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Research Article

Convolutions of Harmonic Mappings Convex in the Horizontal Direction

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In this paper, we establish some results concerning the convolutions of harmonic mappings convex in the horizontal direction with harmonic vertical strip mappings. Furthermore, we provide examples illustrated graphically with the help of Maple to illuminate the results.

1. Introduction

For real-valued harmonic functions u and v in the open unit disk $\mathbb{E}=\{z\in\mathbb{C}:|z|<1\}$, the complex-valued continuous function $f=u+iv$ is said to be harmonic and can be expressed as $f=h+\bar{g}$, where h and g are analytic in $\mathbb{E}$. Let H be the class of harmonic mappings $f=h+\bar{g}$ normalized by $h(0)=g(0)=h'(0)-1=0$ and have the following power series representations:

$$h(z)=z+\sum_{m=2}^{\infty}a_m z^m,\qquad g(z)=\sum_{m=1}^{\infty}b_m z^m. \tag{1}$$

We call h the analytic part and g the coanalytic part of f, respectively. The Jacobian of $f=h+\bar{g}$ is given by $J_f=|h'|^2-|g'|^2$. Lewy's theorem [1] implies that $f\in H$ is locally univalent and sense-preserving if and only if $J_f>0$ in $\mathbb{E}$. The condition $J_f>0$ is equivalent to that dilatation $\omega(z)=g'(z)/h'(z)$ satisfying $|\omega(z)|<1$ for all $z\in\mathbb{E}$ (see [2, 3]).

We denote by $\mathbb{S}_H$ the class of all harmonic, sense-preserving, and univalent mappings $f=h+\bar{g}$ in $\mathbb{E}$, which are normalized by the condition $h(0)=g(0)=0$ and $h'(0)=1$. Let $\mathbb{S}_H^0$ be the subset of all $f\in\mathbb{S}_H$ in which $g'(0)=0$. Further, let $\mathbb{K}_H,\mathbb{C}_H$ (resp., $\mathbb{K}_H^0,\mathbb{C}_H^0$) be the subset of $\mathbb{S}_H$ (resp., $\mathbb{S}_H^0$) whose images are convex and close-to-convex domains. A domain Ω is said to be convex in the horizontal direction (CHD) if the intersection of Ω with each horizontal line is connected (or empty). A function $f=h+\bar{g}\in\mathbb{S}_H$ is said to be a CHD mapping if f maps $\mathbb{E}$ onto a CHD domain. Let $\mathbb{S}_{\text{CHD}}$ be the subset of $\mathbb{C}_H$ which consist of CHD mappings. The following basic theorem of Clunie and Sheil-Small [2] is known as shear construction that constructs harmonic mappings with prescribed dilatations onto a domain convex in one direction.

Theorem 1 (see [2]). *A locally univalent harmonic mapping $f=h+\bar{g}$ in $\mathbb{E}$ is a univalent mapping of $\mathbb{E}$ onto a domain convex in a direction ϕ if and only if $h-e^{2i\phi}g$ is a conformal univalent mapping of $\mathbb{E}$ onto a domain convex in the direction of ϕ.*

Let $f*F=h*H+\overline{g*G}$ be the convolution of two harmonic functions $f=h+\bar{g}$ and $F=H+\bar{G}$ where the operator $*$ is convolution (or Hadamard product) of two power series.

There are several research papers in recent years which investigate the convolution of harmonic univalent functions. In particular, Dorff [4] and Dorff et al. [5] studied the convolution of harmonic univalent mappings in the right half-plane. For some recent investigations involving convolution of harmonic mappings, we refer the reader to [6–13].

Let $F_{\mathfrak{a}} = H_{\mathfrak{a}} + \overline{G_{\mathfrak{a}}}$ sheared by $H_{\mathfrak{a}} - G_{\mathfrak{a}} = z/(1-z)$ with the dilatation $\omega_{\mathfrak{a}} = (\mathfrak{a}+z)/(1+\mathfrak{a}z)$, where $\mathfrak{a} \in (-1,1)$. Using shear construction of Clunie and Sheil-Small [2], we have

$$H_{\mathfrak{a}}(z) = \frac{1/(1-\mathfrak{a})z - 1/2z^2}{(1-z)^2} = \frac{1}{2}\left[\frac{z}{1-z} + \frac{1+\mathfrak{a}}{1-\mathfrak{a}}\frac{z}{(1-z)^2}\right], \tag{2}$$

$$G_{\mathfrak{a}}(z) = \frac{\mathfrak{a}/(1-\mathfrak{a})z + 1/2z^2}{(1-z)^2} = \frac{1}{2}\left[\frac{-z}{1-z} + \frac{1+\mathfrak{a}}{1-\mathfrak{a}}\frac{z}{(1-z)^2}\right]. \tag{3}$$

It is clear that by setting $\mathfrak{a}=0$ in (2) and (3), we obtain $F_0 = H_0 + \overline{G_0}$ which satisfy the conditions $H_0 - G_0 = z/(1-z)$ and $\omega(z) = z$, studied by Liu and Li [8]. Wang et al. [14] also studied convolutions of this mapping. Note that $F_{\mathfrak{a}}$ is a CHD mapping.

Recently, Liu and Li [8] introduced the following generalized harmonic univalent mappings:

$$P_\delta(z) = H_\delta(z) + \overline{G_\delta(z)} = \frac{1}{1+\delta}\left[\frac{\delta z}{(1-z)^2} + \frac{z}{1-z}\right] + \frac{1}{1+\delta}\overline{\left[\frac{\delta z}{(1z)^2}\frac{z}{1z}\right]}, \tag{4}$$

where $\delta > 0$ and $z \in \mathbb{E}$. Obviously, $P_1(z) = F_0(z)$. If $f = h + \bar{g} \in \mathbb{S}_H$, then

$$P_\delta * f = \frac{\delta z h' + h}{1+\delta} + \overline{\frac{\delta z g' g}{1+\delta}}. \tag{5}$$

Also, $P_\delta(z)$ maps $\mathbb{E}$ onto the domain $\{u+iv : v^2 > -[(2\delta)/(1+\delta)u + (1/(1+\delta)^2)],\ \delta > 0\}$ which is a CHD domain. Very recently, Yasar and Ozdemir [15] studied convolutions of these generalized harmonic mappings.

