

## Research Article

# Matrix Structure of Jacobsthal Numbers

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The main scenario of this paper is to introduce a new sequence of Jacobsthal type having a generalized order  $j$ . Some basic properties will be studied concerning it. Also, we will establish the generalized Binet formula.

## 1. Background and Introduction

The Fibonacci sequence is an integer sequence plays a vital role for many fascinating identities. In nature, it shows its presence, even if certain fruits are looked at, the number of little bumps around each ring is counted or the sand on the beach, and how waves hit it is watched out, the Fibonacci sequence is seen there. It was studied by many authors in the well-known systematic manner, and attractive investigations have been witnessed as can be seen in [1–4]. Further, several recurrence sequences of natural numbers have been object of study for many researchers. Illustrations of these are the Fibonacci, Lucas, Pell, Pell-Lucas, Modified Pell, Jacobsthal, and Jacobsthal-Lucas sequences among others as can be seen in [5–12].

It is well known that the Jacobsthal numbers obey attracting structure in many fields of science, engineering and technology as can be seen in [13–15] and many others. The authors in [16, 17] have defined the Jacobsthal numbers  $J_n$  by the following recurrence relation:

$$J_0 = 0, J_1 = 1, J_{n+2} = J_{n+1} + 2J_n, \quad n \geq 0. \quad (1)$$

The author in [18] has shown that some interesting properties of Fibonacci sequence can be obtained from a

matrix description. For a  $j$ th Fibonacci number  $v_j$ , he proved that for

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \quad (2)$$

that

$$A^n \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} v_n \\ v_{n+1} \end{pmatrix}. \quad (3)$$

It is obvious that the Jacobsthal sequence is a particular demonstration of a sequence given recursively as follows:

$$a_{r+j} = c_0 a_r + c_1 a_{r+1} + \dots + c_{j-1} a_{r+j-1}, \quad (4)$$

where  $c_0, c_1, \dots, c_{j-1}$  are real constants. The author in [10] has determined a closed-form formula for the generalized

sequence by companion matrix method as follows:

$$\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ c_0 & c_1 & c_2 & \cdots & c_j & c_{j-1} \end{pmatrix}. \tag{5}$$

Then, by an inductive argument, the generalization of (3) will be obtained, viz.,

$$A^n \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_{j-1} \end{pmatrix} = \begin{pmatrix} a_n \\ a_{n+1} \\ \vdots \\ a_{n+j-1} \end{pmatrix}, \tag{6}$$

where  $a_n$  is the  $n$ th term of the sequence.

It is well established fact that the linear recurrence relations play a vital role of number theory. They show their appearance in almost everywhere in mathematics and computer science as can be found in [17].

In [15], the authors have studied Jacobsthal  $F$ -matrix as follows:

$$F = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}, \tag{7}$$

and proved for any natural numbers that

$$F^n = \begin{pmatrix} J_{n+1} & 2J_n \\ J_n & 2J_{n-1} \end{pmatrix}. \tag{8}$$

Quite recently, for  $n > 0$  and  $1 \leq r \leq j$ , the authors in [19] have defined the  $j$  sequences of the generalized order  $j$ -Jacobsthal numbers as follows:

$$J_n^r = J_{n-1}^r + 2J_{n-2}^r + J_{n-3}^r + \cdots + J_{n-j}^r, \tag{9}$$

with

$$J_n^r = \begin{cases} 1, & \text{if } r = 1 - n, \\ 0, & \text{otherwise,} \end{cases} \tag{10}$$

for  $1 - j \leq n \leq 0$  and  $J_n^r$  is the  $n$ th term of the  $r$ th sequence and was shown that the fundamental recurrence relation (9) can

be defined by the vector recurrence relation

$$\begin{pmatrix} J_{n+1}^r \\ J_n^r \\ J_{n-1}^r \\ \vdots \\ J_{n-j+2}^r \end{pmatrix} = C \begin{pmatrix} J_n^r \\ J_{n-1}^r \\ J_{n-2}^r \\ \vdots \\ J_{n-j+1}^r \end{pmatrix}, \tag{11}$$

for the generalized order  $j$ -Jacobsthal sequences, where  $C$

$$C = (c_{ij})_{j \times j} = \begin{pmatrix} 1 & 2 & 1 & \cdots & 1 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix} \tag{12}$$

and is known as the generalized order  $j$ -Jacobsthal matrix.

## 2. Main Results

In this section, we present some new generalization of Jacobsthal numbers in the form of a matrix and prove some basic properties.

