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Research Article

Orthogonally Biadditive Operators

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In this article, we introduce and study a new class of operators defined on a Cartesian product of ideal spaces of measurable functions. We use the general approach of the theory of vector lattices. We say that an operator $T : E \times F \longrightarrow W$ defined on a Cartesian product of vector lattices E and F and taking values in a vector lattice W is orthogonally biadditive if all partial operators $T_y : E \longrightarrow W$ and $T_x : F \longrightarrow W$ are orthogonally additive. In the first part of the article, we prove that, under some mild conditions, a vector space of all regular orthogonally biadditive operators $\mathcal{OBA}_r(E, F; W)$ is a Dedekind complete vector lattice. We show that the set of all horizontally-to-order continuous regular orthogonally biadditive operators is a projection band in $\mathcal{OBA}_r(E, F; W)$. In the last section of the paper, we investigate orthogonally biadditive operators on a Cartesian product of ideal spaces of measurable functions. We show that an integral Uryson operator which depends on two functional variables is orthogonally biadditive and obtain a criterion of the regularity of an orthogonally biadditive Uryson operator.

1. Introduction and Preliminaries

Orthogonally additive operators in vector lattices first were introduced by Mazón and Segura de León in [1]. Today, the theory of these operators is an active field of the modern analysis (see [2–9]). We note that the study of orthogonally additive operators has useful applications in different areas of modern mathematics, e.g., convex geometry [10, 11], dynamical systems [12], and nonlinear integral equations [13, 14]. In applications, it is often necessary to study integral equations depending on several variables. Nonlinear operators in two variables satisfying the natural condition of the orthogonal additivity with respect to each variable are often appear in applications ([15]). Such operators in the literature are called orthogonally biadditive. We note that this notion is traced back to paper [16] by Mizel and Sundaresan. In present note, we investigate orthogonally biadditive operators in the general setting of the theory of vector lattices. We note that the tools of the theory of vector lattices turned out to be useful and effective in solving a number of problems of the theory of linear integral operators in ideal spaces [17]. The nonlinear integral operators of Uryson and Hammerstein were investigated by methods of the theory of ordered spaces in [14, 18].

Let us describe the content of the article. In the following section, we briefly present a necessary information on vector lattices and orthogonally additive operators. In the next section, we investigate the vector space $\mathcal{OBA}(E, F; W)$ of all orthogonally biadditive operators defined on a Cartesian product of vector lattices E and F and taking values in a vector lattice W. It turned out that there is a natural partial order on $\mathcal{OBA}(E, F; W)$. We get the lattice calculus of orthogonally biadditive operators and prove the first main result of the paper stated that for a Dedekind complete vector lattice W the vector space $\mathcal{OBA}_r(E, F; W)$ of all regular orthogonally biadditive operators defined on a Cartesian product of vector lattices E and F and taking values in a Dedekind complete vector lattice W is a Dedekind complete vector lattice. Then, we explore the special type of horizontally-to-order continuous regular orthogonally biadditive operators. We prove that the vector space of these operators is a projection band in $\mathcal{OBA}_r(E, F; W)$.

In the last section, we investigate orthogonally biadditive operators defined on a Cartesian product of ideal spaces of measurable functions. We show that a nonlinear superposition operator and a Uryson integral operator depending on two variables are orthogonally biadditive in appropriate function spaces. We note that in the classical theory of integral operators, all information concerning an integral operator is encoded by the properties of its kernel. In the final section of the paper, we show that the same is true for integral orthogonally biadditive operators and obtain the second main result of the article which is a criteria for the regularity of an orthogonally biadditive Uryson operator. It is worth noting that the general theory of orthogonally biadditive operators developed below is aimed at getting an additional information on the abovementioned particular operators. This article is the beginning of a project devoted to the study of analytic, algebraic, and order properties of orthogonally biadditive operators.

Now we state our main results. All unexplained notions are defined in next sections.

Theorem 1. *Let E, F be vector lattices and W be a Dedekind complete vector lattice. Then, $\mathcal{OBA}_r(E, F; W)$ is a Dedekind complete vector lattice, and for all $T, T_1, T_2 \in \mathcal{OBA}_r(E, F; W)$ and $(x, y) \in E \times F$, the following relations hold:*

(1) $(T_1 \vee T_2)(x, y) := \sup \{\sum_{i=1}^{n}\sum_{j=1}^{m} T_{k(i,j)}(x_i, y_j): x = \coprod_{i=1}^{n} x_i ; y = \coprod_{j=1}^{m} y_j ; n, m \in N ; k \in \{1, 2\}^{\{1,\cdots,n\}\times\{1,\cdots,m\}}\}$

(2) $(T_1 \wedge T_2)(x, y) := \inf \{\sum_{i=1}^{n}\sum_{j=1}^{m} T_{k(i,j)}(x_i, y_j): x = \coprod_{i=1}^{n} x_i ; y = \coprod_{j=1}^{m} y_j ; n, m \in N ; k \in \{1, 2\}^{\{1,\cdots,n\}\times\{1,\cdots,m\}}\}$

(3) $T^+(x, y) := \sup \{\sum_{i=1}^{n}\sum_{j=1}^{m} T_{k(i,j)}(x_i, y_j): (\coprod_{i=1}^{n} x_i) \sqsubseteq x ; (\coprod_{j=1}^{m} y_j) \sqsubseteq y ; n, m \in N ; k \in \{1, 2\}^{\{1,\cdots,n\}\times\{1,\cdots,m\}}\}$

(4) $T^-(x, y) := -\inf \{\sum_{i=1}^{n}\sum_{j=1}^{m} T_{k(i,j)}(x_i, y_j): (\coprod_{i=1}^{n} x_i) \sqsubseteq x ; (\coprod_{j=1}^{m} y_j) \sqsubseteq y ; n, m \in N ; k \in \{1, 2\}^{\{1,\cdots,n\}\times\{1,\cdots,m\}}\}$

(5) $|T|(x, y) := \sup \{\sum_{i=1}^{n}\sum_{j=1}^{m} (-1)^{k(i,j)} T(x_i, y_j): x = \coprod_{i=1}^{n} x_i ; y = \coprod_{j=1}^{m} y_j ; n, m \in N ; k \in \{1, 2\}^{\{1,\cdots,n\}\times\{1,\cdots,m\}}\}$

(6) $|T(x, y)| \leq |T|(x, y)$

Theorem 2. *Let (C, Θ, λ), (A, Σ, μ), and (B, Ξ, ν) be finite measure spaces; E, F, and J be ideal subspaces of $L_0(\mu)$, $L_0(\nu)$, and $L_0(\lambda)$, respectively; $K : C \times A \times B \times R^2$ be a normalized Carathéodory function; and $T : E \times F \longrightarrow J$ be an integral Uryson operator with the kernel K. Then, the following statements are equivalent:*

(1) T is a regular operator

(2) $|T| : E \times F \longrightarrow J$ is a positive integral Uryson operator with the kernel $|K|$

Here, we provide some necessary facts and notations that we need in the further presentation. The standard reference book on the theory of vector lattices is [19]. All vector lattices we consider below are supposed to be Archimedean. The term "operator" between vector spaces E and W means in general a nonlinear map $T : E \longrightarrow W$. We say that two elements x, y of a vector lattice E are *disjoint* and write $x \perp y$, if $|x| \wedge |y| = 0$. We write $x = \coprod_{i=1}^{n} x_i$ if $x = \sum_{i=1}^{n} x_i$ and $x_i \perp x_j$ for all $i \neq j$. In particular, for $n = 2$, we use the notation $x = x_1 \sqcup x_2$. We say that y is a *fragment* (a *component*) of $x \in E$ and use the symbol $y \sqsubseteq x$, if $y \perp (x - y)$. The set of all fragments of an element $x \in E$ is denoted by $\mathcal{F}_x$. We say that $x_1, x_2 \in \mathcal{F}_x$ are mutually complemented, if $x = x_1 \sqcup x_2$. For vector lattices E and F by $E \times F$, we denote the Cartesian product $E \times F := \{(x, y): x \in E, y \in F\}$ of E and F. We observe that $E \times F$ is a vector lattice with the pointwise algebraic and lattice operations. Namely, for all $x, u \in E$ and $y, v \in F$, we have that

$$\begin{gathered}(x, y) \leq (u, v) \Leftrightarrow x \leq u \text{ and } y \leq v,\\ (x, y) \vee (u, v) = (x \vee u, y \vee v) ; (x, y) \wedge (u, v) = (x \wedge u, y \wedge v),\\ |(x, y)| = (|x|, |y|).\end{gathered} \tag{1}$$

Let (A, Σ, μ) be a finite measure space. By $L_0(A, \Sigma, \mu)$ (or $L_0(\mu)$ for shortness), we denote the vector space of all real valued measurable functions on A. More precisely, $L_0(\mu)$ consists of equivalence classes of such functions, where two functions f_1 and f_2 are said to be equivalent if $f_1(s) = f_2(s)$ for μ-almost all $s \in A$. We note that $L_0(\mu)$ is equipped with the natural partial order, that is

$$f \leq h \Leftrightarrow f(s) \leq h(s) \mu - \text{a.e.} s \in A ; f, h \in L_0(\mu). \tag{2}$$

It is worth noting that $L_0(\mu)$ is a Dedekind complete vector lattice (see [20], page 52). We say that a vector subspace E of $L_0(\mu)$ is an *ideal space* if for every $f \in L_0(\mu)$, $h \in E$ the relation $|f| \leq |h|$ implies that $f \in E$. In particular, the classical $L_p(\mu)$-spaces are typical examples of ideal spaces. For a given $f \in L_0(\mu)$ by supp f, we denote the measurable set $\{t \in A : f(t) \neq 0\}$. The characteristic function of a set D is denoted by 1_D. The union $H \cup D$ of two disjoint sets H and D we denote by $H \sqcup D$. The set of all maps from H to D we denote by D^H.