Let $f_\gamma = h_\gamma + \overline{g_\gamma} \in \mathbb{S}^0_{\mathrm{CHD}}$ with

$$h_\gamma - g_\gamma = \frac{1}{2i\sin\gamma}\log\left(\frac{1+ze^{i\gamma}}{1+ze^{-i\gamma}}\right), \tag{6}$$

where $\pi/2 \le \gamma < \pi$.

In this paper, we investigate the conditions under which the convolutions of harmonic mappings P_δ, f_γ, and F_a with prescribed dilatations are univalent and CHD provided that the convolutions are locally univalent and sense-preserving. Furthermore, we provide two examples illustrated graphically with the help of Maple to illuminate our results.

2. Preliminary Results

Lemma 2 (see [16]). *Let f be an analytic fuction in $\mathbb{E}$ with $f(0) = 0$ and $f'(0) \neq 0$ and let*

$$\varphi(z) = \frac{z}{(1+ze^{i\theta_1})(1+ze^{i\theta_2})}, \tag{7}$$

where $\theta_1, \theta_2 \in \mathbb{R}$. If

$$\operatorname{Re}\left(\frac{zf'(z)}{\varphi(z)}\right) > 0, \tag{8}$$

then f is convex in the horizontal direction.

Lemma 3 (see [17]). *Let φ and G be analytic in $\mathbb{E}$ with $\varphi'(0) = G(0) = 0$. If φ is convex and G is starlike, then for each function F analytic in $\mathbb{E}$ and satisying* Re $(F(z)) > 0$, *we have*

$$\operatorname{Re}\left(\frac{(\varphi * FG)(z)}{(\varphi * G)(z)}\right) > 0 (z \in \mathbb{E}). \tag{9}$$

Lemma 4 ([18], Cohn's rule). *Given a polynomial*

$$p(z) = p_0(z) = a_{k,0}z^k + a_{k-1,0}z^{k-1} + \cdots + a_{1,0}z + a_{0,0}\ (a_{k,0} \neq 0) \tag{10}$$

of degree k, let

$$p^*(z) = p_0^*(z) = z^k\overline{p\left(\frac{1}{\bar{z}}\right)} = \overline{a_{k,0}} + \overline{a_{k1,0}}z + \cdots + \overline{a_{1,0}}z^{k-1} + \overline{a_{0,0}}z^k\ (a_{k,0} \neq 0). \tag{11}$$

Denote by r and s the number of zeros of $p(z)$ inside the unit circle and on it, respectively. If $|a_{0,0}| < |a_{k,0}|$, then

$$p_1(z) = \frac{\overline{a_{k,0}}p(z) - a_{0,0}p^*(z)}{z} \tag{12}$$

is of degree $k-1$ with $r_1 = r-1$ and $s_1 = s$ the number of zeros of $p_1(z)$ inside the unit circle and on it, respectively.

Lemma 5. *Let $P_\delta = H_\delta + \overline{G_\delta}$ be defined by (4) and $f_\gamma = h_\gamma + \overline{g_\gamma}$ be defined by (6) with dilatation $\omega = g'_\gamma/h'_\gamma$. Then the dilatation of $P_\delta * f_\gamma$ is given by*

$$\tilde{\omega}(z) = \frac{\omega(1-\omega)\left[(\delta-1) - 2z\cos\gamma - (\delta+1)z^2\right] + \delta z\omega'\left(1+2z\cos\gamma+z^2\right)}{(1-\omega)[(\delta+1) + 2z\cos\gamma - (\delta-1)z^2] + \delta z\omega'(1+2z\cos\gamma+z^2)}. \tag{13}$$

Proof. Since $h_\gamma - g_\gamma = 1/(2i\sin\gamma)\log((1+ze^{i\gamma})/(1+ze^{-i\gamma}))$ $(\pi/2 \le \gamma < \pi)$ and $g'_\gamma = \omega h'_\gamma$, then $g''_\gamma = \omega' h'_\gamma + \omega h''_\gamma$. We immediately get

$$h'_\gamma = \frac{1}{(1-\omega)(1+ze^{i\gamma})(1+ze^{-i\gamma})}, \tag{14}$$

$$h''_\gamma = -\frac{2(\cos\gamma + z)(1-\omega) - \omega'\left(1+2z\cos\gamma+z^2\right)}{(1-\omega)^2(1+ze^{i\gamma})^2(1+ze^{-i\gamma})^2}. \tag{15}$$

From (4), we have

$$\begin{aligned}\tilde{\omega}(z) &= \frac{\left(G_\delta * g_\gamma\right)'}{\left(H_\delta * h_\gamma\right)'} = \frac{\left(\delta z g'_\gamma - g_\gamma\right)'}{\left(\delta z h'_\gamma + h_\gamma\right)'} = \frac{(\delta-1)g'_\gamma + \delta z g''_\gamma}{(\delta+1)h'_\gamma + \delta z h''_\gamma} = \frac{(\delta-1)\omega h'_\gamma + \delta z\left(\omega' h'_\gamma + \omega h''_\gamma\right)}{(\delta+1)h'_\gamma + \delta z h''_\gamma} \\ &= \frac{\omega(1-\omega)\left[(\delta-1) - 2z\cos\gamma - (\delta+1)z^2\right] + \delta z\omega'\left(1+2z\cos\gamma+z^2\right)}{(1-\omega)\left[(\delta+1) + 2z\cos\gamma - (\delta-1)z^2\right] + \delta z\omega'\left(1+2z\cos\gamma+z^2\right)}.\end{aligned} \tag{16}$$

□

Lemma 6. *Let $P_\delta = H_\delta + \overline{G_\delta}$ be defined by (4) and $f_\gamma = h_\gamma + \overline{g_\gamma}$ be defined by (6). If $P_\delta * f_\gamma$ is locally univalent and sense-preserving, then $P_\delta * f_\gamma$ is univalent and convex in the horizontal direction.*

Proof. Let

$$\begin{aligned}F_1 &= (H_\delta - G_\delta) * \left(h_\gamma + g_\gamma\right) \\ &= H_\delta * h_\gamma + H_\delta * g_\gamma - G_\delta * h_\gamma - G_\delta * g_\gamma, \\ F_2 &= (H_\delta + G_\delta) * \left(h_\gamma - g_\gamma\right) \\ &= H_\delta * h_\gamma - H_\delta * g_\gamma + G_\delta * h_\gamma - G_\delta * g_\gamma.\end{aligned} \tag{17}$$

