Baleanu-Caputo (ABC) type. The fractional homotopy analysis transform technique is used to derive analytic solutions for TFGSE. This method gives the solutions in a series form that converges rapidly in nonlinear time-fractional GS equation. The interval of the convergence by $\hbar_{u, v}$ curves in Figure 1 and the optimal value of $\hbar_{u, v}$ were found by least square error as given Figure 2. Also, the solutions obtained were compared with the exact solution, which were in excellent agreement. The effect of the fractional derivative on the concentration of $\mathscr{U}$ increases when $\delta$ decreases while the concentration of $\mathscr{V}$ is decreases. Moreover, the results obtained using FHATM agree well with the numerical result presented in [5], and the absolute error less than $3 \times 10^{-6}$ as given in Table 6. In conclusion, the FHATM method is a powerful method to handle fractional operators of LC, CFC, and $A B C$ type, generating highly accurate data.

## Data Availability

No data were used to support this study.

## Conflicts of Interest

The authors declare that they have no competing interests.

## Authors' Contributions

All authors equally contributed to this work. All authors read and approved the final manuscript.

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## References

[1] J. M. Amigo and M. Small, "Mathematical methods in medicine: neuroscience, cardiology and pathology," Philosophical Transactions of the Royal Society A: Mathematical, Physical and Engineering Sciences, vol. 375, no. 2096, article 20170016, 2017.
[2] D. S. Jones, M. Plank, and B. D. Sleeman, Differential Equations and Mathematical Biology, CRC Press, 2009.
[3] A. S. Bataineh, M. S. M. Noorani, and I. Hashim, "The homotopy analysis method for Cauchy reaction-diffusion problems," Physics Letters A, vol. 372, no. 5, pp. 613-618, 2008.
[4] S. Kumar, A. Kumar, S. Abbas, M. Al Qurashi, and D. Baleanu, "A modified analytical approach with existence and uniqueness for fractional cauchy reaction-diffusion equations," Advancesin Difference Equations, vol. 2020, pp. 1-18, 2020.
[5] B. Karaagac, "Numerical treatment of Gray-Scott model with operator splitting method," Discrete \& Continuous Dynamical Systems-S, vol. 14, no. 7, p. 2373, 2021.
[6] O. P. Yadav and R. Jiwari, "A finite element approach for analysis and computational modelling of coupled reaction diffusion models," Numerical Methods for Partial Differential Equations, vol. 35, no. 2, pp. 830-850, 2019.
[7] S. Focant and T. Gallay, "Existence and stability of propagating fronts for an autocatalytic reaction- diffusion system," Physica D: Nonlinear Phenomena, vol. 120, no. 3-4, pp. 346368, 1998.
[8] M. Abbaszadeh and M. Dehghan, "A reduced order finite difference method for solving space-fractional reaction-diffusion systems: the Gray-Scott model," The European Physical Journal Plus, vol. 134, no. 12, p. 620, 2019.
[9] T. Wang, F. Song, H. Wang, and G. E. Karniadakis, "Fractional Gray-Scott model: well-posedness, discretization, and simulations," Computer Methods in Applied Mechanics and Engineering, vol. 347, pp. 1030-1049, 2019.
[10] A. K. Alomari, M. S. M. Noorani, R. Nazar, and C. P. Li, "Homotopy analysis method for solving fractional Lorenz system," Communications in Nonlinear Science and Numerical Simulation, vol. 15, no. 7, pp. 1864-1872, 2010.
[11] B. Dumitru, D. Kai, and S. Enrico, Fractional Calculus: Models and Numerical Methods, Volume 3, World Scientific, 2012.
[12] A. Jaradat, M. S. M. Noorani, M. Alquran, and H. M. Jaradat, "A novel method for solving caputo-time-fractional dispersive long wave wu-zhang system," Nonlinear Dynamics and Systems Theory, vol. 18, no. 2, pp. 182-190, 2018.
[13] A. A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, Theory and applications of fractional differential equations, Elsevier Science Limited, 2006.
[14] K. M. Owolabi and A. Atangana, Numerical Methods for Fractional Differentiation, Springer, 2019.
[15] K. M. Saad and J. F. Gomez-Aguilar, "Coupled reactiondiffusion waves in a chemical system via fractional derivatives in Liouville-Caputo sense," Revista Mexicana de Fisica, vol. 64, no. 5, pp. 539-547, 2018.
[16] K. M. Saad, "Comparing the Caputo, Caputo-Fabrizio and Atangana-Baleanu derivative with fractional order: fractional cubic isothermal auto-catalytic chemical system," The European Physical Journal Plus, vol. 133, no. 3, p. 94, 2018.
[17] V. F. Morales-Delgado, J. F. Gomez-Aguilar, K. M. Saad, M. A. Khan, and P. Agarwal, "Analytic solution for oxygen diffusion from capillary to tissues involving external force effects: a fractional calculus approach," Physica A: Statistical Mechanics and its Applications, vol. 523, pp. 48-65, 2019.
[18] S. Ullah, M. Altaf Khan, M. Farooq, Z. Hammouch, and D. Baleanu, "A fractional model for the dynamics of tuberculosis infection using Caputo-Fabrizio derivative," Discrete \& Continuous Dynamical Systems-S, vol. 13, no. 3, pp. 975-993, 2020.