Definition 3. Let E be a vector lattice and let X be a real vector space. An operator $T : E \longrightarrow X$ is said to be orthogonally additive if $T(x + y) = Tx + Ty$ for all disjoint elements $x, y \in E$. It follows from the definition that $T(0) = 0$.

We observe that classical operators of nonlinear analysis such as Uryson, Hammerstein, and Nemytskii operators are orthogonally additive in suitable function spaces (see [1]).

2. The Vector Lattice of Regular Orthogonally Biadditive Operators

In this section, we introduce a notion of an orthogonally biadditive operator and prove that the vector space of all regular orthogonally biadditive operators defined on the Cartesian product $E \times F$ of vector lattice E and F and taking values in a Dedekind complete vector lattice W is a Dedekind complete vector lattice.

Definition 4. Let E, F be vector lattices and W be a vector space. With an operator $T : E \times F \longrightarrow W$ is associated two families of partial operators $T_x : F \longrightarrow W$, $x \in E$, and $T_y : E \longrightarrow W$, $y \in F$ defined by setting:

$$T_x(v) := T(x, v), v \in F ; T_y(u) := T(u, y), u \in E. \quad (3)$$

We say that $T : E \times F \longrightarrow W$ is an orthogonally biadditive operator (OBAO) if all $T_x : F \longrightarrow W$, $x \in E$, and $T_y : E \longrightarrow W$, $y \in F$ are orthogonally additive operators from E to W and F to W, respectively. The vector space of all orthogonally biadditive operators from $E \times F$ to W we denote by $\mathcal{OBA}(E, F ; W)$.

It is clear from the definition that $T(0, y) = T(x, 0) = 0$ for all $x \in E$ and $y \in F$. We note that an OBAO $T : E \times F \longrightarrow W$ need not be orthogonally additive as an operator defined on the vector lattice $E \times F$. Indeed, if $E = F = W = R$, then the operator $T : E \times F \longrightarrow W$ defined by setting

$$T(x, y) := xy, (x, y) \in \mathbb{R}^2, \quad (4)$$

is an OBAO; however, for disjoint elements $s = (0, 1)$ and $t = (1, 0)$ of $E \times F$, one has

$$T(s + t) = 1 \neq 0 = T(s) + T(t). \quad (5)$$

Now we present some examples of OBAOs.

Example 5. Every bilinear operator $T : E \times F \longrightarrow W$ is orthogonally biadditive.

Example 6. Suppose that $E = F = W = R$. Then, $\mathcal{OBA}(E, F ; W)$ coincides with the vector space of all function $f : R^2 \longrightarrow R$ such that $f(0, y) = f(x, 0) = 0$ for all $x, y \in R$.

Definition 7. Let E, F, W be vector lattices. An orthogonally biadditive operator $T : E \times F \longrightarrow W$ is said to be:

(i) *Positive* if $T(x, y) \geq 0$ for all $(x, y) \in E \times F$

(ii) *C -bounded*, if it maps $\mathfrak{F}_{(x,y)}$ to order bounded sets in W for every $(x, y) \in E \times F$

(iii) *Regular*, if $T = S_1 - S_2$, where S_1, S_2 are positive orthogonally biadditive operators from $E \times F$ to W

The sets of all positive, C-bounded, and regular orthogonally biadditive operators from $E \times F$ to W we denote by $\mathcal{OBA}_+(E, F ; W)$, $\mathcal{OBA}_{cb}(E, F ; W)$, and $\mathcal{OBA}_r(E, F ; W)$, respectively. There is a natural partial order on $\mathcal{OBA}_r(E, F ; W)$, namely, $S \leq T \Leftrightarrow (T - S) \in \mathcal{OBA}_+(E, F ; W)$.

Proposition 8. *Let E, F, W be vector lattices. Then, every $T \in \mathcal{OBA}_r(E, F ; W)$ is C -bounded.*

Proof. Suppose that $T = S_1 - S_2$ with $S_1, S_2 \in \mathcal{OBA}_+(E, F ; W)$. Fix $(x, y) \in E \times F$ and take $(v, u) \in \mathfrak{F}_{(x,y)}$. Then, $(x - v, y - u) \perp (v, u)$, and for every, $i \in \{1, 2\}$, we have that

$$S_i(x, y) = S_i(v\text{o}(x - v), u\text{o}(y - u)) = S_i(v, u) + S_i(v, y - u) + S_i(x - v, u) + S_i(x - v, y - u). \quad (6)$$

It follows that $S_i(v, u) \leq S_i(x, y)$, and therefore

$$T(v, u) = S_1(v, u) - S_2(v, u) \leq S_1(x, y) + S_2(x, y), \quad (7)$$

for all $(v, u) \in \mathfrak{F}_{(x,y)}$.□ □

Now we need the following auxiliary statement.

Proposition 9 (see [21], Prop. 3.11). *Let E be a vector lattice and $\coprod_{i=1}^n x_i = \coprod_{k=1}^m y_k$ for some $(x_i)_{i=1}^n$ and $(y_k)_{k=1}^m \subset E$. Then, there exist a family of pairwise disjoint elements $(z_{ik}) \subset E$, where $i \in \{1, \cdots, n\}$ and $k \in \{1, \cdots, m\}$ such that*

(i) $x_i = \coprod_{k=1}^m z_{ik}$ *for any* $i \in \{1, \cdots, n\}$

(ii) $y_k = \coprod_{i=1}^n z_{ik}$ *for any* $k \in \{1, \cdots, m\}$

(iii) $\coprod_{i=1}^n \coprod_{k=1}^m z_{ik} = \coprod_{i=1}^n x_i = \coprod_{k=1}^m y_k$

Now we ready to prove the first main result of the article.

Proof of Theorem 1. First we prove (1). Put, by definition

$$\mathcal{R}(x, y) := \left\{ \sum_{i=1}^{n} \sum_{j=1}^{m} T_{k(i,j)}\left(x_i, y_j\right) : x = \coprod_{i=1}^{n} x_i ; ;y = \coprod_{j=1}^{m} y_j ; n, m \in N ; k \in \{1, 2\}^{\{1,\cdots,n\}\times\{1,\cdots,m\}} \right\}. \quad (8)$$

Since $T_1, T_2 \in \mathcal{OBA}_r(E, F ; W)$, then for all decompositions $x = \coprod_{i=1}^n x_i$, $y = \coprod_{j=1}^m y_j$, and all maps $k : \{1, \cdots, n\} \times \{1, \cdots, m\} \longrightarrow \{1, 2\}$, we have that

$$\sum_{i=1}^{n} \sum_{j=1}^{m} T_{k(i,j)}\left(x_i, y_j\right) \leq \sum_{i=1}^{n} \sum_{j=1}^{m} \left(S_1^1 + S_2^1 + S_1^2 + S_2^2\right)\left(x_i, y_j\right) = \left(S_1^1 + S_2^1 + S_1^2 + S_2^2\right)(x, y), \quad (9)$$

where $S_1^1, S_2^1, S_1^2, S_2^2$ are positive orthogonally biadditive operators such that $T_1 = S_1^1 - S_2^1$ and $T_2 = S_1^2 - S_2^2$. Thus, $\mathcal{R}(x, y)$ is an order bounded subset of W and by the Dedekind completeness of W there exists $R(x,y) := \sup \mathcal{R}(x,y)$. We show that $R: E \times F \longrightarrow W$ is an orthogonally biadditive operator. Fix $y \in F$, disjoint elements $u, v \in E$ and partitions $u \text{o} v = \coprod_{i=1}^{n} x_i$ and $y = \coprod_{j=1}^{m} y_j$. By Proposition 9, for every $i \in \{1, \cdots, n\}$, there exists a decomposition $x_i = x_i^1 \text{o} x_i^2$ such that $u = \coprod_{i=1}^{n} x_i^1$ and $v = \coprod_{i=1}^{n} x_i^2$. Take $\sum_{i=1}^{n}\sum_{j=1}^{m} T_{k(i,j)}(x_i, y_j) \in \mathcal{R}(u \text{o} v, y)$. Then

$$\begin{aligned}\sum_{i=1}^{n}\sum_{j=1}^{m} T_{k(i,j)}\left(x_i, y_j\right) &= \sum_{i=1}^{n}\sum_{j=1}^{m} T_{k(i,j)}\left(x_i^1 \text{o} x_i^2, y_j\right) \\ &= \sum_{i=1}^{n}\sum_{j=1}^{m} T_{k(i,j)}\left(x_i^1, y_j\right) + \sum_{i=1}^{n}\sum_{i=1}^{n} T_{k(i,j)}\left(x_i^2, y_j\right).\end{aligned} \tag{10}$$