Thus,

$$H_\delta * h_\gamma - G_\delta * g_\gamma = \frac{1}{2}(F_1 + F_2). \tag{18}$$

By Theorem 1, we need to prove that $1/2(F_1+F_2)$ is convex in the horizontal direction. Since

$$h_\gamma - g_\gamma = \frac{1}{2i\sin\gamma}\log\left(\frac{1+ze^{i\gamma}}{1+ze^{-i\gamma}}\right)\left(\frac{\pi}{2} \le \gamma < \pi\right), \tag{19}$$

we have

$$\begin{aligned}zF'_1 &= (H_\delta - G_\delta) * \left[z\left(h'_\gamma + g'_\gamma\right)\right] \\ &= (H_\delta - G_\delta) * \left[z\left(h'_\gamma - g'_\gamma\right)\left(\frac{h'_\gamma + g'_\gamma}{h'_\gamma - g'_\gamma}\right)\right] \\ &= \frac{2z}{(1+\delta)(1-z)} * \frac{z}{(1+ze^{i\gamma})(1+ze^{-i\gamma})}\left(\frac{1+\omega_\gamma}{1-\omega_\gamma}\right) \\ &= \frac{2zp_1(z)}{(1+\delta)(1+ze^{i\gamma})(1+ze^{-i\gamma})},\end{aligned} \tag{20}$$

where $p_1(z) = (1+\omega_\gamma)/(1-\omega_\gamma)$ satisfies the condition $\mathrm{Re}\,(p_1(z)) > 0$. Thus, we have

$$\mathrm{Re}\left\{\frac{zF'_1}{2z/[(1+\delta)(1+ze^{i\gamma})(1+ze^{-i\gamma})]}\right\} = \mathrm{Re}\,\{p_1(z)\} > 0. \tag{21}$$

Now, we consider

$$\begin{aligned}zF'_2 &= \left[z\left(H'_\delta + G'_\delta\right) * \left(h_\gamma - g_\gamma\right)\right] \\ &= \left[z\left(H'_\delta - G'_\delta\right)\frac{H'_\delta + G'_\delta}{H'_\delta - G'_\delta}\right] * \left(h_\gamma - g_\gamma\right) \\ &= \left[z\left(H'_\delta - G'_\delta\right)\left(\frac{1+\omega_\delta}{1-\omega_\delta}\right)\right] * \left(h_\gamma - g_\gamma\right) \\ &= \frac{2zp_2(z)}{(1+\delta)(1-z)^2} * \left(h_\gamma - g_\gamma\right),\end{aligned} \tag{22}$$

where $p_2(z) = (1+\omega_\delta)/(1-\omega_\delta)$ satisfies the condition $\mathrm{Re}\,\{p_2(z)\} > 0$. Using the fact that

$$\psi(z) * \frac{z}{(1-z)^2} = z\psi'(z) \tag{23}$$

and $h_\gamma - g_\gamma$ is convex, by Lemma 3, we have

$$\begin{aligned}&\mathrm{Re}\left\{\frac{zF'_2}{z/[(1+ze^{i\gamma})(1+ze^{-i\gamma})]}\right\} \\ &\quad = \mathrm{Re}\left\{\frac{\left(h_\gamma - g_\gamma\right) * p_2(z)\left[2z/((1+\delta)(1-z)^2)\right]}{z\left(h'_\gamma - g'_\gamma\right)}\right\} \\ &\quad = \mathrm{Re}\left\{\frac{\left(h_\gamma - g_\gamma\right) * p_2(z)\left[2z/((1+\delta)(1-z)^2)\right]}{\left(h_\gamma - g_\gamma\right) * z/(1-z)^2}\right\} > 0.\end{aligned} \tag{24}$$

□

Finally, using Lemma 2, we obtain that $F_1 + F_2$ is convex in the horizontal direction.

Lemma 7. *Let $f_\gamma = h_\gamma + g_\gamma \in \mathbb{S}^0_{CHD}$ be given by (6) with dilatation $\omega = g'_\gamma / h'_\gamma$ and $F_\mathfrak{a} = H_\mathfrak{a} + \overline{G_\mathfrak{a}}$ be a mapping defined by (2) and (3). Then the dilatation of $F_\mathfrak{a} * f_\gamma$ is given by*

$$\tilde{W}(z) = \frac{2\omega(1-\omega)\left[\mathfrak{a} - (1-\mathfrak{a})z\cos\gamma - z^2\right] + (1+\mathfrak{a})z\omega'\left(1 + 2z\cos\gamma + z^2\right)}{2(1-\omega)[1 + (1-\mathfrak{a})z\cos\gamma - \mathfrak{a}z^2] + (1+\mathfrak{a})z\omega'(1 + 2z\cos\gamma + z^2)}. \tag{25}$$

Proof. From (2) and (3), we have

$$\tilde{W}(z) = \frac{\left(G_\mathfrak{a} * g_\gamma\right)'}{\left(H_\mathfrak{a} * h_\gamma\right)'} = \frac{\left((1+\mathfrak{a})zg'_\gamma - (1-\mathfrak{a})g_\gamma\right)'}{\left((1+\mathfrak{a})zh'_\gamma + (1-\mathfrak{a})h_\gamma\right)'}$$

$$= \frac{2\mathfrak{a}g'_\gamma + (1+\mathfrak{a})zg''_\gamma}{2h'_\gamma + (1+\mathfrak{a})zh''_\gamma} = \frac{2\mathfrak{a}\omega h'_\gamma + (1+\mathfrak{a})z\left(\omega' h'_\gamma + \omega h''_\gamma\right)}{2h'_\gamma + (1+\mathfrak{a})zh''_\gamma}. \tag{26}$$

Using (14) and (15), then we obtain the dilatation of $F_\mathfrak{a} * f_\gamma$ as follows:

$$\tilde{W}(z) = \frac{2\omega(1-\omega)\left[\mathfrak{a} - (1-\mathfrak{a})z\cos\gamma - z^2\right] + (1+\mathfrak{a})z\omega'\left(1 + 2z\cos\gamma + z^2\right)}{2(1-\omega)[1 + (1-\mathfrak{a})z\cos\gamma - \mathfrak{a}z^2] + (1+\mathfrak{a})z\omega'(1 + 2z\cos\gamma + z^2)}. \tag{27}$$