Following the authors in [2, 5, 20–24], we define, for  $n > 0$ ,  $1 \leq r \leq j$ , and  $s \geq 0$ , the  $j$  sequences of the generalized order  $j$ -Jacobsthal numbers as follows:

$$w_n^r = w_{n-1}^r + 2^s w_{n-2}^r + w_{n-3}^r + \cdots + w_{n-j}^r, \tag{13}$$

with initial conditions as

$$w_n^r = \begin{cases} 1, & \text{if } r = 1 - n, \\ 0, & \text{otherwise,} \end{cases} \tag{14}$$

for  $1 - j \leq n \leq 0$  and  $w_n^r$  is the  $n$ th term of the  $r$ th generalized Jacobsthal sequence.

For different values of  $s$ , we have following deductions:

*Deduction 1.* Choosing  $s = 0$ , the sequence  $w_n^r$  gets reduced to the generalized order- $j$  Fibonacci sequence [25].

*Deduction 2.* Choosing  $s = 1$ , we get the results obtained in [19].

We can redefine it by the vector recurrence relation as follows:

$$\begin{pmatrix} w_{n+1}^r \\ w_n^r \\ w_{n-1}^r \\ \vdots \\ w_{n-j+2}^r \end{pmatrix} = \mathfrak{C} \begin{pmatrix} w_n^r \\ w_{n-1}^r \\ w_{n-2}^r \\ \vdots \\ w_{n-j+1}^r \end{pmatrix}, \tag{15}$$

for the generalized order  $j$ -Jacobsthal sequences, where  $\mathfrak{C}$

$$\mathfrak{C} = (c_{ij})_{j \times j} = \begin{pmatrix} 1 & 2^s & 1 & \cdots & 1 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix} \tag{16}$$

and this matrix  $\mathfrak{C}$  is known as the generalized order  $j$ -Jacobsthal matrix. We now define a  $j \times j$  matrix  $\mathfrak{B}_n = (\mu_{rm})$  as follows:

$$\mathfrak{B}_n = \begin{pmatrix} w_n^1 & w_n^2 & \cdots & w_n^j \\ w_{n-1}^1 & w_{n-1}^2 & \cdots & w_{n-1}^j \\ \vdots & \vdots & \vdots & \ddots \\ w_{n-j+1}^1 & w_{n-j+1}^2 & \cdots & w_{n-j+1}^j \end{pmatrix}. \tag{17}$$

Then, clearly, we get the following matrix equation by expanding (15) to  $j$  columns:

$$\mathfrak{B}_{n+1} = \mathfrak{C} \cdot \mathfrak{B}_n. \tag{18}$$

In this direction, we have following result:

**Theorem 3.** Define  $j \times j$  matrices  $\mathfrak{C}$  and  $\mathfrak{B}_n$ , respectively, given by (16) and (17); then for every  $n \geq 0$ , we have

$$\mathfrak{B}_n = \mathfrak{C}^n. \tag{19}$$

*Proof.* Using (17), we see that  $\mathfrak{B}_n = \mathfrak{C} \cdot \mathfrak{B}_{n-1}$ .

Now employing the inductive argument, we can write  $\mathfrak{B}_n = \mathfrak{C}^{n-1} \cdot \mathfrak{B}_1$ . But by definition of generalized order  $j$ -Jacobsthal numbers,  $\mathfrak{B}_1 = \mathfrak{C}$ , and consequently, we have  $\mathfrak{B}_n = \mathfrak{C}^n$ , as required.  $\square$

**Theorem 4.** Define  $j \times j$  matrix  $\mathfrak{B}_n$  given by (17); then

$$\det \mathfrak{B}_n = \begin{cases} -2^s, & \text{if } j = 2, \\ 1, & \text{if } j \text{ is odd,} \\ -1, & \text{if } j \text{ is even with } k \neq 2. \end{cases} \tag{20}$$

*Proof.* We know by using result Theorem 3 that  $\mathfrak{B}_n = \mathfrak{C}^n$ . Hence, we see that

$$\det \mathfrak{B}_n = \det \mathfrak{C}^n = (\det \mathfrak{C})^n. \tag{21}$$

Consequently, the result follows by using Laplace expansion of determinants along any column.  $\square$

**Lemma 5.** Let  $w_n^r$  be the generalized order  $j$ -Jacobsthal number; then,

$$\begin{aligned} w_{n+1}^1 &= w_n^1 + w_n^2, \\ w_{n+1}^2 &= 2^s w_n^1 + w_n^3, \\ w_{n+1}^r &= w_n^1 + w_n^{r+1}; \quad 3 \leq r \leq j-1, \\ w_{n+1}^j &= w_n^j. \end{aligned} \tag{22}$$

*Proof.* We know by using Theorem 3 that  $\mathfrak{B}_n = \mathfrak{C}^n$ . Hence, we can write

$$\mathfrak{B}_n = \mathfrak{B}_{n-1} \mathfrak{B}_1. \tag{23}$$

Consequently, the result follows by using Laplace expansion of determinants along any column.  $\square$

### 3. Generalized Binet Formula (GBF)

This part of the article deals with the derivation of GBF for generalized order  $j$ -Jacobsthal numbers.