Since $\sum_{i=1}^{n}\sum_{j=1}^{m} T_{k(i,j)}(x_i^1, y_j) \in \mathcal{R}(u, y)$ and $\sum_{i=1}^{n}\sum_{j=1}^{m} T_{k(i,j)}(x_i^2, y_j) \in \mathcal{R}(v, y)$, we have that $\mathcal{R}(u \sqcup v, y) \subset \mathcal{R}(u, y) + \mathcal{R}(v, y)$, and therefore, $R(u \sqcup v, y) \leq R(u, y) + R(v, y)$. Let us prove that converse inequality. Pick $\sum_{i=1}^{n}\sum_{j=1}^{m} T_{k(i,j)}(x_i^1, y_j) \in \mathcal{R}(u, y)$ and $\sum_{t=1}^{l}\sum_{s=1}^{r} T_{k(t,s)}(x_t^2, y_s) \in \mathcal{R}(v, y)$, where

$$u = \coprod_{i=1}^{n} x_i^1\,;\, v = \coprod_{t=1}^{l} x_i^2\,;\, y = \coprod_{j=1}^{m} y_j\,;\, y = \coprod_{s=1}^{r} y_s. \tag{11}$$

Suppose that $l \leq n$. Adding, if necessary zero fragments to the sum $\coprod_{t=1}^{l} x_i^2$, we may assume that $l = n$ and

$$u = \coprod_{i=1}^{n} x_i^1\,;\, v = \coprod_{i=1}^{n} x_i^2. \tag{12}$$

By Proposition 9, there is a family of pairwise disjoint elements $(w_{js}) \subset E$, where $j \in \{1, \cdots, m\}$ and $s \in \{1, \cdots, r\}$ such that

(i) $y_j = \coprod_{s=1}^{r} w_{js}$ for every $j \in \{1, \cdots, m\}$

(ii) $y_s = \coprod_{i=1}^{n} w_{js}$ for every $s \in \{1, \cdots, r\}$

(iii) $y = \coprod_{j=1}^{m}\coprod_{s=1}^{r} w_{js}$

Then, we may write

$$\begin{aligned}\sum_{i=1}^{n}\sum_{j=1}^{m} T_{k(i,j)}\left(x_i^1, y_j\right) &= \sum_{i=1}^{n}\sum_{j=1}^{m} T_{k(i,j)}\left(x_i^1, \coprod_{s=1}^{r} w_{js}\right) = \sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{s=1}^{r} T_{k(i,j)}(x_i^1, w_{js}), \\ \sum_{i=1}^{n}\sum_{s=1}^{r} T_{k(t,s)}(x_t^2, y_s) &= \sum_{t=1}^{l}\sum_{s=1}^{r} T_{k(i,s)}\left(x_i^2, \coprod_{j=1}^{m} w_{js}\right) = \sum_{i=1}^{n}\sum_{s=1}^{r}\sum_{j=1}^{m} T_{k(i,s)}(x_i^2, w_{js}).\end{aligned} \tag{13}$$

Thus

$$\begin{aligned}&\sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{s=1}^{r} T_{k(i,j)}(x_i^1, w_{js}) + \sum_{i=1}^{n}\sum_{s=1}^{r}\sum_{j=1}^{m} T_{k(i,s)}(x_t^2, w_{js}) \\ &\quad = \sum_{i=1}^{n}\sum_{s=1}^{r}\sum_{j=1}^{m} T_{k(i,s)}(x_i^1 \sqcup x_i^2, w_{js}) \in \mathcal{R}(u \sqcup v, y).\end{aligned} \tag{14}$$

Then, $\mathcal{R}(u, y) + \mathcal{R}(v, y) \subset \mathcal{R}(u \sqcup v, y)$, and we have that $R(u, y) + R(v, y) \leq R(u \sqcup v, y)$. Hence, $R(u \sqcup v, y) = R(u, y) + R(v, y)$. Since $T_1(x, y), T_2(x, y) \in \mathcal{R}(x, y)$, we have that $T_1(x, y) \leq R(x, y)$ and $T_2(x, y) \leq R(x, y)$. Suppose $H: E \times F \longrightarrow W$ is an orthogonally biadditive operator with $T_1(x, y) \leq H(x, y)$ and $T_2(x, y) \leq H(x, y)$ for all $(x, y) \in E \times F$. Then

$$H(x, y) = H\left(\coprod_{i=1}^{n} x_i, \coprod_{j=1}^{m} y_j\right) = \sum_{i=1}^{n}\sum_{j=1}^{m} H\left(x_i, y_j\right) \geq \sum_{i=1}^{n}\sum_{j=1}^{m} T_{k(i,j)}\left(x_i, y_j\right), \tag{15}$$

for all disjoint decompositions $x = \coprod_{i=1}^{n} x_i$, $y = \coprod_{j=1}^{m} y_j$, $n, m \in N$, and all functions $k: \{1, \cdots, n\} \times \{1, \cdots, m\} \longrightarrow \{1, 2\}$. Hence, $H \geq R$ and $R = T_1 \vee T_2$. Now we are in the position to derive the other formulas of the lattice calculus.

$$\begin{aligned}(T_1 \wedge T_2)(x, y) &= -((-T_1) \vee (-T_2))(x, y)) \\ &= -\sup\left\{\sum_{i=1}^{n}\sum_{i=1}^{n} -T_{k(i,j)}\left(x_i, y_j\right) : x = \coprod_{i=1}^{n} x_i\,;\, y\right. \\ &\quad \left.= \coprod_{j=1}^{m} y_j\,;\, n, m \in N\,;\, k \in \{1, 2\}^{\{1, \cdots, n\} \times \{1, \cdots, m\}}\right\} \\ &= \inf\left\{\sum_{i=1}^{n}\sum_{i=1}^{n} T_{k(i,j)}\left(x_i, y_j\right) : x = \coprod_{i=1}^{n} x_i\,;\, y\right. \\ &\quad \left.= \coprod_{j=1}^{m} y_j\,;\, n, m \in N\,;\, k \in \{1, 2\}^{\{1, \cdots, n\} \times \{1, \cdots, m\}}\right\}.\end{aligned} \tag{16}$$

Assuming that $T_1 = T$ and $T_2 = 0$ we get formulas for the positive and the negative parts of T. The formula for the modulus T is obvious. Now we prove inequality (6). Take trivial decomposition $x = x$, $y = y$, and $k, k' \in \{1, 2\}^{\{1\} \times \{1\}}$ with $k(1, 1) = 1$ and $k'(1, 1) = 2$. Then

$$\begin{aligned}|T(x, y)| &= T(x, y) \vee (-T(x, y)) = (-1)^{k'(1,1)} T(x, y) \vee (-1)^{k(1,1)} T(x, y) \\ &\leq \sup\left\{\sum_{i=1}^{n}\sum_{i=1}^{n} (-1)^{k(i,j)} T\left(x_i, y_j\right) : x = \coprod_{i=1}^{n} x_i\,;\, y\right. \\ &\quad \left.= \coprod_{j=1}^{m} y_j\,;\, n, m \in N\,;\, k \in \{1, 2\}^{\{1, \cdots, n\} \times \{1, \cdots, m\}}\right\} = |T|(x, y).\end{aligned} \tag{17}$$

It remains to show the Dedekind completeness of the vector lattice $\mathcal{OBA}_r(E, F; W)$. Take a family $(T_\alpha)_{\alpha \in A}$ of positive OBAOs with $0 \leq T_\alpha \leq T \in \mathcal{OBA}_+(E, F; W)$. Without a loss of generality, we may assume that $(T_\alpha)_{\alpha \in A}$ is an upward directed set. Define an operator $G: E \times F \longrightarrow W$ as $G(x, y) = \sup_{\alpha \in A} T_\alpha(x, y)$. Since the vector lattice W is Dedekind complete, the operator G is well defined. Let us show the orthogonal biadditivity of G. Fix $y \in F$ and $x, v \in E$ with

$x \perp v$. Then, we have that

$$G(x \sqcup v, y) = \sup_{\alpha} T_{\alpha}(x \sqcup v, y) = \sup_{\alpha} T_{\alpha}(x, y) + \sup_{\alpha} T_{\alpha}(v, y) = G(x, y) + G(v, y), \tag{18}$$

and we deduce that G_y is an orthogonally additive operator. Similar arguments are valid for G_x for all $x \in E$. Clearly, $G = \sup_{\alpha \in A} T_{\alpha}$. □

Definition 10. Let E be a vector lattice. A net $(e_{\alpha})_{\alpha \in A}$ in E horizontally converges (or laterally converges in another terminology) to an element $e \in E$ (notation $e_{\alpha} \xrightarrow{h} e$) if the net $(e_{\alpha})_{\alpha \in A}$ order converges to e and $e_{\alpha} \sqsubseteq e_{\beta} \sqsubseteq e$ for all $\alpha, \beta \in A$ with $\alpha \leq \beta$.

