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Research Article

Some Results on Fractional m-Point Boundary Value Problems

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In this paper, we will apply some fixed-point theorems to discuss the existence of solutions for fractional m-point boundary value problems $D_{0^+}^q(u''(t))=h(t)f(u(t)), t\in[0,1], 1<q\le 2, u'(0)=u''(0)=u(1)=0, u''(1)-\sum_{i=1}^{m-2}\alpha_i u'''(\xi_i)=0$. In addition, we also present Lyapunov's inequality and Ulam-Hyers stability results for the given m-point boundary value problems.

1. Introduction

Mathematical models due to fractional differential equations can describe the natural phenomenon in physics, population dynamics, chemical technology, biotechnology, aerodynamics, electrodynamics of complex medium, polymer rheology, and control of dynamical systems (see [1–4]). Due to the nonlocal characteristics and the rapid development of the theory of fractional operators, some authors have investigated different aspects of fractional differential equations including existence of solutions, Lyapunov's inequality, and Hyers-Ulam stability for fractional differential equations by different mathematical techniques. For example, first, many authors have discussed the existence of nontrivial solutions of fractional differential equations in nonsingular case as well as singular case. Usually, the proof is based on either the method of upper and lower solutions, fixed-point theorems, alternative principle of Leray-Schauder, topological degree theory, or critical point theory. We refer the readers to [5–20]. Second, Lyapunov, during his study of general theory of stability of motion in 1892, introduced the stability criterion for second-order differential equations, which yielded a counter inequality be called Lyapunov inequality (see [21, 22]). Since then, we can find considerable modifications of Lyapunov-type inequality of differential equations, such as linear differential-algebraic equations, fractional differential equations, extreme Pucci equations, and dynamic equations, which are applied to study the stability and disconjugacy or oscillatory criterion for the mentioned problems, and we refer the readers to [23–32]. Finally, the stability of functional equations was originally raised by Hyers in 1941 (see [33, 34]). Thereafter, the stability properties of all kinds of equations have attracted the attention of many mathematicians. To see more details on the Ulam-Hyers stability and Ulam-Hyers-Rassias of differential equations, we refer the readers to [35–38].

Inspired by the references, this paper is mainly concerned with the existence, Lyapunov's inequality, and Ulam-Hyers stability results for the m-point boundary value problems.

$$\begin{cases} D_{0^+}^q\left(u''(t)\right)=h(t)f(u(t)), t\in[0,1], 1<q\le 2, \\ u'(0)=u''(0)=u(1)=0, u''(1)-\sum_{i=1}^{m-2}\alpha_i u'''(\xi_i)=0, \end{cases} \quad (1)$$

where α_i, ξ_i, h, and f satisfy the following assumptions:

(H1) $\sum_{i=1}^{m-2}\alpha_i\xi_i^{q-2}>1/q-1$ and $\sum_{i=2}^{m-2}\alpha_i\le(1-\xi_{m-2})^{q-1}/(q-1)\xi_{m-2}^{q-2}((1-\xi_{m-2})^{q-1}/(q-1)\xi_{m-2}^{q-2})$

(H2) $h:[0,1]\longrightarrow\mathbb{R}$ is Lebesgue integral

(H3) $f:\mathbb{R}\longrightarrow\mathbb{R}$ is continuous

For these goals, we first convert problem (1) into an integral equation via Green function. Furthermore, we study the properties and estimates of the Green function. Then, on the basis of these properties, we apply some fixed-point theorems to establish some existence results of problem (1) under some suitable conditions. In addition, the Lyapunov inequality and Hyers-Ulam stability of the proposed problem are also considered.

2. Preliminaries

Before beginning the main results, we state some classic and modified definitions and lemmas from fractional calculus.

Definition 1 [4]. The fractional integral of order $q>0$ of a function $u:(0,+\infty)\longrightarrow R$ is given by

$$I_{0^+}^q u(t)=\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}u(s)ds, \tag{2}$$

provided the right-hand side is pointwise defined on $(0,+\infty)$.

Definition 2 [4]. The fractional derivative of order $q>0$ of a continuous function $u:(0,+\infty)\longrightarrow R$ is given by

$$D_{0^+}^q u(t)=\frac{1}{\Gamma(n-q)}\left(\frac{d}{dt}\right)^n\int_0^t \frac{u(s)}{(t-s)^{q-n-1}}ds, \tag{3}$$

where $n=[q]+1$, provided that the right-hand side is pointwise defined on $(0,+\infty)$.

Definition 3 [21]. Assume that $q>0$, then

$$I_{0^+}^q D_{0^+}^q u(t)=u(t)+\sum_{i=1}^{n} C_i t^{q-i}, \tag{4}$$

for some $C_i\in R$, $i=1,2,\cdots,n$, where n is the smallest integer greater than or equal to q.