In 1843, it was Binet who derived the interesting formula using Fibonacci numbers:

$$F_n = \frac{a^n - b^n}{a - b}, \tag{24}$$

where the values of  $a$  and  $b$  are  $(1 \pm \sqrt{5})/2$ . Moreover, the attractive Binet formula for generalized Fibonacci numbers is studied in [12].

It is well know from the concept of companion matrices that the characteristic equation of the matrix  $\mathfrak{C}$  defined by (16) is

$$\zeta^j - \zeta^{j-1} - 2^s \zeta^{j-2} - \zeta^{j-3} - \cdots - \zeta - 1 = 0, \tag{25}$$

which is also the characteristic equation of generalized order  $j$ -Jacobsthal numbers.

In order to prove our main result of this section, we first define the following lemma without proof:

**Lemma 6.** For  $s \geq 0$ , the equation

$$\zeta^j - \zeta^{j-1} - 2^s \zeta^{j-2} - \zeta^{j-3} - \cdots - \zeta - 1 = 0 \tag{26}$$

does not have multiple roots with  $j \geq 3$ .

Let  $f(\eta)$  be the characteristic polynomial of the generalized order  $j$ -Jacobsthal matrix  $\mathfrak{C}$ . Let  $\eta_1, \eta_2, \dots, \eta_j$  be the eigenvalues of  $\mathfrak{C}$ , which are clearly all distinct by Lemma 6. Let  $j$

$\times j$  Vandermonde matrix  $\mathcal{V}$  be given by

$$\mathcal{V} = \begin{pmatrix} \eta_1^{j-1} & \eta_1^{j-2} & \cdots & \eta_1 & 1 \\ \eta_2^{j-1} & \eta_2^{j-2} & \cdots & \eta_2 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \eta_j^{j-1} & \eta_j^{j-2} & \cdots & \eta_j & 1 \end{pmatrix}. \quad (27)$$

Also, let  $\theta_j^r$  be a  $j \times 1$  matrix as follows:

$$\theta_j^r = \begin{pmatrix} \eta_1^{n+j-r} \\ \eta_2^{n+j-r} \\ \vdots \\ \eta_j^{n+j-r} \end{pmatrix}. \quad (28)$$

and  $\mathcal{V}_m^{(r)}$  be a  $j \times j$  matrix obtained from  $\mathcal{V}$  by replacing  $m$ th column of  $\mathcal{V}$  by  $\theta_j^r$ .

**Theorem 7.** For  $1 \leq r \leq j$ , let  $w_n^r$  be the  $n$ th term of  $r$ th Jacobsthal sequence; then,

$$w_{n-r+1}^k = \frac{\det(\mathcal{V}_m^{(r)})}{\det \mathcal{V}}. \quad (29)$$

*Proof.* As we know that matrix  $\mathfrak{C}$  has all distinct eigenvalues, consequently, it is diagonalizable. Denote  $\mathcal{V}^T = \Omega$ , and obviously,  $\Omega$  is invertible; then,  $\Omega^{-1}\mathfrak{C}\Omega = \mathfrak{G}$ , where  $\mathfrak{G}$  is given by

$$\mathfrak{G} = \text{diag}(\eta_1, \eta_2, \dots, \eta_j). \quad (30)$$

Consequently,  $\mathfrak{C}$  is similar to  $\Omega$  and hence yields  $\mathfrak{C}^n \Omega = \Omega \mathfrak{G}^n$ . But using Theorem 3, we know that  $\mathfrak{B}_n = \mathfrak{C}^n$ . We thus have the following system of equations:

$$\begin{aligned} \mu_{r1} \eta_1^{j-1} + \mu_{r2} \eta_1^{j-2} + \cdots + \mu_{rj} &= \eta_1^{n+j-r} \\ \mu_{r1} \eta_2^{j-1} + \mu_{r2} \eta_2^{j-2} + \cdots + \mu_{rj} &= \eta_2^{n+j-r} \\ &\vdots \\ \mu_{r1} \eta_j^{j-1} + \mu_{r2} \eta_j^{j-2} + \cdots + \mu_{rj} &= \eta_j^{n+j-r} \end{aligned}, \quad (31)$$

where  $\mathfrak{B} = (\mu_{rm})_{j \times j}$ . Hence, for each  $m = 1, 2, \dots, j$  giving

$$\mu_{rm} = \frac{\det(\mathcal{V}_m^{(r)})}{\det \mathcal{V}}, \quad (32)$$

by observing that  $\mu_{rm} = w_{n-r+1}^m$  and the result follows.  $\square$

## Data Availability

In this manuscript, we have approached generalized Binet formula in a different way, and we concerned  $j$ -Jacobsthal matrix given in the paper in a generalized way.

## Conflicts of Interest

The authors declare that they have no conflicts of interest.

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