Definition 11. Let E and F be vector lattices. An operator $T : E \longrightarrow F$ is said to be:

(i) *Horizontally-to-order continuous* (or *laterally-to-order continuous*) if every horizontally convergent net $(e_{\alpha})_{\alpha \in A}$ in E with $e_{\alpha} \xrightarrow{h} eT$ maps to an order convergent net $(Te_{\alpha})_{\alpha \in A}$ in F with $Te_{\alpha} \xrightarrow{o} Te$

(ii) *Horizontally-to-order σ -continuous* (or *laterally-to-order σ -continuous*) if every horizontally convergent sequence $(e_n)_{n \in N}$ in E with $e_n \xrightarrow{h} eT$ maps to an order convergent sequence $(Te_n)_{n \in N}$ in F with $Te_n \xrightarrow{o} Te$

The vector space of all horizontally-to-order continuous (σ-continuous) orthogonally additive operators from E to F is denoted by $\mathcal{OA}_c(E, F)$ ($\mathcal{OA}_{\sigma c}(E, F)$).

We observe that this class of operators has been studied in [22–25]. It is worth noting that the Dedekind completeness of a vector lattice F implies the relation $\mathcal{OA}_c(E, F) \subset \mathcal{OA}_r(E, F)$ ([24], Theorem 3.6., Lemma 3.12).

Definition 12. Let E, F, and W be vector lattices. An orthogonally biadditive operator $T : E \times F \longrightarrow W$ is called separately horizontally-to-order continuous (σ-continuous), if partial operators T_x and T_y are horizontally-to-order continuous (σ-continuous), for all $x \in E$, $y \in F$. The sets of all horizontally-to-order continuous (σ -continuous) and separately horizontally-to-order continuous regular OBAOs we denote by $\mathcal{OBA}_c(E, F ; W)$ ($\mathcal{OBA}_{\sigma c}(E, F ; W)$) and $\mathcal{OBA}_{sc}(E, F ; W)$ ($\mathcal{OBA}_{s\sigma c}(E, F ; W)$), respectively.

Example 13. Suppose that $E = F = W = R$. Then, $\mathcal{OBA}_r(E, F ; W) = \mathcal{OBA}_{sc}(E, F ; W) = \mathcal{OBA}_{sc}(E, F ; W)$. Indeed, since $\mathfrak{F}_e = \{0, e\}$, we have that every horizontally convergent net $(e_{\alpha})_{\alpha \in A}$ in E with $e_{\alpha} \xrightarrow{h} e$ is the constant one, that is $e_{\alpha} = e$ for all $\alpha \geq \alpha_0$ where $\alpha_0 \in A$ is some index.

The next theorem has its own interest.

Theorem 14. *Let E, F, and W be vector lattices with W Dedekind complete. Then, $\mathcal{OBA}_{sc}(E, F ; W)$ and $\mathcal{OBA}_{s\sigma c}(E, F ; W)$ are projection bands in $\mathcal{OBA}_r(E, F ; W)$.*

Proof. We prove the assertion for $\mathcal{OBA}_{sc}(E, F ; W)$; the proof for $\mathcal{OBA}_{s\sigma c}(E, F ; W)$ is similar. It is clear that $\mathcal{OBA}_{sc}(E, F ; W)$ is a vector space. We show that $\mathcal{OBA}_{sc}(E, F ; W)$ is an order ideal of $\mathcal{OBA}_r(E, F ; W)$. Suppose that $T \in \mathcal{OBA}_{sc}(E, F ; W)$. We show that $|T| \in \mathcal{OBA}_{sc}(E, F ; W)$ too. Indeed, take $y \in F$ and a horizontally convergent net $(x_{\alpha})_{\alpha \in A}$ in E with $x_{\alpha} \xrightarrow{h} x$. We need to prove that $|T|_y x_{\alpha} \xrightarrow{o} |T|_y x$. Since $|T| \in \mathcal{OBA}_+(E, F ; W)$, we have that $|T|_y \in \mathcal{OA}_+(E, W)$, and therefore

$$o - \lim_{\alpha} |T|_y x_{\alpha} = \sup_{\alpha} |T|_y x_{\alpha} \leq |T|_y x. \tag{19}$$

On the other hand, by Theorem 1 we have that

$$|T|_y x = |T|(x, y) = \sup \left\{ \sum_{i=1}^{n} \sum_{i=1}^{n} (-1)^{k(i,j)} T\left(x_i, y_j\right) : x = \coprod_{i=1}^{n} x_i \, ; y = \coprod_{j=1}^{m} y_j, \; n, m \in N \, ; k \in \{1, 2\}^{\{1, \cdots, n\} \times \{1, \cdots, m\}} \right\}. \tag{20}$$

Put, by definition

$$\mathcal{R} := \left\{ \sum_{i=1}^{n} \sum_{j=1}^{m} (-1)^{k(i,j)} T\left(x_i, y_j\right) : x = \coprod_{i=1}^{n} x_i \, ; y = \coprod_{j=1}^{m} y_j, \; n, m \in N \, ; k \in \{1, 2\}^{\{1, \cdots, n\} \times \{1, \cdots, m\}} \right\}. \tag{21}$$

Pick $\sum_{i=1}^{n} \sum_{j=1}^{m} (-1)^{k(i,j)} T(x_i, y_j) \in \mathcal{R}$. By Proposition 9, for every $\alpha \in A$, there exists a decomposition $x_{\alpha} = \coprod_{i=1}^{n} x_{\alpha}^i$ such that $x_{\alpha}^i \xrightarrow{h} x_i$ for all $\{1, \cdots, n\}$. Since by assumptions above all partial operators $T_{y_j} : F \longrightarrow W$ are horizontally-to-order continuous, we have that

$$\begin{aligned} \sum_{i=1}^{n} \sum_{i=1}^{n} (-1)^{k(i,j)} T\left(x_i, y_j\right) &= \sum_{i=1}^{n} \sum_{j=1}^{m} (-1)^{k(i,j)} T_{y_j} x_i \\ &= o - \lim_{\alpha \in A} \sum_{i=1}^{n} \sum_{j=1}^{m} (-1)^{k(i,j)} T_{y_j} x_{\alpha}^i = \sum_{i=1}^{n} \sum_{j=1}^{m} (-1)^{k(i,j)} T_{y_j} \left(h - \lim_{\alpha \in A} x_{\alpha}^i\right) \\ &\leq \sum_{i=1}^{n} \sum_{j=1}^{m} \sup_{\alpha \in A} |T|_{y_j} x_{\alpha}^i = \sup_{\alpha \in A} |T|_y x_{\alpha}. \end{aligned} \tag{22}$$

Passing to the supremum in the left-hand side of the above inequality over all elements of $\mathcal{R}$, we deduce that

$$\left| T \right|_y x \leq \sup_{\alpha \in A} \left| T \right|_y x_{\alpha} = o - \lim_{\alpha \in A} |T|_y x_{\alpha}, \tag{23}$$

and therefore, $|T|_y x = o - \lim_{\alpha \in A} |T|_y x_{\alpha}$. The horizontal-to-order continuity of a partial operator $|T|_x$ for $x \in E$ can be proved by the same way. Now we prove that $\mathscr{OBA}_{sc}(E, F; W)$ is an order ideal of $\mathscr{OBA}_r(E, F; W)$. Suppose that $0 \le T \in \mathscr{OBA}_{sc}(E, F; W)$, $0 \le S \in \mathscr{OBA}_r(E, F; W)$, and $0 \le S \le T$. Then, $0 \le S_y \le T_y$ $(0 \le S_x \le T_x)$ for every $y \in F$ $(x \in E)$ and by ([24], Theorem 3.13) we have that $S \in \mathscr{OBA}_{sc}(E, F; W)$. It remains to show that $\mathscr{OBA}_{sc}(E, F; W)$ is a band in $\mathscr{OBA}_r(E, F; W)$. Pick a net $(T_\alpha)_{\alpha\in A}$ in $\mathscr{OBA}_{sc}(E, F; W)$ with $0 \le T_\alpha \uparrow T$ for some $T \in \mathscr{OBA}_+(E, F; W)$. Then, we have that $0 \le T_{\alpha y} \uparrow T_y$ $(0 \le T_{\alpha x} \uparrow T_x)$ for every $y \in F$ $(x \in E)$. Now, applying ([24], Theorem 3.13), we obtain that $T \in \mathscr{OBA}_{sc}(E, F; W)$. Finally, taking into account the Dedekind completeness of $\mathscr{OBA}_r(E, F; W)$, we deduce that $\mathscr{OBA}_{sc}(E, F; W)$ is the projection band in $T \in \mathscr{OBA}_r(E, F; W)$.