□

Lemma 8 ([14], Lemma 2.4). *Let $F_\mathfrak{a} = H_\mathfrak{a} + \overline{G_\mathfrak{a}}$ be a mapping defined by (2), (3) and $f_\gamma = h_\gamma + \overline{g_\gamma} \in \mathbb{S}^0_{CHD}$ be defined by (6). If $F_\mathfrak{a} * f_\gamma$ is locally univalent and sense-preserving, then $F_\mathfrak{a} * f_\gamma$ is univalent and convex in the horizontal direction.*

Lemma 9 ([19], Gauss-Lucas theorem). *Let $T(z)$ be a nonconstant polynomial with complex coefficients. Then, the zeros of the derivative $T'(z)$ are contained in the convex hull of the set of the zeros of $T(z)$.*

Lemma 10. *Let*

$$\begin{aligned} q(z) = z^{k+1} &+ \frac{2(k+1)\cos\gamma}{k+2} z^k + \frac{k}{k+2} z^{k-1} \\ &- \frac{2}{k+2} e^{-i\theta} z - \frac{2\cos\gamma}{k+2} e^{-i\theta} \end{aligned} \tag{28}$$

be a complex polynomial of degree $k+1$, where $\theta \in \mathbb{R}$, $k \in \mathbb{N}^+$, and $\pi/2 \le \gamma < \pi$. Then, all zeros of $q(z)$ lie in the closed unit disk $|z| \le 1$.

Proof. Note that $q(z) = 1/(k+2)T'(z)$, where

$$T(z) = \left(z^k - e^{-i\theta}\right)\left(1 + 2z\cos\gamma + z^2\right). \tag{29}$$

It is obvious that the roots of $(z^k - e^{-i\theta})$ lie on the unit circle. Also, $-\cos\gamma \pm i\sin\gamma$ which are the roots of $(1 + 2z\cos\gamma + z^2)$ lie on the unit circle as well. Hence, the result follows from Lemma 9. □

3. Main Results

Theorem 11. *Let $P_\delta = H_\delta + \overline{G_\delta} \in \mathbb{S}_{CHD}$ be a mapping given by (4) and $f_\gamma = h_\gamma + \overline{g_\gamma} \in \mathbb{S}^0_{CHD}$ be given by (6) with the dilatation $\omega_k = g'_\gamma / h'_\gamma = e^{i\theta} z^k (\theta \in \mathbb{R}, k \in \mathbb{N}^+)$. Then $P_\delta * f_\gamma$ is univalent and convex in the horizontal direction.*

Proof. By Lemma 6, we need to prove that the dilatation $\tilde{\omega}$ of $P_\delta * f_\gamma$ satisfies $|\tilde{\omega}| < 1$ for all $z \in \mathbb{E}$. Substituting $\omega = e^{i\theta} z^k$ in (13), we yield

$$\tilde{\omega}(z) = \frac{e^{i\theta}z^k\left(1 - e^{i\theta}z^k\right)\left[(\delta-1) - 2z\cos\gamma - (\delta+1)z^2\right] + \delta k e^{i\theta} z^k\left(1 + 2z\cos\gamma + z^2\right)}{\left(1 - e^{i\theta}z^k\right)\left[(\delta+1) + 2z\cos\gamma - (\delta-1)z^2\right] + \delta k e^{i\theta} z^k (1 + 2z\cos\gamma + z^2)} = e^{2i\theta} z^k \frac{t(z)}{t^*(z)}, \tag{30}$$

where

$$t(z) = z^{k+2} + \frac{2\cos\gamma}{1+\delta} z^{k+1} + \frac{1-\delta}{1+\delta} z^k + \frac{\delta(k-1)-1}{1+\delta} e^{-i\theta} z^2 + \frac{2(k\delta-1)\cos\gamma}{1+\delta} e^{-i\theta} z + \frac{\delta(1+k)-1}{1+\delta} e^{-i\theta}, \quad (31)$$

$$t^*(z) = 1 + \frac{2\cos\gamma}{1+\delta} z + \frac{1-\delta}{1+\delta} z^2 + \frac{\delta(k-1)-1}{1+\delta} e^{i\theta} z^k + \frac{2(k\delta-1)\cos\gamma}{1+\delta} e^{i\theta} z^{k+1} + \frac{\delta(1+k)-1}{1+\delta} e^{i\theta} z^{k+2}. \quad (32)$$

If we substitute $\delta = 2/k$ into (30), then $t(z)/t^*(z) = e^{-i\theta}$, and it is clear that $|\tilde{\omega}| < 1$ for all $z \in \mathbb{E}$. Now, we need to show that $|\tilde{\omega}| < 1$ for $0 < \delta < 2/k$. Obviously, if z_0 is a zero $t(z)$, then $1/\overline{z_0}$ is zero of $t^*(z)$. Then, we may write

$$\tilde{\omega}(z) = e^{2i\theta} z^k \frac{(z+A_1)(z+A_2)\cdots(z+A_{k+2})}{(1+\overline{A_1}z)(1+\overline{A_2}z)\cdots(1+\overline{A_{k+2}}z)}. \quad (33)$$

Using Lemma 4, we only need to show that all zeros of (31) lie in the closed unit disk for $0 < \delta < 2/k$. Since $|a_{0,0}| = |(\delta(1+k)-1)/(1+\delta)e^{-i\theta}| = |(\delta(1+k)-1)/(1+\delta)| < |a_{k+2,0}| = 1$ for $0 < \delta < 2/k$, thus we have

$$t_1(z) = \frac{\overline{a_{k+2,0}}t(z) - a_{0,0}t^*(z)}{z} = \frac{\delta(k+2)(2-k\delta)}{(1+\delta)^2} \cdot \left(z^{k+1} + \frac{2(k+1)\cos\gamma}{k+2} z^k + \frac{k}{k+2} z^{k-1} - \frac{2}{k+2} e^{-i\theta} z - \frac{2\cos\gamma}{k+2} e^{-i\theta} \right). \quad (34)$$

By Lemma 10, we know that all zeros of

$$q(z) = z^{k+1} + \frac{2(k+1)\cos\gamma}{k+2} z^k + \frac{k}{k+2} z^{k-1} - \frac{2}{k+2} e^{-i\theta} z - \frac{2\cos\gamma}{k+2} e^{-i\theta} \quad (35)$$

lie inside the closed disk. Then, by Cohn's rule, $t(z)$ given by (31) has all its zeros in the closed unit disk. The proof is complete. ☐

