Research Article

Fractional Operators in $p$-adic Variable Exponent Lebesgue Spaces and Application to $p$-adic Derivative

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In this paper, we prove the boundedness of the fractional maximal and the fractional integral operator in the $p$-adic variable exponent Lebesgue spaces. As an application, we show the existence and uniqueness of the solution for a nonhomogeneous Cauchy problem in the $p$-adic variable exponent Lebesgue spaces.

1. Introduction

The field of $p$-adic numbers are an interesting and useful tool to study phenomena in physics, biology, and medicine, among other sciences; see, e.g., [1–4] and references therein. For this reason, the study of operators that allows us to describe such phenomena is essential. Even more so when in the $p$-adic setting it is not possible to define the derivative in the classical sense.

Variable exponent Lebesgue spaces generalize the notion of $q$-integrability in the classical Lebesgue spaces, allowing the exponent to be a measurable function. These spaces were introduced in 1931 by Orlicz [5] but lay essentially dormant for more than 50 years. They received a thrust in the paper [6] and are now an active area of research having many known applications, e.g., in the modeling of thermorheological fluids [7] as well as electrorheological fluids [8–11], in differential equations with nonstandard growth [12, 13], and in the study of image processing [14–20]. For a thorough history, theory, and applications of variable exponent Lebesgue spaces, see [6, 21–24].

In this article, we are interested in the boundedness of the fractional integral and maximal fractional operator on the $p$-adic Lebesgue spaces with a variable exponent. The corresponding result for classical $p$-adic Lebesgue space is known (cf. [25]). These operators play an important role in such areas as Sobolev spaces, potential theory, PDEs, and integral geometry, to name a few.

This work is divided as follows. Section 2 contains a quick description of the preliminary on the topic of the $p$-adic analysis and variable exponent Lebesgue spaces on the $p$-adic numbers, necessary for the development of this work. In Section 3, the boundedness of the fractional maximal operator

$$M_{\alpha}f(x) = \sup_{y \in \mathbb{Z}} \frac{1}{p^{y(n-\alpha)}} \int_{F_p^\alpha(x)} |f(y)| dy, \quad x \in \mathbb{Q}_p^n,$$

is studied in the framework of variable exponent $p$-adic Lebesgue spaces. The boundedness of the special case $M_0$, the so-called Hardy-Littlewood maximal operator, was obtained in [26] under appropriate conditions on the exponent function. We prove, using a suitable pointwise estimate, the boundedness of the fractional maximal operator from $L^{q_{\#}}(\mathbb{Q}_p^n)$ to $L^{q_{\#}}(\mathbb{Q}_p^n)$, where $q_{\#}$ is the Sobolev limiting exponent; see (31) for the corresponding definition. The boundedness of the fractional integral operator

$$I_{\alpha}f(x) = \int_{\mathbb{Q}_p^n} \frac{f(y)}{\|x-y\|_{p}^{-\alpha}} dy, \quad x \in \mathbb{Q}_p^n,$$

is obtained from the boundedness of the fractional maximal operator and Welland’s pointwise inequality tailored for the
In the $p$-adic setting; this approach is inspired from [27]. In the literature, it is customary to prove first the boundedness of the fractional potential operator and, as a corollary, the boundedness of the fractional maximal operator is obtained using the lattice property of the norm and the elementary estimate

$$M_a f(x) \leq I_a |f| (x).$$

As already mentioned, we will use a reverse approach. Finally, in Section 4, we define the Talbot operator in $p$-adic Lebesgue spaces with variable exponent, which is the analogue of the derivative in the spatial variable $x (x \in \mathbb{Q}_p^n)$, and study the nonhomogeneous Cauchy problem (72) associated with this operator.

The notation $a \leq b$ denotes the existence of a constant $C$ for which $a \leq Cb$, $a=b$ means that $a \leq b$ and $b \leq a$.

2. Preliminaries

For an exposition on the $p$-adic analysis, see [25, 28].

2.1. The Field of $p$-adic Numbers. By $p$ we denote a prime number. The field $\mathbb{Q}_p$ is given as the completion of $\mathbb{Q}$ with respect to the $p$-adic norm $|\cdot |_p$, given by

$$|x|_p := \begin{cases} 0, & \text{if } x = 0, \\ p^{-\gamma}, & \text{if } x = p^{-\gamma} b, \end{cases}$$

where $a,b$ are integers coprime with $p$. The integer $\gamma := \text{ord } (x)/(\text{ord} (0) = +\infty)$ is denoted as the $p$-adic order of $x$. This norm can be extended to $\mathbb{Q}_p^n$ as

$$\|x\|_p := \max_{1 \leq i \leq n} |x_i|_p$$

for $x = (x_1, \cdots, x_n) \in \mathbb{Q}_p^n$,

and satisfies the so-called strong triangular inequality

$$\|x + y\|_p \leq \max \{\|x\|_p, \|y\|_p\},$$

with equality when $\|x\|_p = \|y\|_p$. If $\text{ord}(x) = \min_{1 \leq i \leq n} \{\text{ord} (x_i)\}$, it follows that $\|x\|_p = p^{-\text{ord}(x)}$. The set $(\mathbb{Q}_p^n, \|\cdot\|_p)$ is a complete ultrametric space and, as a topological space, $\mathbb{Q}_p$ is homeomorphic to a Cantor-like subset of the real line. A $p$-adic number $x \neq 0$ has a unique series expansion, viz.,