Now we are ready to prove that the Dedekind completeness of a vector lattice W implies the horizontal-to-order continuity (σ-continuity) of a regular separately horizontal-to-order continuous (σ-continuous) operator $T : E \times W$.□ □

Proposition 15. *Let E and F be vector lattices, W be a Dedekind complete vector lattice, and $T \in \mathscr{OBA}_r(E, F; W)$. Then, the following statements hold:*

(1) $T \in \mathscr{OBA}_{sc}(E, F; W) \Leftrightarrow T \in \mathscr{OBA}_c(E, F; W)$

(2) $T \in \mathscr{OBA}_{s\sigma c}(E, F; W) \Leftrightarrow T \in \mathscr{OBA}_{\sigma c}(E, F; W)$

Proof. We prove statement (1). The implication $T \in \mathscr{OBA}_c(E, F; W) \Rightarrow T \in \mathscr{OBA}_{sc}(E, F; W)$ is obvious. Suppose that $0 \le T \in \mathscr{OBA}_{sc}(E, F; W)$. We need to show horizontal-to-order continuity of T. Pick a horizontally convergent net $((x, y)_\alpha)_{\alpha\in A}$ with $(x, y)_\alpha \xrightarrow{h} (x, y)$. Then, $x_\alpha \xrightarrow{h} x$ and $y_\alpha \xrightarrow{h} y$. Now we may write

$$\begin{aligned} |T(x, y) - T(x_\alpha, y_\alpha)| &= |T((x - x_\alpha) + x_\alpha, (y - y_\alpha) + y_\alpha) \\ &\quad - T(x_\alpha, y_\alpha)| \le |T(x - x_\alpha, y_\alpha)| + |T(x_\alpha, y - y_\alpha)| + |T(x - x_\alpha, y - y_\alpha)| \\ &= T(x - x_\alpha, y_\alpha) + T(x_\alpha, y - y_\alpha) + T(x - x_\alpha, y - y_\alpha) \\ &\le T(x - x_\alpha, y) + T(x, y - y_\alpha) + T(x, y - y_\alpha) \\ &= T(x - x_\alpha, y) + 2T(x, y - y_\alpha). \end{aligned} \tag{24}$$

Taking into account the separate horizontal-to-order continuity of T, we have that $T(x, y)_\alpha \xrightarrow{o} T(x, y)$. Now, suppose that T is an arbitrary element of $\mathscr{OBA}_{sc}(E, F; W)$. Then, by Theorem 14, every $T \in \mathscr{OBA}_{sc}(E, F; W)$ has the representation $T = T^+ - T^-$, where $0 \le T^+, -T^- \in \mathscr{OBA}_{sc}(E, F; W)$. Hence, by above, we have that $T \in \mathscr{OBA}_c(E, F; W)$.□ □

3. Orthogonally Biadditive Operators on a Cartesian Product of Ideal Spaces of Measurable Functions

In this section, we consider orthogonally biadditive operators in lattices of measurable functions and obtain a criteria of the regularity of an integral Uryson operator.

Definition 16. Suppose that (A, Σ, μ) and (B, Ξ, ν) are finite measure space and $\mu \otimes \nu$ is the product measure on $\Sigma \otimes \Xi$. We say that $N : A \times B \times \mathbb{R} \longrightarrow \mathbb{R}$ is a superpositionally measurable (or sup-measurable for shortness) function, if $N(\cdot, \cdot, f(\cdot, \cdot))$ is $\mu \otimes \nu$ -measurable for each $f \in L_0(\mu \otimes \nu)$. A sup-measurable function N is said to be normalized if $N(s, t, 0) = 0$ for $\mu \otimes \nu$-almost all $(s, t) \in A \times B$.

The following proposition provides an important example of an orthogonally biadditive operator.

Proposition 17. *Let $N : A \times B \times \mathbb{R} \longrightarrow \mathbb{R}$ be a normalized sup-measurable function and E and F be order ideals of $L_0(\mu)$ and $L_0(\mu)$, respectively. Then, the map $\mathscr{N}$ defined by*

$$\mathscr{N}(f, g)(s, t) := N(s, t, f(s)g(t)), f \in E, g \in F, \tag{25}$$

is an orthogonally biadditive operator from $E \times F$ to $L_0(\lambda)$.

Proof. Take $f \in E$ and $g \in F$. Put $\tilde{f} := f1_B$ and $\tilde{g} := g1_A$. We note that the relations $f \in L_0(\mu)$ and $g \in L_0(\nu)$ imply that $\tilde{f}, \tilde{g} \in L_0(\mu \otimes \nu)$. Then, $N(s, t, f(s)g(t)) = N(s, t, \tilde{f}(s, t)\tilde{g}(s, t))$, and therefore, the operator $\mathscr{N}$ is well defined. Fix $g \in F$. We show that the partial operator $\mathscr{N}_g : E \longrightarrow W$ is orthogonally additive. Indeed, take disjoint $f, e \in E$. Then, $\tilde{f}$ and $\tilde{e}$ are disjoint elements of $L_0(\mu \otimes \nu)$, and we may write

$$\begin{aligned} N(\cdot, \cdot, (f \sqcup e)g) &= N\left(\cdot, \cdot, \left(\tilde{f} \sqcup \tilde{e}\right)\tilde{g}\right) \\ &= N\left(\cdot, \cdot, \left(\tilde{f} \sqcup \tilde{e}\right)1_{\text{supp }\left(\tilde{f} \sqcup \tilde{e}\right)}\tilde{g}\right) \\ &= N\left(\cdot, \cdot, \left(\tilde{f} \sqcup \tilde{e}\right)\tilde{g}\right)\left(1_{\text{supp }\left(\tilde{f} \sqcup \tilde{e}\right)}\right) \\ &= N\left(\cdot, \cdot, \left(\tilde{f} \sqcup \tilde{e}\right)\tilde{g}\right)1_{\text{supp }\tilde{f}} + N\left(\cdot, \cdot, \left(\tilde{f} \sqcup \tilde{e}\right)\tilde{g}\right)1_{\text{supp }\tilde{e}} \\ &= N\left(\cdot, \cdot, \tilde{f}\tilde{g}\right) + N(\cdot, \cdot, \tilde{e}\tilde{g}) = N(\cdot, \cdot, fg) + N(\cdot, \cdot, eg). \end{aligned} \tag{26}$$

Noting that similar arguments are valid for a partial operator $\mathscr{N}_f : F \longrightarrow W$, for all $f \in E$, we finish the proof.□ □

We observe that $\mathscr{N}$ is known as the *nonlinear superposition operator* or Nemytskii operator. The basic constructions of the theory of Nemytskii operators are presented in [26]. Recently, nonlinear superposition operators were investigated in [2, 27, 28].

Definition 18. Let (C, Θ, λ), (A, Σ, μ), and (B, Ξ, ν) be finite measure spaces. By $(C \times A \times B, \lambda \otimes \mu \otimes \nu)$, we denote the completion of their product measure space. A map $K : C \times A \times B \times R^2 \longrightarrow R$ is said to be a Carathéodory function if it is satisfies the following conditions:

(C_1) $K(\cdot, \cdot, \cdot, r, q)$ is $\mu \otimes \nu \otimes \lambda$-measurable for all $(r, q) \in R^2$

(C_2) $K(p, s, t, \cdot, \cdot)$ is continuous on R^2 for $\lambda \otimes \mu \otimes \nu$ -almost all $(p, s, t) \in C \times A \times B$

We say that a Carathéodory function K is *normalized* if $K(p,s,t,0,q)=K(p,s,t,r,0)=0$ for $\lambda\otimes\mu\otimes\nu$-almost all $(p,s,t)\in C\times A\times B$ and all $q,r\in R$.

Proposition 19. *Let $K: C\times A\times B\times\mathbb{R}^2$ be a normalized Carathéodory function, $f\in L_0(\mu)$ and $g\in L_0(\nu)$. Then, $K(\cdot,\cdot,\cdot,f(\cdot),g(\cdot))\in L_0(\lambda\otimes\mu\otimes\nu)$.*

Proof. First we note that $\lambda\otimes\mu\otimes\nu$-measurable null sets

$$D_q := \{(p,s,t)\in C\times A\times B : K(p,s,t,0,q)\neq 0\}, q\in\mathbb{R},$$
$$D^r := \{(p,s,t)\in C\times A\times B : K(p,s,t,r,0)\neq 0\}, r\in\mathbb{R}, \tag{27}$$

depend of q and r, respectively. We claim that there exists $\lambda\otimes\mu\otimes\nu$-measurable set $H\subset C\times A\times B$ such that

$$\lambda\otimes\mu\otimes\nu(C\times A\times B)=\lambda\otimes\mu\otimes\nu(H),$$
$$K(p,s,t,0,q)=K(p,s,t,r,0)=0, \tag{28}$$

for all $(p,s,t)\in H$ and all $q,r\in\mathbb{R}$. Indeed, consider two sequences of $\lambda\otimes\mu\otimes\nu$-measurable sets $(D_q)_{q\in Q}$ and $(D^r)_{r\in Q}$ and put

$$G := \{(p,s,t)\in C\times A\times B : K(p,s,t,\cdot,\cdot)\text{is not continuous}\},$$
$$H = C\times A\times B\setminus\left(G\cup\bigcup_{q\in Q}D_q\cup\bigcup_{r\in Q}D^r\right). \tag{29}$$

Since $\lambda\otimes\mu\otimes\nu(G\cup\bigcup_{q\in\mathbb{Q}}D_q\cup\bigcup_{r\in\mathbb{Q}}D^r)=0$, we have that $\lambda\otimes\mu\otimes\nu(C\times A\times B)=\lambda\otimes\mu\otimes\nu(H)$. Fix $(p,s,t)\in H$ and $q\in\mathbb{R}$ $(r\in\mathbb{R})$. Then, there exists a sequence $(q_n)_{n\in N}$ $((r_k)_{k\in N})$ in $\mathbb{Q}$ that converges to q (r) with $K(p,s,t,0,q_n)=0$ $(K(p,s,t,r_k,0)=0)$ for all $n\in\mathbb{N}$ $(n\in\mathbb{N})$. Then, by (C_2), we have that $K(p,s,t,0,q)=0$ $(K(p,s,t,r,0)=0)$.