Theorem 12. *Let $F_{\mathfrak{a}}$ be a mapping given by (2) and $f_\gamma = h_\gamma + g_\gamma \in \mathbb{S}^0_{CHD}$ be a mapping given by (6) with the dilatation $\omega_k = g'_\gamma/h'_\gamma = e^{i\theta} z^k (\theta \in \mathbb{R}, k \in \mathbb{N}^+)$. Then, $F_{\mathfrak{a}} * f_\gamma$ is univalent and convex in the horizontal direction for $-1 < \mathfrak{a} \le (2-k)/(2+k)$.*

Proof. By Lemma 8, we need to prove that $F_{\mathfrak{a}} * f_\gamma$ is locally univalent and sense-preserving, i.e., the dilatation $\tilde{W}$ of $F_{\mathfrak{a}} * f_\gamma$ satisfies $|\tilde{W}(z)| < 1$ for all $z \in \mathbb{E}$. Substituting $\omega = e^{i\theta} z^k$ in (25),

$$\tilde{W}(z) = e^{2i\theta} z^k \times \frac{z^{k+2} + (1-\mathfrak{a})\cos\gamma z^{k+1} - \mathfrak{a} z^k + ((1+\mathfrak{a})k-2)/2e^{-i\theta} z^2 + [(k-1)+\mathfrak{a}(k+1)]\cos\gamma e^{-i\theta} z + ((k+2)\mathfrak{a}+k)/2e^{-i\theta}}{1 + (1-\mathfrak{a})\cos\gamma z - \mathfrak{a} z^2 + ((1+\mathfrak{a})k-2)/2e^{i\theta} z^k + [(k-1)+\mathfrak{a}(k+1)]\cos\gamma e^{i\theta} z^{k+1} + ((k+2)\mathfrak{a}+k)/2e^{i\theta} z^{k+2}} = e^{2i\theta} z^k \frac{u(z)}{u^*(z)}, \quad (36)$$

where

$$u(z) = z^{k+2} + (1-\mathfrak{a})\cos\gamma z^{k+1} - \mathfrak{a} z^k + ((1+\mathfrak{a})k-2)/2e^{-i\theta} z^2 + [(k-1)+\mathfrak{a}(k+1)]\cos\gamma e^{-i\theta} z + ((k+2)\mathfrak{a}+k)/2e^{-i\theta},$$

$$u^*(z) = 1 + (1-\mathfrak{a})\cos\gamma z - \mathfrak{a} z^2 + ((1+\mathfrak{a})k-2)/2e^{i\theta} z^k + [(k-1)+\mathfrak{a}(k+1)]\cos\gamma e^{i\theta} z^{k+1} + ((k+2)\mathfrak{a}+k)/2e^{i\theta} z^{k+2}. \quad (37)$$

If we substitute $\mathfrak{a} = (2-k)/(2+k)$ into (36), we yield

$$\tilde{W}(z) = e^{2i\theta} z^k \frac{z^{k+2} + 2k/(k+2)\cos\gamma z^{k+1} - (2-k)/(k+2) z^k - (2-k)/(k+2) e^{-i\theta} z^2 + 2k/(k+2)\cos\gamma e^{-i\theta} z + e^{-i\theta}}{1 + 2k/(k+2)\cos\gamma z - (2-k)/(k+2) z^2 - (2-k)/(k+2) e^{i\theta} z^k + 2k/(k+2)\cos\gamma e^{i\theta} z^{k+1} + e^{i\theta} z^{k+2}} = e^{i\theta} z^k. \quad (38)$$

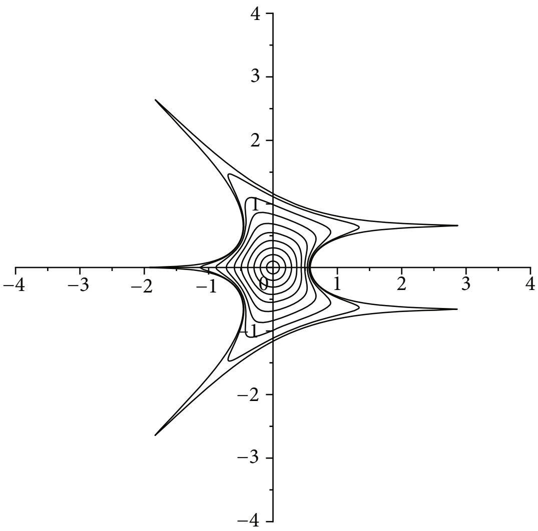

Figure 1: Image of $f_{\Pi/2}$.

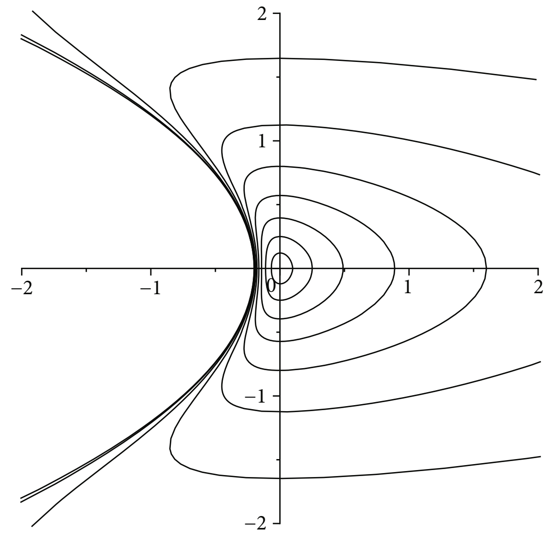

Figure 2: Image of $P_{2/3}$.

Hence, $|\tilde{W}(z)| = |e^{i\theta}z^k| < 1$.