$$x = p^{\text{ord}(x)} \sum_{j=0}^{\infty} x_j p^j,$$

where $x_j \in \{0, 1, 2, \cdots, p - 1\}$ and $x_0 \neq 0$. For $\gamma \in \mathbb{Z}$, we denote by

$$B_x^\gamma = \{x \in \mathbb{Q}_p^n : \|x - a\|_p \leq p^\gamma\}$$

the ball of radius $p^\gamma$ with center at $a = (a_1, \cdots, a_n) \in \mathbb{Q}_p^n$ and by

$$S_x^\gamma (a) = \{x \in \mathbb{Q}_p^n : \|x - a\|_p = p^\gamma\} = B_x^\gamma (a) \setminus B_x^{\gamma-1} (a),$$

the corresponding sphere. We denote

$$B^{\gamma}_x (0) = B_x^\gamma, \quad S^{\gamma}_x (0) = S_x^\gamma,$$

and note that

$$\mathbb{Q}_p^n \setminus \{0\} = \bigcup_{\gamma \in \mathbb{Z}} S_x^\gamma.$$

Note that $B_x^\gamma (a) = B_y (a_1) \times \cdots \times B_y (a_n)$, where $B_y (a_i) := \{x \in \mathbb{Q}_p : \|x - a_i\|_p \leq p^\gamma\}$ is the one-dimensional ball. In $\mathbb{Q}_p^n$ there exists the additive Haar measure $d^n x = dx$ (by $|E|$ we denote the Haar measure of the set $E$), since the field $\mathbb{Q}_p$ is a locally compact commutative group with respect to addition. Normalizing the measure $dx$ by $\int_{\mathbb{Q}_p^n} dx = 1$, we get a unique measure. From here onwards, we use the normalized Haar measure; thus,

$$\text{Vol} (B_x^\gamma (a)) = p^\gamma, \quad \text{Vol} (S_x^\gamma (a)) = p^\gamma (1 - p^{-\gamma}),$$

for any $a \in \mathbb{Q}_p^n$.

Note that the collection of all disjoint balls of the same radius $\gamma$ forms a partition of $\mathbb{Q}_p^n$, since inequality (6) implies that any two balls in $\mathbb{Q}_p^n$ with the same radius are either identical or disjoint.

2.2. Some Function Spaces. A complex-valued function $\varphi$ defined on $\mathbb{Q}_p^n$ is called locally constant if for any $x \in \mathbb{Q}_p^n$, there exists an integer $l(x) \in \mathbb{Z}$ such that

$$\varphi (x + x') = \varphi (x) \quad \text{for } x' \in B_x^{l(x)}.$$

A function $\varphi : \mathbb{Q}_p^n \rightarrow \mathbb{C}$ is called a Schwartz-Bruhat function (or a test function) if it is locally constant with the compact support. The $\mathbb{C}$-vector space of Schwartz-Bruhat functions is denoted by $S(\mathbb{Q}_p^n)$.

A measurable function $f : \mathbb{Q}_p^n \rightarrow \mathbb{C}$ belongs to the Lebesgue space $L^q(\mathbb{Q}_p^n)$, $1 \leq q < \infty$, when

$$\|f\|_q := \int_{\mathbb{Q}_p^n} |f(x)|^q dx < \infty, \quad 1 < q < \infty,$$

where

$$\int_{\mathbb{Q}_p^n} |f(x)|^q dx = \lim_{r \rightarrow \infty} \int_{B_r(x)} |f(x)|^q dx,$$

if the limit exists.
We now introduce the notion of $p$-adic Lebesgue spaces with a variable exponent and give some properties needed in the sequel; see [26] for the respective proofs.

We say that a measurable function is a variable exponent if $f : \Omega_p^p \longrightarrow [1, \infty)$. By $\mathcal{O}(\Omega_p^p)$ we denote the set of all variable exponent satisfying $q^{-} < \infty$, where $q^{-} = \text{ess inf}_{x \in \Omega_p^p} q(x)$ and $q^{+} = \text{ess inf}_{x \in \Omega_p^p} q(x)$.

For $q \in \mathcal{O}(\Omega_p^p)$ by $L^{q^{-}}(\Omega_p^p)$, we denote the space of measurable functions $f : \Omega_p^p \longrightarrow \mathbb{R}$ such that

$$\|f\|_{L^{q^{-}}(\Omega_p^p)} = \inf \left\{ \lambda > 0 : \mathcal{Q}_\lambda(f) \left( \frac{x}{\lambda} \right) \leq 1 \right\} < \infty,$$

where $\mathcal{Q}_\lambda(f) = \int_{\Omega_p^p} |f(x)|^{q_{\lambda}(x)} \, dx$.