Lemma 4. *Assume that (H1) holds. Then, for any $y(t)\in L^1[0,1]$, the boundary value problem*

$$\begin{cases} D_{0^+}^q\left(u''(t)\right)=y(t), t\in[0,1], 1<q\le 2,\\ u'(0)=u''(0)=u(1)=0, u''(1)-\sum_{i=1}^{m-2}\alpha_i u'''(\xi_i)=0, \end{cases} \tag{5}$$

has a unique solution $u(t)=\int_0^1 G(t,s)y(s)ds$. Let $p=1-(q-1)\sum_{i=1}^{m-2}\alpha_i\xi_i^{q-2}<0$,, and we have

(i) for $s\le t$, $s\le\xi_1$

$$G(t,s)=\frac{1}{\Gamma(q+2)}\left[(t-s)^{q+1}-(1-s)^{q+1}+1-t^{q+1}-\frac{\left(1-t^{q+1}\right)\left[1-(1-s)^{q-1}\right]}{p}\right] \tag{6}$$

(ii) for $t\le s\le\xi_1$

$$G(t,s)=\frac{1}{\Gamma(q+2)}\left[-(1-s)^{q+1}+1-t^{q+1}-\frac{\left(1-t^{q+1}\right)\left[1-(1-s)^{q-1}\right]}{p}\right] \tag{7}$$

(iii) for $s\le t$, $\xi_j\le s\le\xi_{j+1}$, $j=1,2,\cdots,m-3$

$$G(t,s)=\frac{1}{\Gamma(q+2)}\left[\frac{1-t^{q+1}}{p}\left[(1-s)^{q-1}-(q-1)\cdot\sum_{i=j+1}^{m-2}\alpha_i(\xi_i-s)^{q-2}\right]+(t-s)^{q+1}-(1-s)^{q+1}\right] \tag{8}$$

(iv) for $t\le s$, $\xi_j\le s\le\xi_{j+1}$, $j=1,2,\cdots,m-3$

$$G(t,s)=\frac{1}{\Gamma(q+2)}\left[\frac{1+t^{q+1}}{p}\left[(1-s)^{q-1}-(q-1)\cdot\sum_{i=j+1}^{m-2}\alpha_i(\xi_i-s)^{q-2}\right]-(1-s)^{q+1}\right] \tag{9}$$

(v) for $\xi_{m-2}\le s\le t$

$$G(t,s)=\frac{1}{\Gamma(q+2)}\left[(t-s)^{q+1}-(1-s)^{q+1}+\frac{1-t^{q+1}}{p}(1-s)^{q-1}\right] \tag{10}$$

(vi) for $\xi_{m-2}\le s$, $t\le s$

$$G(t,s)=\frac{1}{\Gamma(q+2)}\left[-(1-s)^{q+1}+\frac{1-t^{q+1}}{p}(1-s)^{q-1}\right] \tag{11}$$

Proof. From Definition 3, it follows that

$$u''(t)=I_{0^+}^q y(t)-C_1t^{q-1}-C_2t^{q-2}. \tag{12}$$

Since $u''(0)=0$, it is clear that $C_2=0$. Then,

$$u''(t)=I_{0^+}^{q}y(t)-C_1t^{q-1}=\frac{1}{\Gamma(q)}\int_0^t(t-s)^{q-1}y(s)ds-C_1t^{q-1}. \tag{13}$$

On one hand, taking the derivative of $u''(t)$, we can get

$$u'''(t)=\frac{q-1}{\Gamma(q)}\int_0^t(t-s)^{q-2}y(s)ds-(q-1)C_1t^{q-2}. \tag{14}$$

On the other hand, combining the boundary conditions $u(1)=u'(0)=0$, we have

$$\begin{aligned}u'(t)&=\frac{1}{\Gamma(q)}\int_0^t\int_0^\tau(\tau-s)^{q-1}y(s)d\tau-\frac{C_1}{q}t^q\\&=\frac{1}{\Gamma(q+1)}\int_0^t(t-s)^{q}y(s)ds-\frac{C_1}{q}t^q.\end{aligned} \tag{15}$$

Furthermore, we have

$$\begin{aligned}u(t)&=-\frac{1}{\Gamma(q+1)}\int_t^1\int_0^\tau(\tau-s)^{q}y(s)dsd\tau+\frac{C_1}{q}\int_t^1\tau^q d\tau\\&=-\frac{1}{\Gamma(q+2)}\left[\int_0^t(1-s)^{q+1}y(s)ds-\int_0^t(t-s)^{q+1}y(s)ds\right]\\&\quad+\frac{C_1\left(1-t^{q+1}\right)}{q(q+1)}.\end{aligned} \tag{16}$$