Now, we show that $K(\cdot,\cdot,\cdot,r1_{A_1}(\cdot),q1_{B_1}(\cdot))\in L_0(\lambda\otimes\mu\otimes\nu)$ for arbitrary $r,q\in\mathbb{R}$, $A_1\in\Sigma$, and $B_1\in\Xi$. We claim that

$$K\left(p,s,t,r1_{A_1}(s),q1_{B_1}(t)\right)=K(p,s,t,r,q)1_{A_1}(s)1_{B_1}(t), \tag{30}$$

for $\lambda\otimes\mu\otimes\nu$ almost all $(p,s,t)\in C\times A\times B$. Indeed, pick $(p,s,t)\in H$. If $s\in A_1$ and $t\in B_1$, we have

$$K\left(p,s,t,r1_{A_1}(s),q1_{B_1}(t)\right)=K(p,s,t,r,q)=K(p,s,t,r,q)1_{A_1}(s)1_{B_1}(t). \tag{31}$$

If either $s\notin A_1$ or $t\notin B_1$, then

$$K\left(p,s,t,r1_{A_1}(s),q1_{B_1}(t)\right)=0=K(p,s,t,r,q)1_{A_1}(s)1_{B_1}(t). \tag{32}$$

Suppose that $A_1,\cdots,A_n$ and $B_1,\cdots,B_m$ are pairwise disjoint measurable subsets of A and B, respectively, $f=\sum_{i=1}^n r_i 1_{A_i}$ and $g=\sum_{j=1}^m q_j 1_{B_j}$ are simple functions in $L_0(\mu)$ and $L_0(\nu)$, respectively. Then

$$K(p,s,t,f(s),q(t))=K\left(p,s,t,\sum_{i=1}^n r_i 1_{A_i}(s),\sum_{j=1}^m q_j 1_{B_j}(t)\right)$$
$$=\sum_{i=1}^n\sum_{j=1}^m K\left(p,s,t,r_i 1_{A_i}(s),q_j 1_{B_j}(t)\right), \tag{33}$$

and we deduce that $K(\cdot,\cdot,\cdot,f(\cdot),q(\cdot))\in L_0(\lambda\otimes\mu\otimes\nu)$. Finally, assume that f and g are arbitrary elements of $L_0(\mu)$ and $L_0(\nu)$, $(f_n)_{n\in N}$ and $(g_k)_{k\in N}$ are sequences of simple functions in $L_0(\mu)$ and $L_0(\nu)$, respectively, such that $(f_n)_{n\in N}$ converges to $f\mu$-a.e. and $(g_n)_{k\in N}$ converges to $g\nu$-a.e. Put by definition

$$A_0 := \{s\in A : f_n(s)\text{does not converges to} f\},$$
$$B_0 := \{t\in B : g_k(t)\text{does not converges to} g\}. \tag{34}$$

Clearly,

$$\lambda\otimes\mu\otimes\nu(G)=\lambda\otimes\mu\otimes\nu(C\times A_0\times B)$$
$$=\lambda\otimes\mu\otimes\nu(C\times A\times B_0)=0. \tag{35}$$

Then, $K(p,s,t,f_n(s),g_k(t))$ converges to $K(p,s,t,f(s),g(t))$ for all $(p,s,t)\in C\times A\times B\setminus(G\cup(C\times A_0\times B)\cup(C\times A\times B_0))$, and therefore, $K(\cdot,\cdot,\cdot,f(\cdot),g(\cdot))\in L_0(\lambda\otimes\mu\otimes\nu)$.□ □

Remark 20. Using similar arguments as above, we get the following useful equalities:

$$K(p,s,t,f(s),q)=K(p,s,t,f(s),q)1_{\text{supp } f}(s),$$
$$K(p,s,t,r,g(t))=K(p,s,t,r,g(t))1_{\text{supp } g}(t), \tag{36}$$

for $\lambda\otimes\mu\otimes\nu$ almost all $(p,s,t)\in C\times A\times B$ and all $q\in R$ $(r\in R)$.

The next proposition provides an important example of an orthogonally biadditive operator.

Proposition 21. *Let $K: C\times A\times B\times\mathbb{R}^2$ be a normalized Carathéodory function, E and F be order ideals of $L_0(\mu)$ and $L_0(\nu)$, respectively, and $K(p,\cdot,\cdot,f(\cdot),g(\cdot))\in L_1(\mu\otimes\nu)$ for all $f\in E$, $g\in F$, and λ -almost all $p\in C$. Then, the map T defined by setting*

$$T(f,g)(p) := \int_{A\times B}K(p,s,t,f(s),g(t))\,d(\mu\otimes\nu)\,; f\in E, g\in F, \tag{37}$$

is an orthogonally biadditive operator from $E\times F$ to $L_0(\lambda)$.

Proof. By Proposition 19, T is a well-defined operator from $E\times F$ to $L_0(\lambda)$. We show the orthogonal additivity of a partial operator T_g, where $g \in F$. Fix disjoint $f_1, f_2 \in E$. Then, taking into account considerations above, we may write

$$\begin{aligned} T(f_1 \sqcup f_2, g)(p) &= \int_{A\times B} K(p,s,t,f_1(s)\sqcup f_2(s), g(t)) d(\mu\otimes\nu) \\ &= \int_{A\times B} K(p,s,t,f_1(s)+f_2(s), g(t)) 1_{\text{supp }(f_1\sqcup f_2)} d(\mu\otimes\nu) \\ &= \int_{A\times B} \Big(K(p,s,t,f_1(s)\sqcup f_2(s), g(t)) 1_{\text{supp } f_1} d(\mu\otimes\nu) \\ &\quad + \int_{A\times B} K(p,s,t,f_1(s)\sqcup f_2(s), g(t)) 1_{\text{supp } f_2} d(\mu\otimes\nu) \\ &= \int_{A\times B} (K(p,s,t,f_1(s), g(t)) d(\mu\otimes\nu) \\ &\quad + \int_{A\times B} (K(p,s,t,f_2(s), g(t)) d(\mu\otimes\nu) \\ &= T(f_1, g)(p) + T(f_2, g)(p). \end{aligned} \tag{38}$$

□ □

The orthogonal additivity of a partial operator T_f, $f \in E$ can be proved analogously.

We observe that an operator T above can be considered as the Uryson integral operator that depends on two variables. We say that a function K is a *kernel* of an operator T. Classical integral Uryson operators were investigated by many mathematicians (see for instance monograph [29]).

The following example of an OBAO is a Hammerstein operator which depends on two variables.

Example 22. Let (C, Θ, λ), (A, Σ, μ), and (B, Ξ, ν) be as above, $N : A\times B\times \mathbb{R} \longrightarrow \mathbb{R}$ be a normalized sup-measurable function, E and F be order ideals in $L_0(\mu)$ *and* $L_0(\nu)$, respectively, $L : C\times A\times B \longrightarrow R$ be a $\lambda\otimes\mu\otimes\nu$-measurable function, and $L(p,\cdot,\cdot)N(\cdot,\cdot,f(\cdot)g(\cdot)) \in L_1(\mu\otimes\nu)$ for all $f \in E$, $g \in F$, and λ -almost all $p \in C$. Then, the following formula defines a *OBAO* $T : E\times F \longrightarrow L_0(\tau)$

$$T(f,g)(p) \coloneqq \int_{A\times B} L(p,s,t) N(s,t,f(s)g(t)) d(\mu\otimes\nu), f \in E, g \in F. \tag{39}$$

We note that biorthogonal additivity of a superposition operator $\mathcal{N}(f,g) \coloneqq N(\cdot,\cdot,f(\cdot)g(\cdot))$ implies that $T \in \mathcal{OBA}(E,F;L_0(\lambda))$. The operator T can be treated as an integral Hammerstein operator that depends on two variables.

Now we are ready to prove the second main result of the article.

Proof of Theorem 2. $(2) \Rightarrow (1)$. Since $S \in \mathcal{OBA}_+(E,F;W)$ and $T \le S$, we have that $T = S-(S-T)$, and therefore, $T \in \mathcal{OBA}_r(E,F;W)$.