For the Lebesgue space with a variable exponent, we have

$$\|f\|_{L^{q^{-}}(\Omega_p^p)} \leq \mathcal{Q}_1(f) + 1,$$

$$\mathcal{Q}_1(f) \leq \left( 1 + \|f\|_{L^{q^{-}}(\Omega_p^p)} \right)^{q^{+}},$$

$$\|f\|_{L^{q^{-}}(\Omega_p^p)} = \|f\|_{L^{q^{+}}(\Omega_p^p)}^{q^{+}}, \quad s \in (0, q^{-}].$$

The Hölder inequality is valid, up to a multiplicative constant, in the framework of Lebesgue spaces with variable exponent, viz.,

$$\int_{\Omega_p^p} |f(x)g(x)| \, dx \leq C \|f\|_{L^{q^{-}}(\Omega_p^p)} \|g\|_{L^{q^{+}}(\Omega_p^p)},$$

where $q$ and $q^{+}$ are conjugate exponents, viz., $1 = 1/q(x) + 1/q^{+}(x)$.

For $q \in \mathcal{O}(\Omega_p^p)$, we say that $q \in W_0(\Omega_p^p)$ when there is a positive constant $C$, for which

$$\gamma \left( q^{+} \left( B_p^p(x) \right) - q^{-} \left( B_p^p(x) \right) \right) \leq C,$$

for all $\gamma > 0$ and any $x \in \Omega_p^p$. We say that $q \in W^{\alpha(\Omega_p^p)}$ when there is a positive constant $C$, for which

$$|q(x) - q(y)| \leq C \frac{1}{\log_x (p + \min \left\{ \|y\|_p, \|x\|_p \right\})},$$

for any $x, y \in \Omega_p^p$.

The class $W^{\alpha(\Omega_p^p)}$ is defined as $W^{\alpha(\Omega_p^p)} = W_0(\Omega_p^p) \cap W^{\alpha(\Omega_p^p)}$. The importance of the class $W^{\alpha(\Omega_p^p)}$ stems from the fact that it is a sufficient condition for the boundedness of the maximal operator in $L^{q^{-}}(\Omega_p^p)$: if $q \in W^{\alpha(\Omega_p^p)}$, then

$$\|\text{Mf}\|_{L^{q^{-}}(\Omega_p^p)} \leq C \|f\|_{L^{q^{-}}(\Omega_p^p)},$$

see Theorem 5.2 of [26] for the corresponding proof.

In the case where $\Omega_p^p$ is a bounded subset of $\Omega_p^p$, we have the following: if $q \in W_0(\Omega_p^p)$, then

$$\|\text{Mf}\|_{L^{q^{-}}(\Omega_p^p)} \leq C \|f\|_{L^{q^{-}}(\Omega_p^p)},$$

see Theorem 5.1 of [26].

**Lemma 1.** Let $q \in \mathcal{O}(\Omega_p^p)$ be a $L$-Lipschitz function, for some $L \geq 0$. Then, $q \in W_0(\Omega_p^p)$.

**Proof.** We give the proof only for the case $\gamma \leq 0$ since the other case is immediate. Since $q$ is a continuous function, for any ball $B_p^p(x) \subset \Omega_p^p$, there exists a maximum (respectively, minimum) point $\bar{x} \in B_p^p(x)$ (respectively, $\underline{x} \in B_p^p(x)$). From the Lipschitzianity of $q$, we have

$$-\gamma \left( q^{+} \left( B_p^p(\bar{x}) \right) - q^{-} \left( B_p^p(\bar{x}) \right) \right) = -\gamma (q(\bar{x}) - q(\underline{x}))$$

$$\leq -Ly \|\bar{x} - \underline{x}\|_p \leq 2L \frac{|\gamma|}{p|\gamma|} \leq C,$$

which completes the proof.

We now show an extension result via the well-known McShane extension technique (a similar approach was used in the Euclidean framework; see [23]).

**Lemma 2.** Let $q \in W_0(\Omega_p^p)$, where $\Omega_p^p$ is a bounded subset of $\Omega_p^p$. Then, there exists an extension function $\tilde{q} \in W_0(\Omega_p^p)$ which is constant outside some fixed ball.

**Proof.** The proof will be divided into two steps. **First step:** we show that there exists an extension function $\tilde{q} \in W_0(\Omega_p^p)$. Let us define $\tilde{q}$ as

$$\tilde{q}(x) = \sup_{\xi \in B_p^p} \left[ q(\xi) - \omega \left( \|x - \xi\|_p \right) \right],$$

with

$$\omega(t) = \begin{cases} \frac{C}{\log_p (p + 1/t)} & t > 0, \\ 0 & t = 0, \end{cases}$$

where $C$ comes from equation (21). Since $\omega(t)$ is an increasing and concave function for $t \geq 0$ and approaches zero with $t$, then from Theorem 2 of [29], we have that $|\tilde{q}(x) - \tilde{q}(y)| \leq \omega(\|x - y\|_p)$. In order to prove that $\tilde{q} \in W_0(\Omega_p^p)$, it suffices to check for $\gamma < 0$. Since $\omega$ is an increasing function and taking $\tilde{x}$ and $\tilde{y}$ as in the proof of Lemma 1, we see that
\[ y(\bar{q}(x)) = y(\tilde{q}(x)) \]
\[ = |y| |\bar{q}(x) - \tilde{q}(x)| \leq |y| |\alpha| \|x - \tilde{x}\|_p \]
\[ \leq C \frac{|y|}{\log_p (p + p^\alpha)} \leq C, \]

which ends the first step.