Next, we will show that $|\tilde{W}(z)| <1$ for all $-1 < \mathfrak{a} < (2-k)/(2+k)$. If z_0 is a zero of $u(z)$, then $1/\overline{z_0}$ is zero of $u^*(z)$; hence,

$$\tilde{W}(z) = e^{2i\theta}z^k \frac{u(z)}{u^*(z)} = e^{2i\theta}z^k \cdot \frac{(z+A_1)(z+A_2)\cdots(z+A_{k+2})}{(1+\overline{A_1}z)(1+\overline{A_2}z)\cdots(1+\overline{A_{k+2}}z)}. \tag{39}$$

By Lemma 4, we need to show that all zeros of $u(z)$ lie inside or on the unit disk for $-1 < \mathfrak{a} < (2-k)/(2+k)$. Since

$$|a_{0,0}| = \left|\frac{(k+2)\mathfrak{a}+k}{2}e^{-i\theta}\right| < 1 = |a_{k+2,0}| \text{ for } -1 < \mathfrak{a} < \frac{2-k}{2+k}, \tag{40}$$

from (12), we have

$$\begin{aligned} u_1(z) &= \frac{\overline{a_{k+2,0}}u(z) - a_{0,0}u^*(z)}{z} \\ &= -\frac{(k+2)(1+\mathfrak{a})[(k+2)\mathfrak{a}+k-2]}{4} \cdot \left(z^{k+1} + \frac{2(k+1)\cos\gamma}{k+2}z^k + \frac{k}{k+2}z^{k-1} - \frac{2}{k+2}e^{-i\theta}z^k - \frac{2\cos\gamma}{k+2}e^{-i\theta}\right) \\ &= -\frac{(k+2)(1+\mathfrak{a})[(k+2)\mathfrak{a}+k-2]}{4}q(z), \end{aligned} \tag{41}$$

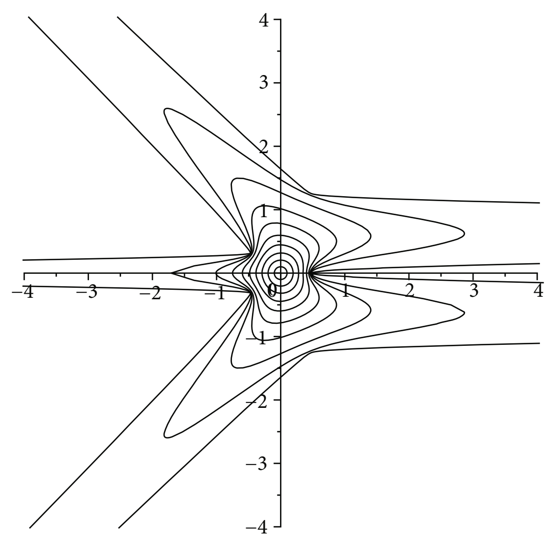

Figure 3: Image of $P_{2/3} * f_{\Pi/2}$.

where

$$q(z) = z^{k+1} + \frac{2(k+1)\cos\gamma}{k+2}z^k + \frac{k}{k+2}z^{k-1} - \frac{2}{k+2}e^{-i\theta}z - \frac{2\cos\gamma}{k+2}e^{-i\theta}. \tag{42}$$

Because $(k+2)(1+\mathfrak{a})[(k+2)\mathfrak{a}+k-2]/4 \neq 0$ for $-1 < \mathfrak{a} < (2-k)/(2+k)$, it follows that both $u_1(z)$ and $q(z)$ have the same zeros. By Lemma 10, we know that all zeros of $q(z)$ lie inside the closed unit disk. Then, by Cohn's rule, we know that all zeros $u(z)$ lie inside or on the boundary of the unit disk. The proof is completed. □

Theorem 13. *Let $F_0 = H_0 + \overline{G_0} \in \mathbb{S}^0_{CHD}$ be a harmonic mapping with $H_0 - G_0 = z/(1-z)$ and dilatation $G_0'(z)/H_0'(z) = z$. Let $f_{\pi/2} = h_{\pi/2} + \overline{g_{\pi/2}} \in \mathbb{S}_{CHD}$ be a mapping defined by (6)*

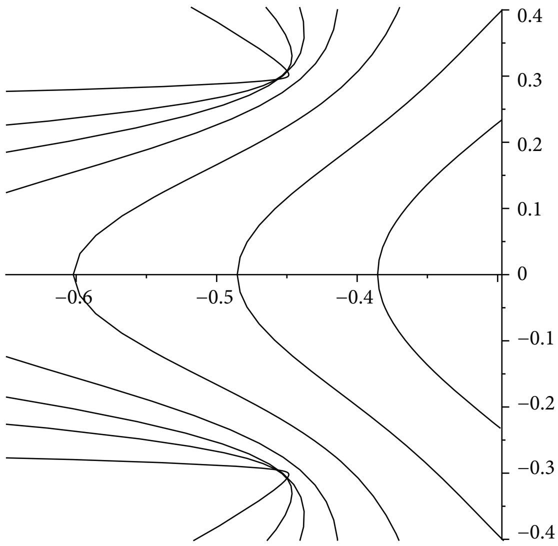

Figure 4: Image of $P_{3/4} * f_{\Pi/2}$.

with $\gamma = \pi/2$ *and dilatation* $\omega_\mu(z) = (\mu + z^2)/(1 + \mu z^2)$, $-1 < \mu < 1$. *Then the mapping* $F_0 * f_{\pi/2}$ *is univalent and convex in the horizontal direction.*

Proof. Since $f_{\pi/2} = h_{\pi/2} + \overline{g_{\pi/2}} \in \mathbb{S}_{\text{CHD}}$ is a mapping defined by (6) with $\gamma = \pi/2$, we have

$$h_{\pi/2}(z) - g_{\pi/2}(z) = \frac{1}{2i} \log\left(\frac{1 + iz}{1 - iz}\right). \quad (43)$$

Therefore, we know that

$$\begin{aligned} \widetilde{W}(z) &= \frac{\left(G_0 * g_{\pi/2}\right)'}{\left(H_0 * h_{\pi/2}\right)'} = \frac{\left(zg'_{\pi/2} - g_{\pi/2}\right)'}{\left(zh'_{\pi/2} + h_{\pi/2}\right)'} \\ &= \frac{zg''_{\pi/2}}{2h'_{\pi/2} + zh''_{\pi/2}} = z\frac{\omega_\mu h''_{\pi/2} + \omega'_\mu h'_{\pi/2}}{2h'_{\pi/2} + zh''_{\pi/2}}. \end{aligned} \quad (44)$$

Substituting

$$\begin{aligned} h'_{\pi/2}(z) &= \frac{1}{\omega_\mu (1 + z^2)}, \\ h''_{\pi/2}(z) &= \frac{\omega'_\mu (1 + z^2) - 2z\omega_\mu}{\left(1 - \omega_\mu\right)^2 (1 + z^2)^2}, \end{aligned} \quad (45)$$

into (44) yields

$$\widetilde{W}(z) = z\frac{\omega_\mu^2 - \left(\omega_\mu - 1/2\omega'_\mu z\right) + 1/2\omega'_\mu 1/z}{1/z - \left(\omega_\mu - 1/2\omega'_\mu z\right)1/z + 1/2\omega'_\mu z^2}. \quad (46)$$