[19] S. Ullah and M. A. Khan, "Modeling the impact of nonpharmaceutical interventions on the dynamics of novel coronavirus with optimal control analysis with a case study," Chaos, Solitons \& Fractals, vol. 139, article 110075, 2020.
[20] M. Altaf Khan, S. Ullah, S. Ullah, and M. Farhan, "Fractional order seir model with generalized incidence rate," AIMS Mathematics, vol. 5, no. 4, pp. 2843-2857, 2020.
[21] M. A. Khan and A. Atangana, "Modeling the dynamics of novel coronavirus (2019-ncov) with fractional derivative," Alexandria Engineering Journal, vol. 59, no. 4, pp. 23792389, 2020.
[22] R. L. Magin, Fractional Calculus in Bioengineering, Volume 2, Begell House Redding, 2006.
[23] M. Caputo and M. Fabrizio, "A new definition of fractional derivative without singular kernel," Progress in Fractional Differentiation and Applications, vol. 1, no. 2, pp. 1-13, 2015.
[24] A. Shaikh, A. Tassaddiq, K. S. Nisar, and D. Baleanu, "Analysis of differential equations involving caputo-fabrizio fractional operator and its applications to reaction-diffusion equations," Advances in Difference Equations, vol. 2019, Article ID 178, 2019.
[25] A. I. Aliyu, A. S. Alshomrani, Y. Li, M. Inc, and D. Baleanu, "Existence theory and numerical simulation of HIV-I cure model with new fractional derivative possessing a nonsingular kernel," Advances in Difference Equations, vol. 2019, no. 1, 2019.
[26] A. Atangana and D. Baleanu, "New fractional derivatives with nonlocal and non-singular kernel: theory and application to heat transfer model," 2016, https://arxiv.org/abs/1602.03408.
[27] W. Gao, P. Veeresha, D. G. Prakasha, and H. M. Baskonus, "Novel dynamic structures of 2019-ncov with nonlocal operator via powerful computational technique," Biology, vol. 9, no. 5, p. 107, 2020.
[28] W. Gao, P. Veeresha, D. G. Prakasha, and H. M. Baskonus, "New numerical simulation for fractional Benney-Lin equation arising in falling film problems using two novel techniques," Numerical Methods for Partial Differential Equations, vol. 37, no. 1, pp. 210-243, 2021.
[29] W. Gao, P. Veeresha, D. G. Prakasha, B. Senel, and H. M. Baskonus, "Iterative method applied to the fractional nonlinear systems arising in thermoelasticity with Mittag-Leffler kernel," Fractals, vol. 28, no. 8, article 2040040, 2020.
[30] W. Gao, P. Veeresha, D. G. Prakasha, H. M. Baskonus, and G. Yel, "New numerical results for the time-fractional phifour equation using a novel analytical approach," Symmetry, vol. 12, no. 3, p. 478, 2020.
[31] K. M. Owolabi and Z. Hammouch, "Spatiotemporal patterns in the Belousov-Zhabotinskii reaction systems with AtanganaBaleanu fractional order derivative," Physica A: Statistical Mechanics and its Applications, vol. 523, pp. 1072-1090, 2019.
[32] M. Khan, M. A. Gondal, I. Hussain, and S. Karimi Vanani, "A new comparative study between homotopy analysis transform method and homotopy perturbation transform method on a semi infinite domain," Mathematical and Computer Modelling, vol. 55, no. 3-4, pp. 1143-1150, 2012.
[33] M. M. Khader, S. Kumar, and S. Abbasbandy, "New homotopy analysis transform method for solving the discontinued problems arising in nanotechnology," Chinese Physics B, vol. 22, no. 11, article 110201, 2013.
[34] S. Kumar and M. M. Rashidi, "New analytical method for gas dynamics equation arising in shock fronts," Computer Physics Communications, vol. 185, no. 7, pp. 1947-1954, 2014.
[35] A. K. Alomari, M. S. M. Noorani, and R. Nazar, "Comparison between the homotopy analysis method and homotopy perturbation method to solve coupled Schrodinger-KDV equation," Journal of Applied Mathematics and Computing, vol. 31, no. 1-2, pp. 1-12, 2009.
[36] S. Liao, Beyond Perturbation: Introduction to the Homotopy Analysis Method, Chapman and Hal-l/CRC, 2003.
[37] J. Singh, D. Kumar, and D. Baleanu, "On the analysis of fractional diabetes model with exponential law," Advances in Difference Equations, vol. 2018, Article ID 231, 2018.
[38] P. Gray and S. K. Scott, "Autocatalytic reactions in the isothermal, continuous stirred tank reactor," Chemical Engineering Science, vol. 39, no. 6, pp. 1087-1097, 1984.
[39] J. Losada and J. J. Nieto, "Properties of a new fractional derivative without singular kernel," Progress in Fractional Differentiation and Applications, vol. 1, no. 2, pp. 87-92, 2015.
[40] A. Atangana and I. Koca, "Chaos in a simple nonlinear system with Atangana-Baleanu derivatives with fractional order," Chaos, Solitons \& Fractals, vol. 89, pp. 447-454, 2016.
[41] S. Liao, "Notes on the homotopy analysis method: some definitions and theorems," Communications in Nonlinear Science and Numerical Simulation, vol. 14, no. 4, pp. 983-997, 2009.
[42] Z. Niu and C. Wang, "A one-step optimal homotopy analysis method for nonlinear differential equations," Communications in Nonlinear Science and Numerical Simulation, vol. 15, no. 8, pp. 2026-2036, 2010.