Second step: we show that there exists an extension function \( \tilde{q} \in \mathcal{W}_{\mathcal{O}}^\infty (\mathbb{Q}_p^n) \). Since \( \mathcal{O}_p^n \) is a bounded set, let us take \( \gamma > 0 \) such that \( \mathcal{O}_p^n \subset B_p^\gamma (0) \). We define two Urysohn functions, \( u_0 \) and \( u_{\infty} \), as follows:

\[ u_0(x) = \frac{d(x, \mathbb{Q}_p^n \setminus B_{p+1}^\gamma (0))}{d(x, \mathbb{Q}_p^n \setminus B_{p+1}^\gamma (0)) + d(x, \mathcal{O}_p^n)} \]
\[ = \frac{d(x, \mathbb{Q}_p^n \setminus B_{p+1}^\gamma (0))}{d(x, \mathcal{O}_p^n)} + d(x, \mathcal{O}_p^n). \]

The functions \( u_0 \) and \( u_{\infty} \) are \( L \)-Lipschitz with \( L \leq 1/p \), due to the fact that \( d(\mathcal{O}_p^n, \mathbb{Q}_p^n \setminus B_{p+1}^\gamma (0)) \geq p \), see Prop. 2.1.1 of [30]. Defining the exponent \( \tilde{q} \) as,

\[ \tilde{q}(x) = \bar{q}(x) u_0(x) + q^* u_{\infty}(x), \]

we see that \( \tilde{q} \in \mathcal{W}_{\mathcal{O}}^\infty (\mathbb{Q}_p^n) \) since the class \( \mathcal{W}_{\mathcal{O}}^\infty (\mathbb{Q}_p^n) \) is closed under addition and multiplication, and \( \tilde{q} \in \mathcal{W}_{\mathcal{O}}^\infty (\mathbb{Q}_p^n) \) because \( \bar{q}(x) \equiv q^* \) in the exterior of the ball \( B_{p+1}^\gamma (0) \).

3. Main Results

3.1. Boundedness in \( \mathbb{Q}_p^n \). In this section, we study the boundedness of the operators in the case of \( \mathbb{Q}_p^n \).

3.1.1. Fractional Maximal Operator. The classical result regarding boundedness of the fractional maximal operator says that if \( 1 < q < n/\alpha \) and \( 1/s = 1/q - \alpha/n \), then \( M_\alpha : L^s (\mathbb{Q}_p^n) \to L^r(\mathbb{Q}_p^n) \) is bounded (this follows at once from inequality (3) and the boundedness of the operator \( I_\alpha : L^s (\mathbb{Q}_p^n) \to L^r(\mathbb{Q}_p^n) \), cf. [25]). For further goals, we need estimate (32).

Lemma 3. Let \( 0 < \alpha < n \), \( q \) be an exponent function such that \( 1 < q^* \leq q < n/\alpha \), and we define the Sobolev limiting exponent \( q^# \) by

\[ \frac{1}{q^#} = \frac{1}{q} - \frac{\alpha}{n}. \]

Then,

\[ M_{a\alpha f}(x) \leq C \left( M \left( \left| f(x) \right| \left| \bar{q}(x) \right| \right)(a(n-a)) \right)^{1-\alpha/n} \left( \int_{\mathbb{Q}_p^n} \left| f(x) \right|^n \right)^{1/n} \]

\[ \leq C \left( M \left( \left| f(x) \right| \right)(a(n-a)) \right)^{1-\alpha/n} \left( \int_{\mathbb{Q}_p^n} \left| f(x) \right|^n \right)^{1/n} \]

\[ \leq C \left( M \left( \left| f(x) \right| \right)(a(n-a)) \right)^{1-\alpha/n} \left( \int_{\mathbb{Q}_p^n} \left| f(x) \right|^n \right)^{1/n} \]

Taking the supremum over all \( \gamma \in \mathbb{Z} \), we establish the desired inequality.

Proof. From \( q(x)/q^#(x) + aq(x)/n = 1 \) and Hölder’s inequality (20), we have

\[ \frac{1}{p^{ \alpha/n - n}} \int_{B_p(x)} |f(y)| dy = \frac{1}{p^{ \alpha/n - n}} \int_{B_p(x)} |f(y)|^{q(y)/q^#(y)} |f(y)|^{q#(y)/n} dy \]

\[ \leq C \left( \int_{B_p(x)} |f(y)|^{q(y)/q^#(y)} dy \right)^{1-\alpha/n} \]

\[ \leq C \left( \int_{B_p(x)} |f(y)|^{q(y)/q^#(y)} dy \right)^{1-\alpha/n} \]

\[(33)\]

The general case follows from homogeneity considerations.

3.1.2. Fractional Potential Operator. The well-known Sobolev theorem states that the fractional potential operator (2), sometimes introduced with a normalizing factor, is bounded from \( L^q(\mathbb{Q}_p^n) \) to \( L^{p^#}(\mathbb{Q}_p^n) \) where \( 1/q^# = 1/q - \alpha/n \) is the so-called Sobolev limiting exponent; see, for instance, [25].

In order to obtain the boundedness result in the variable exponent framework, we first obtain the validity of a Welfand-type estimate in the \( p \)-adic setting; see [31] for the Euclidean counterpart.