$(1) \Rightarrow (2)$. By Theorem 1, the modulus $|T|$ exists and can be calculated by the formula

$$\begin{aligned} |T|(f,g) \coloneqq \sup \Bigg\{ &\sum_{i=1}^{n}\sum_{j=1}^{m} (-1)^{k(i,j)} T\left(f_i, g_j\right) : f = \coprod_{i=1}^{n} f_i \,;\, g \\ &= \coprod_{j=1}^{m} g_j \,;\, n, m \in N, \\ &k \in \{1,2\}^{\{1,\cdots,n\}\times\{1,\cdots,m\}} \Bigg\} ; f \in E ; g \in F. \end{aligned} \tag{40}$$

Fix $f \in E$, $g \in F$. By Remark 20, for almost all $p \in C$, we have that

$$\begin{aligned} &\int_{A\times B} K(p,s,t,f(s),g(t))\, d(\mu\otimes\nu) \\ &\quad = \int_{\text{supp } f\times\text{supp } g} K(p,s,t,f(s),g(t))\, d(\mu\otimes\nu). \end{aligned} \tag{41}$$

We also note that

$$\begin{aligned} &\int_{\text{supp } f\times\text{supp } g} K(p,s,t,f(s),g(t))\, d(\mu\otimes\nu) \\ &\quad = \int_{\text{supp } f\times\text{supp } g} K\big(p,s,t,f(s)1_{\text{supp } g}(t)1_C(p), g(t)1_{\text{supp } f} \\ &\qquad \times (s)1_C(p)\big)\, d(\mu\otimes\nu), \end{aligned} \tag{42}$$

for λ-almost all $p \in C$. Put by definition

$$\begin{aligned} \mathfrak{A} \coloneqq \{&(p,s,t) \in C\times\text{supp } f\times\text{supp } g : K \\ &\cdot \big(p,s,t,f(s)1_{\text{supp } g}(t)1_C(p), g(t)1_{\text{supp } f}(s)1_C(p)\big) > 0\}, \end{aligned} \tag{43}$$

$$\begin{aligned} \mathfrak{B} \coloneqq \{&(p,s,t) \in C\times A\times B : K \\ &\cdot \big(p,s,t,f(s)1_{\text{supp } g}(t)1_C(p), g(t)1_{\text{supp } f}(s)1_C(p)\big) < 0\} \\ &= C\times\text{supp } f\times\text{supp } g \setminus \mathfrak{A}. \end{aligned} \tag{44}$$

Clearly, $\mathfrak{A}$ and $\mathfrak{B}$ are $\lambda\otimes\mu\otimes\nu$-measurable sets and

$$\lambda\otimes\mu\otimes\nu\{(p,s,t) \in \mathfrak{A}\cap\mathfrak{B}\} = 0. \tag{45}$$

We may assume that $\lambda\otimes\mu\otimes\nu(\mathfrak{A}) > 0$ and $\lambda\otimes\mu\otimes\nu(\mathfrak{B}) > 0$. Otherwise, if $\lambda\otimes\mu\otimes\nu(\mathfrak{A}) = 0$ $(\lambda\otimes\mu\otimes\nu(\mathfrak{B}) = 0)$, then the following equalities

$$
\begin{aligned}
T(f,g)(p) &= \int_{\operatorname{supp} f\times \operatorname{supp} g} \left|K\left(p,s,t,f(s)1_{\operatorname{supp} g}(t)1_C(p), g\cdot(t)1_{\operatorname{supp} f}(s)1_C(p)\right)\right| d(\mu\otimes\nu) \\
&= \int_{\operatorname{supp} f\times \operatorname{supp} g} |K(p,s,t,f(s),g(t))| d(\mu\otimes\nu),
\end{aligned}
$$

$$
\begin{aligned}
(-T(f,g)(p) &= \int_{\operatorname{supp} f\times \operatorname{supp} g} \left|K\left(p,s,t,f(s)1_{\operatorname{supp} g}(t)1_C(p), g\cdot(t)1_{\operatorname{supp} f}(s)1_C(p)\right)\right| d(\mu\otimes\nu) \\
&= \int_{\operatorname{supp} f\times \operatorname{supp} g} \mid K(p,s,t,f(s),g(t)\mid d(\mu\otimes\nu)),
\end{aligned}
\tag{46}
$$

hold for λ-almost all $p\in C$, and it is nothing to prove. We observe that

$$
\begin{aligned}
&K\left(p,s,t,f(s)1_{\operatorname{supp} g}(t)1_C(p), g(t)1_{\operatorname{supp} f}(s)1_C(p)\right)1_{\mathfrak{A}}(p,s,t) \\
&\quad = K\left(p,s,t,f(s)1_{\operatorname{supp} g}(t)1_C(p)1_{\mathfrak{A}}(p,s,t), g(t)1_{\operatorname{supp} f}(s)1_C(p)1_{\mathfrak{A}}(p,s,t)\right) \\
&\quad = K(p,s,t,f(s)1_{\mathscr{A}}(s)1_{\mathscr{B}}(t)1_{\mathscr{C}}(p), g(t)1_{\mathscr{A}}(s)1_{\mathscr{B}}(t)1_{\mathscr{C}}(p)),
\end{aligned}
\tag{47}
$$

where $\mathscr{A}\in\Sigma$, $\mathscr{A}\subset \operatorname{supp} f$, $\mathscr{B}\in\Xi$, $\mathscr{B}\subset\operatorname{supp} g$, and $\mathscr{C}\in\Theta$. First, we assume that $\mu(\mathscr{A})=0$. Then, we see that

$$
\begin{aligned}
&K(p,s,t,f(s),g(t))<0 \text{ for,} \\
&\lambda\otimes\mu\otimes\nu \text{ almost all } (p,s,t)\in C\times A\times B.
\end{aligned}
\tag{48}
$$

Consider the trivial decompositions $f=f$ and $g=g$. Then, for λ-almost all $p\in C$, we have that

$$
\begin{aligned}
-T(f,g))(p) &= \int_{\operatorname{supp} f\times \operatorname{supp} g} -K(p,s,t,f(s),g(t))\, d(\mu\otimes\nu) \\
&= \int_{\operatorname{supp} f\times \operatorname{supp} g} |K(p,s,t,f(s),g(t))|\, d(\mu\otimes\nu).
\end{aligned}
\tag{49}
$$

The same arguments are valid for the case $\nu(\mathscr{B})=0$. Now, suppose that $\mu(\mathscr{A})>0$ and $\nu(\mathscr{B})>0$. Then, there is a decomposition $C=\mathscr{C}o\mathscr{C}_1 o\mathscr{C}_2$, where $\mathscr{C}\times\mathscr{A}\times\mathscr{B}\subset\mathfrak{A}$, $\mathscr{C}_1\times\mathscr{A}\times\mathscr{B}\subset\mathfrak{B}$, and $\mathscr{C}_2\times\mathscr{A}\times\mathscr{B}\subset\mathfrak{A}\cap\mathfrak{B}$. Clearly $\lambda(\mathscr{C}_2)=0$. Put

$$
\begin{aligned}
&f_1 := f1_{\mathscr{A}};\, f_2 := f - f1_{\mathscr{A}}, \\
&g_1 := g1_{\mathscr{B}};\, g_2 := g - g1_{\mathscr{B}}.
\end{aligned}
\tag{50}
$$

We observe that $C\times(\operatorname{supp} f\setminus\mathscr{A})\times\mathscr{B}\subset\mathfrak{B}$, $C\times\mathscr{A}\times(\operatorname{supp} g\setminus\mathscr{B})\subset\mathfrak{B}$, and $C\times(\operatorname{supp} f\setminus\mathscr{A})\times(\operatorname{supp} g\setminus\mathscr{B})\subset\mathfrak{B}$. Now we may write

$$
\begin{aligned}
&T(f_1,g_1)(p)-T(f_1,g_2)(p)-T(f_2,g_1)(p)-T(f_2,g_2)(p) \\
&\quad = \int_{\operatorname{supp} f\times\operatorname{supp} g} K(p,s,t,f_1(s),g_1(t))d(\mu\otimes\nu) \\
&\qquad - \int_{\operatorname{supp} f\times\operatorname{supp} g} K(p,s,t,f_1(s),g_2(t))d(\mu\otimes\nu) \\
&\qquad - \int_{\operatorname{supp} f\times\operatorname{supp} g} K(p,s,t,f_2(s),g_1(t))d(\mu\otimes\nu) \\
&\qquad - \int_{\operatorname{supp} f\times\operatorname{supp} g} K(p,s,t,f_2(s),g_2(t))d(\mu\otimes\nu) \\
&\quad = \int_{\mathscr{A}\times\mathscr{B}} K(p,s,t,f(s),g(t))\, d(\mu\otimes\nu) \\
&\qquad + \int_{\mathscr{A}\times(\operatorname{supp} g\setminus\mathscr{B})} -K(p,s,t,f(s),g(t))\, d(\mu\otimes\nu) \\
&\qquad + \int_{(\operatorname{supp} f\setminus\mathscr{A})\times\mathscr{B}} -K(p,s,t,f(s),g(t))d(\mu\otimes\nu) \\
&\qquad + \int_{(\operatorname{supp} f\setminus\mathscr{A})\times(\operatorname{supp} g\setminus\mathscr{B})} -K(p,s,t,f(s),g(t))d(\mu\otimes\nu).
\end{aligned}
\tag{51}
$$