Setting $\omega_\mu(z) = (\mu + z^2)/(1 + \mu z^2)$ in the above equation, we get $\widetilde{W}(z) = z^2$, and hence, $|\widetilde{W}(z)| < 1$ for all $z \in \mathbb{E}$. □

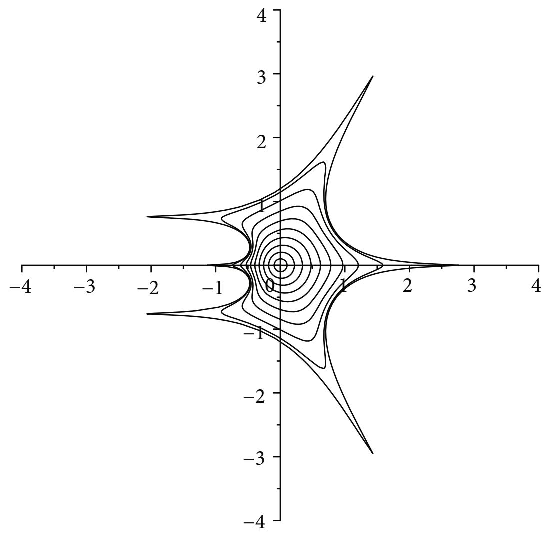

Figure 5: Image of $f_{2\Pi/3}$.

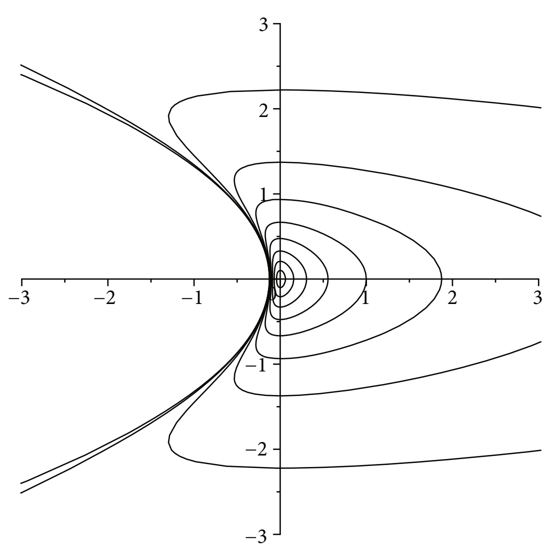

Figure 6: Image of $F_{-1/3}$.

Example 14. Suppose $f_\gamma = h_\gamma + \overline{g_\gamma} \in \mathbb{S}^0_{\text{CHD}}$ *be given by ((6)). If we set* $\gamma = \pi/2$ *and* $\omega_1 = -z^3$ *then by shear construction of Clunie and Sheil-Small* [2], *we have*

$$\begin{aligned} h_\gamma(z) &= \frac{1}{6}\log(1 + z) - \frac{i}{4}\log\left(\frac{1 + iz}{1 - iz}\right) \\ &\quad + \frac{1}{4}\log\left(1 + z^2\right) - \frac{1}{3}\log\left(1 - z + z^2\right), \\ g_\gamma(z) &= \frac{1}{6}\log(1 + z) + \frac{i}{4}\log\left(\frac{1 + iz}{1 - iz}\right) \\ &\quad + \frac{1}{4}\log\left(1 + z^2\right) - \frac{1}{3}\log\left(1 - z + z^2\right). \end{aligned} \quad (47)$$

Recall that, if $f = h + \overline{g} \in \mathbb{S}_H$, then

$$P_\delta * f = \frac{\delta z h' + h}{1 + \delta} + \overline{\frac{\delta z g' g}{1 + \delta}}. \quad (48)$$

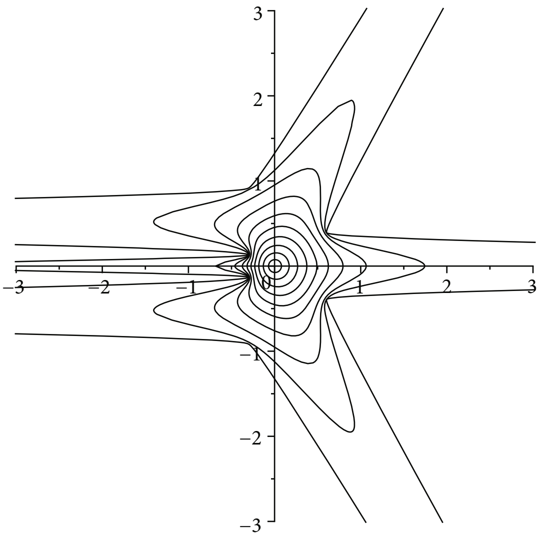

Figure 7: Image of $F_{-1/3} * f_{2\Pi/3}$.

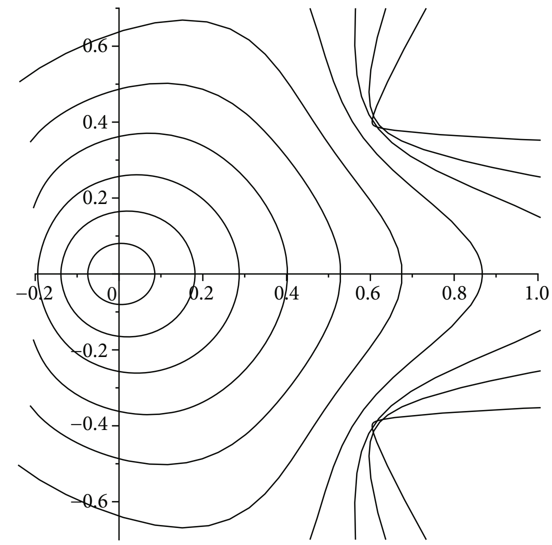

Figure 8: Image of $F_{-1/4} * f_{2\Pi/3}$.