Lemma 5. Let \( 0 < \alpha < n \), \( 0 < \varepsilon < \max (a, a - \alpha) \), and \( f \in L^1_{\text{loc}} (\mathbb{Q}_p^n) \). Then,

\[ \left| I_{a\alpha f}(x) \right| \leq C \left( M_{a\alpha f}(x) \right)^{1/2} \left( M_{a\alpha f}(x) \right)^{1/2}. \]

\[(35)\]
Let $\gamma \in \mathbb{Z}$, then

$$\int_{B^c_k(x)} \frac{|f(y)|}{|x-y|^n} dy = \sum_{k=0}^{\infty} \frac{1}{p_k^{(n-\alpha)}} \int_{B^c_k(x)} |f(y)| dy \leq \sum_{k=0}^{\infty} p_k^{\alpha} M_{\alpha-f}(x) \leq a_{n,p} p_k^{\alpha} M_{\alpha-f}(x).$$

(36)

On the other hand,

$$\int_{B^c_k(x)} \frac{|f(y)|}{|x-y|^n} dy = \sum_{k=0}^{\infty} \frac{1}{p_k^{(n-\alpha)}} \int_{B^c_k(x)} |f(y)| dy \leq \sum_{k=0}^{\infty} p_k^{\alpha} M_{\alpha-f}(x) \leq a_{n,p} p_k^{\alpha} M_{\alpha-f}(x).$$

(37)

Taking the previous estimates into consideration, we have

$$|I_a f(x)| \leq a_{n,p} p^\gamma M_{\alpha-f}(x) + p^{-\gamma} M_{\alpha+f}(x).$$

(38)

The inequality (35) is obtained taking into account (38) with $\gamma$ given by

$$\gamma = \left[ \frac{\log[M_{\alpha+f}(x)/M_{\alpha-f}(x)]^{1/2}}{C} \right],$$

(39)

since $x \leq [x] < x + 1$. \hfill $\square$

**Theorem 6.** Let $1 < q^* \leq q \leq n/\alpha$ and $q \in W^n_0(Q^p_\alpha)$. Then, the fractional potential operator,

$$I_a : L^q(\mathbb{Q}^p_\alpha) \longrightarrow L^{q^*}(\mathbb{Q}^p_\alpha),$$

is bounded, where $q^*$ is the Sobolev limiting exponent (31).

**Proof.** From the definition of the variable exponent Lebesgue norm and homogeneity, it suffices to show that

$$\int_{Q^p_\alpha} |I_a f(x)|^{q^*(x)} dx \leq C,$$

(41)

when $\|f\|_{L^q(Q^p_\alpha)} \leq 1$, see Lemma 3.3 of [26] for more details.

From (35) and H"older’s inequality, we obtain

$$\int_{Q^p_\alpha} |I_a f(x)|^{q^*(x)} dx \leq a_{n,p} \|M_{\alpha-f} \|_{L^{q^*}(\mathbb{Q}^p_\alpha)} \|M_{\alpha+f} \|_{L^{q^*}(\mathbb{Q}^p_\alpha)} dx \leq a_{n,p} \|I_a f \|_{L^{q^*}(\mathbb{Q}^p_\alpha)} \|I_a f \|_{L^{q^*}(\mathbb{Q}^p_\alpha)} \leq C I_1 I_2,$$

(42)

where $q_1$ and $q_2$ are conjugate exponents and will be chosen appropriately. To estimate $I_1$, from (17), we have

$$I_1 \leq \mathcal{E}_{q_1}(\|M_{\alpha-f} \|^2_{L^{q^*}(\mathbb{Q}^p_\alpha)}) + 1 = \int_{Q^p_\alpha} (M_{\alpha-f}(x))^{q^*(x)} dx + 1.$$

(43)

Defining $q_1$ by $1/q(x) - 2/q^*(x)q_1(x) = (\alpha - \epsilon)/n$, we have, by Lemma 3, that the operator $M_{\alpha - \epsilon} : L^{q_1}(\mathbb{Q}^p_\alpha) \longrightarrow L^{q_1^*}(\mathbb{Q}^p_\alpha)$ is bounded, where $q(x) = q^*(x)q_1(x)/2$. From (43) and (18), it follows that

$$\|M_{\alpha-f} \|^2_{L^{q^*}(\mathbb{Q}^p_\alpha)} \leq C \left( 1 + \|I_a f \|_{L^{q^*}(\mathbb{Q}^p_\alpha)} \right)^{q^*} + 1 \leq C \left( 1 + \|f \|_{L^{q^*}(\mathbb{Q}^p_\alpha)} \right)^{q^*} + 1,$$

(44)

which end the estimate for $I_1$.

The estimate for $I_2$ follows, mutatis mutandis, as $I_1$ taking into account that defining $q_2$ as $1/q(x) - 2/[q^*(x)q_2(x)] = (\alpha + \epsilon)/n$, we get that $q_1$ and $q_2$ are indeed conjugate exponents and $q_2$ is the right exponent for the boundedness of $M_{\alpha+\epsilon}$, the details are left to the reader.

3.2. Boundedness in $Q^p_\alpha$. The fractional integral operator $I_a$ can be defined for an open set $Q^p_\alpha \subset \mathbb{Q}^p_\alpha$ in the following way:

$$I_a f(x) = \int_{Q^p_\alpha} \frac{f(y)}{|x-y|^n} dy, \quad x \in Q^p_\alpha.$$

(45)

We are interested in proving the $L^{q^*}(\mathbb{Q}^p_\alpha) \longrightarrow L^{q^*}(\mathbb{Q}^p_\alpha)$ boundedness for the operator $I_a$, where $q^*$ is defined by (31). We begin with two lemmas, which are important on their own.