Thus, for all $p\in\mathscr{C}$, we have that

$$
\begin{aligned}
&T(f_1,g_1)(p)-T(f_1,g_2)(p)-T(f_2,g_1)(p)-T(f_2,g_2)(p) \\
&\quad = \int_{\mathscr{A}\times\mathscr{B}} |K(p,s,t,f(s),g(t))|d(\mu\otimes\nu) \\
&\qquad + \int_{\mathscr{A}\times(\operatorname{supp} g\setminus\mathscr{B})} |K(p,s,t,f(s),g(t))|d(\mu\otimes\nu) \\
&\qquad + \int_{(\operatorname{supp} f\setminus\mathscr{A})\times\mathscr{B}} |K(p,s,t,f(s),g(t))|d(\mu\otimes\nu) \\
&\qquad + \int_{(\operatorname{supp} f\setminus\mathscr{A})\times(\operatorname{supp} g\setminus\mathscr{B})} |K(p,s,t,f(s),g(t))|d(\mu\otimes\nu).
\end{aligned}
\tag{52}
$$

On the other hand, for all $p\in\mathscr{C}_1$, we have

$$
\begin{aligned}
&T(f_1,g_1)(p)-T(f_1,g_2)(p)-T(f_2,g_1)(p)-T(f_2,g_2)(p) \\
&\quad = \int_{\mathscr{A}\times\mathscr{B}} -|K(p,s,t,f(s),g(t))|d(\mu\otimes\nu) \\
&\qquad + \int_{\mathscr{A}\times(\operatorname{supp} g\setminus\mathscr{B})} |K(p,s,t,f(s),g(t))|d(\mu\otimes\nu) \\
&\qquad + \int_{(\operatorname{supp} f\setminus\mathscr{A})\times\mathscr{B}} |K(p,s,t,f(s),g(t))|d(\mu\otimes\nu) \\
&\qquad + \int_{(\operatorname{supp} f\setminus\mathscr{A})\times(\operatorname{supp} g\setminus\mathscr{B})} |K(p,s,t,f(s),g(t))|d(\mu\otimes\nu).
\end{aligned}
\tag{53}
$$

Consider the second sum $-T(f_1,g_1)-T(f_1,g_2)-T(f_2$

$, g_1) - T(f_2, g_2)$. Then, for all $p \in \mathscr{C}$, we have

$$\begin{aligned} -T(f_1, g_1)(p) - T(f_1, g_2)(p) - T(f_2, g_1)(p) - T(f_2, g_2)(p) &= \int_{\mathscr{A}\times\mathscr{B}} -|K(p,s,t,f(s),g(t))|d(\mu\otimes\nu) \\ &+ \int_{\mathscr{A}\times(\text{supp } g\backslash\mathscr{B})} |K(p,s,t,f(s),g(t))|d(\mu\otimes\nu) \\ &+ \int_{(\text{supp } f\backslash\mathscr{A})\times\mathscr{B}} |K(p,s,t,f(s),g(t))|d(\mu\otimes\nu) \\ &+ \int_{(\text{supp } f\backslash\mathscr{A})\times(\text{supp } g\backslash\mathscr{B})} |K(p,s,t,f(s),g(t))|d(\mu\otimes\nu), \end{aligned} \tag{54}$$

and for all $p \in \mathscr{C}_1$, we have that

$$\begin{aligned} -T(f_1, g_1)(p) - T(f_1, g_2)(p) - T(f_2, g_1)(p) - T(f_2, g_2)(p) &= \int_{\mathscr{A}\times\mathscr{B}} |K(p,s,t,f(s),g(t))|d(\mu\otimes\nu) \\ &+ \int_{\mathscr{A}\times(\text{supp } g\backslash\mathscr{B})} |K(p,s,t,f(s),g(t))|d(\mu\otimes\nu) \\ &+ \int_{(\text{supp } f\backslash\mathscr{A})\times\mathscr{B}} |K(p,s,t,f(s),g(t))|d(\mu\otimes\nu) \\ &+ \int_{(\text{supp } f\backslash\mathscr{A})\times(\text{supp } g\backslash\mathscr{B})} |K(p,s,t,f(s),g(t))|d(\mu\otimes\nu). \end{aligned} \tag{55}$$

Put by definition

$$\begin{aligned} R_1 &= T(f_1, g_1) - T(f_1, g_2) - T(f_2, g_1) - T(f_2, g_2), \\ R_2 &= -T(f_1, g_1) - T(f_1, g_2) - T(f_2, g_1) - T(f_2, g_2). \end{aligned} \tag{56}$$

Then, for all $p \in C$, we have that

$$\begin{aligned} |T|(f,g)(p) \geq (R_1 \vee R_2)(p) &= \int_{\mathscr{A}\times\mathscr{B}} |K(p,s,t,f(s),g(t))|d(\mu\otimes\nu) \\ &+ \int_{\mathscr{A}\times(\text{supp } g\backslash\mathscr{B})} |K(p,s,t,f(s),g(t))|d(\mu\otimes\nu) \\ &+ \int_{(\text{supp } f\backslash\mathscr{A})\times\mathscr{B}} |K(p,s,t,f(s),g(t))|d(\mu\otimes\nu) \\ &+ \int_{(\text{supp } f\backslash\mathscr{A})\times(\text{supp } g\backslash\mathscr{B})} |K(p,s,t,f(s),g(t))|d(\mu\otimes\nu) \\ &= \int_{\text{supp } f\times\text{supp } g} |K(p,s,t,f(s),g(t))|d(\mu\otimes\nu). \end{aligned} \tag{57}$$

Thus, $S \leq |T|$ and S map $E \times F$ to J. On the other hand, since $S \in \mathscr{OBA}_+(E, F; W)$, $S \geq T$, and $S \geq -T$, we have that $S \geq T \vee (-T) = |T|$. □

Corollary 23. *Let (C, Θ, λ), (A, Σ, μ), and (B, Ξ, ν) be finite measure spaces and E and F be ideal subspaces of $L_0(\mu)$ and $L_0(\nu)$, respectively. Then, every integral Uryson operator $T : E \times F \longrightarrow L_0(\lambda)$ is regular.*

Proof. Suppose T is an integral Uryson operator with kernel K. Taking into account that the function

$$p \mapsto \int_{A\times B} |K(p,s,t,f(s),g(t))|d(\mu\otimes\nu), \tag{58}$$

is λ-measurable for each $f \in E$ and $g \in F$ by Theorem 2 we deduce that T is a regular operator.□ □

Proposition 24. *Let (C, Θ, λ), (A, Σ, μ), and (B, Ξ, ν) be finite measure spaces, E, F, and J be ideal subspaces of $L_0(\mu)$, $L_0(\nu)$, and $L_0(\lambda)$, respectively, and $T : E \times F \longrightarrow J$ be a regular integral Uryson operator $T : E \times F \longrightarrow L_0(\lambda)$ with a kernel K. Then, T is a horizontally-to-order continuous operator.*

Proof. By Proposition 15, it is enough to prove the separate horizontal-to-order continuity of T. Fix $g \in F$. We show the horizontal-to-order continuity of the partial operator $T_g : E \longrightarrow J$. It is worth noting that the countable sup property of $L_0(\mu)$ (see [20], page 52) implies that the concept of a sequentially order continuity for functionals and operators coincides with the concept of order continuity. Pick a sequence $(f_n)_{n\in N}$ which horizontally converges to f. We need to show that the sequence $(T_g(f_n))_{n\in N}$ order converges to $T_g(f)$. Taking into the account the regularity of T, we may write

$$\begin{aligned} &\left|\int_{A\times B} K(p,s,t,f(s),g(t))d(\mu\otimes\nu) - \int_{A\times B} K(p,s,t,f_n(s),g(t))d(\mu\otimes\nu)\right| \\ &= \left|\int_{A\times B} K(p,s,t,((f-f_n)\sqcup(f_n))(s),g(t))d(\mu\otimes\nu) - \int_{A\times B} K(p,s,t,f_n(s),g(t))\,d(\mu\otimes\nu)\right| \\ &= \left|\int_{A\times B} K(p,s,t,(f-f_n)(s),g(t))d(\mu\otimes\nu)\right| \\ &\leq \int_{A\times B} |K(p,s,t,(f-f_n)(s),g(t))|d(\mu\otimes\nu) \\ &= \int_{\text{supp } (f-f_n)\times\text{supp } g} |K(p,s,t,f(s),g(t))|d(\mu\otimes\nu). \end{aligned} \tag{59}$$

□ □

Since $\mu(\text{supp } (f - f_n))$ converges to 0 by ([30], Theorem 2.5.7), we have that $(T_g(f - f_n))_{n\in N}$ order converges to 0, and therefore, T_g is a horizontally-to-order continuous operator. Similar arguments are valid for a partial operator T_f, $f \in E$.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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