So, we have

$$\begin{aligned} P_\delta * f_\gamma &= \frac{1}{1+\delta}\left[\delta z h_\gamma'(z) + h_\gamma(z)\right] + \frac{1}{1+\delta}\overline{\left[\delta z g_\gamma'(z) g_\gamma(z)\right]} \\ &= \frac{1}{1+\delta}\left[\frac{\delta z}{(1+z^3)(1+z^2)} + \frac{1}{6}\log(1+z) - \frac{i}{4}\log\right. \\ &\quad \left.\cdot\left(\frac{1+iz}{1-iz}\right) + \frac{1}{4}\log\left(1+z^2\right) - \frac{1}{3}\log\left(1-z+z^2\right)\right] \\ &\quad + \frac{1}{1+\delta}\overline{\left[\frac{\delta z^4}{(1+z^3)(1+z^2)}\frac{1}{6}\log(1+z)\frac{i}{4}\log\left(\frac{1+iz}{1iz}\right)\right.} \\ &\quad \overline{\left.\cdot\frac{1}{4}\log\left(1+z^2\right) + \frac{1}{3}\log\left(1z+z^2\right)\right]} \\ &= \operatorname{Re}\left\{\frac{1}{1+\delta}\left[\frac{\delta z\left(1-z^3\right)}{(1+z^3)(1+z^2)} - \frac{i}{2}\log\left(\frac{1+iz}{1-iz}\right)\right]\right\} \\ &\quad + i\operatorname{Im}\left\{\frac{1}{1+\delta}\left[\frac{\delta z}{1+z^2} + \frac{1}{3}\log(1+z)\right.\right. \\ &\quad \left.\left. + \frac{1}{2}\log\left(1+z^2\right) - \frac{2}{3}\log\left(1-z+z^2\right)\right]\right\}. \end{aligned} \tag{49}$$

Now, in view of Theorem 11, if we set the parameter $\delta = 2/3$, then $P_\delta * f_\gamma$ is univalent and CHD. Also, if we choose $\delta = 3/4$, then $P_\delta * f_\gamma$ is not guaranteed to be univalent. The images of $|z| = r < 1$ under $f_{\Pi/2}, P_{2/3}, P_{2/3} * f_{\Pi/2}$ and $P_{3/4} * f_{\Pi/2}$ are shown in Figures 1–4.

Example 15. Suppose $f_\gamma = h_\gamma + \overline{g_\gamma} \in \mathbb{S}^0_{\mathrm{CHD}}$ be given by (6). If we set $\gamma = 2\pi/3$ and $\omega_2 = z^4$, then calculations lead to

$$\begin{aligned} h_\gamma(z) &= \frac{1}{12}\log(1+z) + \frac{1}{4}\log\left(\frac{1+z^2}{1-z}\right) \\ &\quad - \frac{1}{6}\log\left(1-z+z^2\right) - \frac{i\sqrt{3}}{6}\log\left(\frac{1+ze^{(2\pi/3)i}}{1+ze^{(-2\pi/3)i}}\right), \\ g_\gamma(z) &= \frac{1}{12}\log(1+z) + \frac{1}{4}\log\left(\frac{1+z^2}{1-z}\right) \\ &\quad - \frac{1}{6}\log\left(1-z+z^2\right) + \frac{i\sqrt{3}}{6}\log\left(\frac{1+ze^{(2\pi/3)i}}{1+ze^{(-2\pi/3)i}}\right). \end{aligned} \tag{50}$$

If $f = h + \overline{g} \in \mathbb{S}_H$, then

$$F_\mathfrak{a} * f = \frac{1}{2}\left[\frac{(1+\mathfrak{a})zh'}{1-\mathfrak{a}} + h\right] + \frac{1}{2}\overline{\left[\frac{(1+\mathfrak{a})zg'}{1\mathfrak{a}}g\right]}. \tag{51}$$

So, we have

$$\begin{aligned} F_\mathfrak{a} * f_\gamma &= \frac{1}{2}\left[\frac{(1+\mathfrak{a})zh_\gamma'}{1-\mathfrak{a}} + h_\gamma\right] + \frac{1}{2}\overline{\left[\frac{(1+\mathfrak{a})zg_\gamma'}{1\mathfrak{a}}g_\gamma\right]} \\ &= \frac{1}{2}\left[\frac{(1+\mathfrak{a})z}{(1-\mathfrak{a})(1-z+z^2)(1-z^4)} + \frac{1}{12}\log(1+z)\right. \\ &\quad + \frac{1}{4}\log\left(\frac{1+z^2}{1-z}\right) - \frac{1}{6}\log\left(1-z+z^2\right) \\ &\quad \left. - \frac{i\sqrt{3}}{6}\log\left(\frac{1+ze^{(2\pi/3)i}}{1+ze^{(-2\pi/3)i}}\right)\right] \\ &\quad + \frac{1}{2}\overline{\left[\frac{(1+\mathfrak{a})z^5}{(1\mathfrak{a})(1z+z^2)(1z^4)}\frac{1}{12}\log(1+z)\frac{1}{4}\log\left(\frac{1+z^2}{1z}\right)\right.} \\ &\quad \overline{\left. + \frac{1}{6}\log\left(1z+z^2\right)\frac{i\sqrt{3}}{6}\log\left(\frac{1+ze^{(2\pi/3)i}}{1+ze^{(2\pi/3)i}}\right)\right]} \\ &= \operatorname{Re}\left\{\frac{1}{2}\left[\frac{(1+\mathfrak{a})z\left(1+z^4\right)}{(1-\mathfrak{a})(1-z+z^2)(1-z^4)}\right.\right. \\ &\quad \left.\left. - \frac{i\sqrt{3}}{3}\log\left(\frac{1+ze^{(2\pi/3)i}}{1+ze^{(-2\pi/3)i}}\right)\right]\right\} \\ &\quad + i\operatorname{Im}\left\{\frac{1}{2}\left[\frac{(1+\mathfrak{a})z}{(1-\mathfrak{a})(1-z+z^2)} + \frac{1}{6}\log(1+z)\right.\right. \\ &\quad \left.\left. + \frac{1}{2}\log\left(\frac{1+z^2}{1-z}\right) - \frac{1}{3}\log\left(1-z+z^2\right)\right]\right\}. \end{aligned} \tag{52}$$

Now, if we set the parameter $\mathfrak{a} = -1/3$, in view of Theorem 11, $F_{\mathfrak{a}} * f_{\gamma}$ is univalent and CHD. If we choose $\mathfrak{a} = -1/4$, then $F_{\mathfrak{a}} * f_{\gamma}$ is not guaranteed to be univalent (see Figures 5–8).

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there is no conflict of interest.

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