Although we need Lemma 7 for bounded set $Q^p_\alpha$, we give the lemma for general measurable sets $\Omega^p_\alpha$ which include, as a particular case, $\Omega^p_\alpha = Q^p_\alpha$.

**Lemma 7.** Let $q \in W^n_0(\Omega^p_\alpha)$. Then,

$$\|X_{\alpha}(x)\|_{L^{q^*}(\Omega^p_\alpha)} \leq C p^{n[q(x), y]},$$

(46)

where
\[ q(x,y) = \begin{cases} q(x), \quad y < 0, \\ q(\infty), \quad y \geq 0. \end{cases} \tag{47} \]

**Proof.** We split the proof into three cases: (1) \( y < 0 \), (2) \( y \geq 0 \) and \( p^y \leq \|x\|_p \), and (3) \( y \geq 0 \) and \( p^y > \|x\|_p \).

**Case 1.** \( y < 0 \). Since \( q \in W_1^0(\Omega^y_p) \), we see that
\[ p^{\gamma y}(q(x)/q(\infty)) \leq p^{\gamma y}((q(x)/q(\infty)) \geq p^{\gamma y}((q(x)/q(\infty)) \leq p^{-y}p^C, \tag{48} \]

thus, \( p^{-\gamma y}(q(x)/q(\infty)) \leq p^{-\gamma y}p^C \). Integrating the last inequality over \( B^y_\gamma(x) \), we obtain
\[ \int_{B^y_\gamma(x)} \frac{1}{p^{\gamma y}(q(x)/q(\infty))} \, dy \leq \int_{B^y_\gamma(x)} p^{-\gamma y}p^C \, dy \leq p^C, \tag{49} \]

from which it follows, using the definition of variable exponent norm and (49), that
\[ \left\| X_{B^y_\gamma(x)}(\cdot) \right\|_{L^{\gamma y}(\Omega^y_p)} \leq p^{\gamma y}(q(x)). \tag{50} \]

**Case 2.** \( y \geq 0 \) and \( p^y \leq \|x\|_p \). In this case, we have \( \|y\|_p \geq \|x\|_p - \|x - y\|_p \geq p^y(p - 1) \) (because \( y \in B^y_\gamma(x) \)). Since \( q \in W^{\gamma y}(\Omega^y_p) \), then taking limit when \( \|x\|_p \to \infty \), we get
\[ |q(\infty) - q(y)| \leq \frac{C}{\log_p \left( p + \|y\|_p \right)}, \tag{51} \]

from which we obtain
\[ p^{y(n-\gamma)(q(\infty))} = p^{y(n-q(\infty))} \leq p^{y(n-\gamma)(q(\infty))} \leq p^C, \tag{52} \]

consequently, \( p^{-\gamma y}(q(x)/q(\infty)) \leq p^{-\gamma y}p^C \). Integrating the last inequality over \( B^y_\gamma(x) \), we have
\[ \int_{B^y_\gamma(x)} \frac{1}{p^{\gamma y}(q(x)/q(\infty))} \, dy \leq \int_{B^y_\gamma(x)} p^{-\gamma y}p^C \, dy \leq p^C, \tag{53} \]

from which, by the definition of variable exponent norm, we get
\[ \left\| X_{B^y_\gamma(x)}(\cdot) \right\|_{L^{\gamma y}(\Omega^y_p)} \leq p^{\gamma y}(q(\infty)). \tag{54} \]

**Case 3.** \( y \geq 0 \) and \( \|x\|_p < p^y \). By the ultrametricity and the condition on \( \|x\|_p \), we have \( B^y_\gamma(x) \subset B^y_\gamma(0) \). Then,
\[ \int_{B^y_\gamma(x)} \frac{1}{p^{\gamma y}(q(x)/q(\infty))} \, dy \leq \int_{B^y_\gamma(0)} \frac{1}{p^{\gamma y}(q(y)/q(\infty))} \, dy \leq \int_{B^y_\gamma(0)} \frac{1}{p^{\gamma y}} \, dy \leq \frac{1}{p^{\gamma y}} \int_{B^y_\gamma(0)} dy \leq 1, \tag{55} \]

from which, by the definition of variable exponent norm, we get
\[ \left\| X_{B^y_\gamma(x)}(\cdot) \right\|_{L^{\gamma y}(\Omega^y_p)} \leq p^{\gamma y}(q(\infty)), \tag{56} \]

which completes the proof.

**Lemma 8.** Let \( q \in W^{\gamma y}_0(\Omega^y_p) \). Then,
\[ \left\| X_{Q^y_p} \right\|_{L^{\gamma y}(\Omega^y_p)} \leq C p^{\gamma y(x,y) - y^p}, \tag{57} \]

where \( q(x,y) \) is defined in (47).

**Proof.** We first prove the theorem for the case \( \beta = n \). When \( y \notin B^y_\gamma(x) \), by the ultrametric condition, we have \( B^y_\gamma(x) \subset B^{n-\gamma y}(x-y) \); thus,
\[ M \left( \frac{X_{B^y_\gamma(x)}(\cdot)}{p^{\gamma y}} \right)(\cdot) \geq \frac{1}{p^{\gamma n-\gamma y}} \int_{p^{\gamma n-\gamma y}(x)} X_{B^y_\gamma(x)}(\cdot) \, dz \]
\[ \geq \frac{C}{p^{\gamma n}} \left\| X_{B^y_\gamma(x)}(\cdot) \right\|_{L^{\gamma y}(\Omega^y_p)} \leq Cp^{\gamma y(x,y) - y^p}, \tag{58} \]

From the pointwise estimate (58), the boundedness of the maximal operator \( M \) (see (23)), and (46), we see that
\[ \left\| X_{Q^y_p} \right\|_{L^{\gamma y}(\Omega^y_p)} \leq \frac{C}{p^{\gamma n}} \left\| X_{B^y_\gamma(x)}(\cdot) \right\|_{L^{\gamma y}(\Omega^y_p)} \leq Cp^{\gamma y(x,y) - y^p}, \tag{59} \]

which proves the estimate for \( \beta = n \). The case for general \( \beta \) follows from the case \( \beta = n \) and the identity \( \|f\|_{L^{\gamma y}(\Omega^y_p)} = \|f\|_{L^{\gamma y}(\Omega^y_p)} \), where \( \Omega^y_p \) is a bounded open subset of \( \Omega^y_p \). Then,
\[ I_{\alpha}(x) \leq C(Mf(x))^{q(x)/q^\alpha(x)}, \quad x \in \Omega^y_p, \tag{60} \]
where $q^*$ is the Sobolev limiting exponent defined by (31) and the constant does not depend on $f$ and $x$.

**Proof.** Assuming $f(x) = 0$ for $x \in \Omega^n_p \setminus \Omega^q_p$, we have

$$
\int_{\Omega^n_p \setminus \Omega^q_p} \frac{|f(y)|}{|x-y|^{\alpha n}} \, dy \leq c^n \left( \int_{\Omega^n_p \setminus \Omega^q_p} |f(y)| |x-y|^{-\beta n} \, dy \right)^{\alpha n} \left( \int_{\Omega^n_p \setminus \Omega^q_p} |f(y)| \, dy \right)^{-\beta n}.
$$

where the estimate in the second inequality follows taking $\varepsilon = \alpha$ in (36) (we are indeed allowed to take $\varepsilon = \alpha$ in that estimate).

To estimate the integral over the exterior of the ball, using the extension $\tilde{\eta}$ exponent (30), Hölder’s inequality, and the estimate (57), we have

$$
\int_{\Omega^n_p \setminus \Omega^q_p} \frac{|f(y)|}{|x-y|^{\alpha n}} \, dy \leq c^n \left( \int_{\Omega^n_p \setminus \Omega^q_p} |f(y)| |x-y|^{-\beta n} \, dy \right)^{\alpha n} \left( \int_{\Omega^n_p \setminus \Omega^q_p} |f(y)| \, dy \right)^{-\beta n} \leq p^{\beta n q - \alpha \gamma n + \alpha n},
$$

where, in the last inequality, we use the fact that $\tilde{\eta}(\infty) \geq q(x)$ for $x \in \Omega^q_p$. From the previous estimates, when $\|f\|_{L^{q/n}(|\alpha^n|)} \leq 1$, we have

$$
P^n f(x) \leq p^{\beta n Mf(x)} + p^{\beta n q - \alpha n q(x)}.
$$

Choosing

$$
y = \left[ -\log_p \left( \left( \frac{Mf(x)}{q(x)} \right)^{\alpha/n} \right) \right],
$$

and replacing it in (63), we obtain (60).

**Theorem 10.** Let $q \in W_0(\Omega^n_p)$ with $1 \leq q^- \leq q^* < \alpha n$ and $\Omega^q_p$ be a bounded open set in $\Omega^n_p$. Then, the operator

$$
I_{\alpha} : L^{q}(\Omega^n_p) \longrightarrow L^{q^{*}}(\Omega^q_p),
$$

is bounded, where $q^*$ is defined by (31).

**Proof.** Let us take $\|f\|_{L^{q^{*}/(\alpha^n)}} \leq 1$. From (60), (18), and the boundedness of the maximal operator (24), we have

$$
\int_{\Omega^n_p} |f(x)|^{q^{*}(x)} \, dx \leq \int_{\Omega^n_p} |Mf(x)|^{q^{*}(x)} \, dx \leq \left( 1 + \|Mf\|_{L^{q^{*}}(\Omega^n_p)} \right)^{q^-} \leq \left( 1 + \|f\|_{L^{q^{*}}(\Omega^n_p)} \right)^{q^-} \leq 1.
$$

The result now follows from homogeneity of the norm.

## 4. Applications to $p$-adic Derivative

When we work with functions of a $p$-adic variable, $f : \Omega^n_p \longrightarrow \mathbb{C}$, it is not possible to define the derivative in the classical sense (as a limit); therefore, we must resort to the pseudodifferential operators to supply such need. The most popular operator in the $p$-adic numbers that plays the role of the derivative is the Tableson operator (not local operator).

First, we need the Fourier transform. We set $\chi_p(y) = \exp(2\pi i \{y\})$ for $y \in \Omega^n_p$. The map $\chi_p(\cdot)$ is an additive character on $\Omega^n_p$, i.e., a continuous map from $\Omega^n_p$ into the unit circle satisfying $\chi_p(y_0 + y_1) = \chi_p(y_0) \chi_p(y_1)$, $y_0, y_1 \in \Omega^n_p$.

Given $\xi = (\xi_1, \ldots, \xi_n)$ and $x = (x_1, \ldots, x_n) \in \Omega^n_p$, we set $\xi \cdot x = \sum_{i=1}^n \xi_i x_i$. The Fourier transform of $\varphi \in S(\Omega^n_p)$ is defined as

$$
\mathcal{F} \varphi(\xi) = \int_{\Omega^n_p} \chi_p(-\xi \cdot x) \varphi(x) \, dx \quad \text{for } \xi \in \Omega^n_p,
$$

where $dx = d^n x$ is the Haar measure on $\Omega^n_p$ normalized by the condition $\text{Vol}(B^n_1) = 1$.

The Fourier transform is a linear isomorphism from $S(\Omega^n_p)$ onto itself satisfying $(\mathcal{F}(\mathcal{F} \varphi))(\xi) = \varphi(-\xi)$. We will also use the notation $\mathcal{F}x$ for $\mathcal{F} \varphi$ and $\mathcal{F} \varphi$ for the Fourier transform of $\varphi$.

**Definition 11.** The Tableson pseudodifferential is given by operator

$$
(D^a_p \varphi)(x) = \mathcal{F}^{-1} \left( \|\xi\|^{a n} \mathcal{F} \varphi \right), \quad \varphi \in S(\Omega^n_p), \quad a > 0.
$$

This operator can be expressed in the following way:

$$
(D^a_p \varphi)(x) = (R_{-a} \ast \varphi)(x) = \frac{1 - p^{-a}}{1 - p^{-\alpha n}} \int_{\Omega^n_p} \frac{\varphi(x - y) - \varphi(x)}{|y|^{\alpha n}} \, dy,
$$

where $R_{-a}$ is the $p$-adic Riesz kernel; see, e.g., [25]. The right-hand side of the last equation makes sense for a wider class of functions, for example, for locally constant functions $\varphi(x)$ satisfying...
The operator \( D^p_f \) given by Definition 11 can be extended to the maximal domain \( \{ \varphi \in L^p(\Omega^n_p) : \| \xi \|_p^p \varphi(\xi) \in L^p(\Omega^n_p) \} \); thus, this operator is not bounded, and also, its spectrum is \( \{ p^i \} \in \mathbb{Z} \cup \{ 0 \} \).

Theorem 13. Let \(-n < \alpha < 0, 1 < q^- \leq q^+ < n/|\alpha|, \) and \( q \in W^n_0(\Omega^n_p) \). Then, the Taibleson operator

\[
D^p_f : L^{\theta(\cdot)}(\Omega^n_p) \longrightarrow L^{\theta(\cdot)}(\Omega^n_p),
\]

is well-defined and bounded, where \( 1/q^+ = 1/q(x) + \alpha/n \). If \( q \in W^n_0(\Omega^n_p) \), then \( D^p_f : L^{\theta(\cdot)}(\Omega^n_p) \longrightarrow L^{\theta(\cdot)}(\Omega^n_p) \) is bounded.

Proof. Note that, under the hypothesis of the theorem, we have \( D^p_f f(x) = \int L_{\alpha}(x) f(x) \) The result now follows from Theorem 6 and Theorem 10.\hfill \( \square \)

In what follows, we are interested to study the following semilinear Cauchy problem \((\alpha < 0)\):

\[
\begin{aligned}
\frac{\partial u(x, t)}{\partial t} + D^p_f u(x, t) &= f(u(x, t)), \\
u(x, 0) &= u_0(x),
\end{aligned}
\]

where \( f : \mathbb{R} \longrightarrow \mathbb{R} \) is a Lipschitz function.

Lemma 14. Let \(-n < \alpha < 0, 1 < q^- \leq q^+ < n/|\alpha|, \) and \( q \in W^n_0(\Omega^n_p) \), where \( |\Omega^n_p| < \infty \). Then, the operator \( T u(x, t) = -D^p_f u(x, t) + f(u(x, t)) \) is Lipschitz, i.e.,

\[
\| Tu(x, t) - Tv(x, t) \|_{L^{\theta(\cdot)}(\Omega^n_p)} \leq C \| u(x, t) - v(x, t) \|_{L^{\theta(\cdot)}(\Omega^n_p)},
\]

Proof. From the embedding \( L^{\theta(\cdot)}(\Omega^n_p) \rightarrow L^{\theta(\cdot)}(\Omega^n_p) \) and Theorem 10, we see that

\[
\| Tu(x, t) - Tv(x, t) \|_{L^{\theta(\cdot)}(\Omega^n_p)} \leq \| Tu(x, t) - Tv(x, t) \|_{L^{\theta(\cdot)}(\Omega^n_p)} + \| f(u(x, t)) - f(v(x, t)) \|_{L^{\theta(\cdot)}(\Omega^n_p)}
\]

\[
\leq \| D^p_f u(x, t) - v(x, t) \|_{L^{\theta(\cdot)}(\Omega^n_p)} + C_1 \| u(x, t) - v(x, t) \|_{L^{\theta(\cdot)}(\Omega^n_p)}
\]

\[
\leq C \| u(x, t) - v(x, t) \|_{L^{\theta(\cdot)}(\Omega^n_p)},
\]

which ends the proof.\hfill \( \square \)