According to these above expressions, we have

$$\begin{aligned}u''(1)&=\frac{1}{\Gamma(q)}\int_0^1(1-s)^{q-1}y(s)ds-C_1,\\\sum_{i=1}^{m-2}\alpha_iu'''(\xi_i)&=\frac{q-1}{\Gamma(q)}\sum_{i=1}^{m-2}\alpha_i\int_0^{\xi_i}(\xi_i-s)^{q-2}y(s)ds\\&\quad-(q-1)C_1\sum_{i=1}^{m-2}\alpha_i\xi_i^{q-2}.\end{aligned} \tag{17}$$

Then, from $u''(1)-\sum_{i=1}^{m-2}\alpha_iu'''(\xi_1)=0$, it follows that

$$\begin{aligned}C_1&=\frac{1}{p\Gamma(q)}\left[\int_0^1(1-s)^{q-1}y(s)ds\right.\\&\quad\left.-(q-1)\sum_{i=1}^{m-2}\alpha_i\int_0^{\xi_i}(\xi_i-s)^{q-2}y(s)ds\right],\end{aligned} \tag{18}$$

which yields

$$\begin{aligned}u(t)&=\frac{1}{\Gamma(q+2)}\left[\int_0^t(t-s)^{q+1}y(s)ds-\int_0^1(1-s)^{q+1}y(s)ds\right.\\&\quad+\frac{1-t^{q+1}}{p}\left(\int_0^1(1-s)^{q-1}y(s)ds\right.\\&\quad\left.\left.-(q-1)\sum_{i=1}^{m-2}\alpha_i\int_0^{\xi_i}(\xi_i-s)^{q-2}y(s)ds\right)\right].\end{aligned} \tag{19}$$

If $s\le t$, $s\le\xi_1$, we have

$$\begin{aligned}G(t,s)&=\frac{1}{\Gamma(q+2)}\left[(t-s)^{q+1}-(1-s)^{q+1}+\frac{1-t^{q+1}}{p}\right.\\&\quad\left.\cdot\left((1-s)^{q-1}-(q-1)\sum_{i=1}^{m-2}\alpha_i(\xi_i-s)^{q-2}\right]\right.\\&=\frac{1}{\Gamma(q+2)}\left[(t-s)^{q+1}-(1-s)^{q+1}+1\right.\\&\quad\left.-t^{q+1}-\frac{1-t^{q+1}}{p}\left(1-(1-s)^{q-1}\right)\right].\end{aligned} \tag{20}$$

□

In the similar way, we also can get the expression of $G(t,s)$ on other intervals.

Lemma 5. *Assume that (H1) holds. Then, $G(t,s)$ satisfies the following properties:*

(I) Sign of $G(t,s)$

(i) $G(t,s)\ge0$, for $0\le s\le\xi_1$

(ii) $G(t,s)\le0$, for $\xi_1<s\le1$

(II) The range of $G(t,s)$

(1) For $0\le s\le\xi_1$

$$0\le G(t,s)<\frac{1}{\Gamma(q+2)}\left[1-(1-s)^{q+1}-\frac{1-(1-s)^{q-1}}{p}\right] \tag{21}$$

(2) For $\xi_j\le s\le\xi_{j+1}$, $j=1,2,\cdots,m-3$

$$\begin{aligned}&\frac{1}{\Gamma(q+2)}\left[\frac{1}{p}\left((1-s)^{q-1}-(q-1)\sum_{i=j+1}^{m-2}\alpha_i(\xi_i-s)^{q-2}\right)\right.\\&\quad\left.-(1-s)^{q+1}\right]\le G(t,s)\le0\end{aligned} \tag{22}$$

(3) For $\xi_{m-2} \le s \le 1$

$$\frac{1}{\Gamma(q+2)}\left[\frac{(1-s)^{q-1}}{p}-(1-s)^{q+1}\right] \le G(t,s) \le 0 \quad (23)$$

Proof. For $0 \le s \le \xi_1$, by the definition of $G(t,s)$, it is clear that $G(t,s)$ is continuous and derivativable with respect to t at $[0,1]$. On one hand, if $s \le t \le 1$, we have

$$\frac{\partial G}{\partial t} = \frac{1}{\Gamma(q+1)}\left((t-s)^q - t^q + \frac{t^q\left[1-(1-s)^{q-1}\right]}{p}\right) \le 0. \quad (24)$$

On the other hand, if $0 \le t < s$, we have

$$\frac{\partial G}{\partial t} = \frac{t^q}{\Gamma(q+1)}\left(-1+\frac{1-(1-s)^{q-1}}{p}\right) \le 0. \quad (25)$$

Then, $G(t,s)$ is nonincreasing on t, which yields that

$$\begin{aligned} &\min\{G(t,s) : t \in [0,1]\} = G(1,s) = 0, \\ &\max\{G(t,s) : t \in [0,1]\} \\ &\quad = G(0,s) = \frac{1}{\Gamma(q+2)}\left[1-(1-s)^{q+1}-\frac{1-(1-s)^{q-1}}{p}\right]. \end{aligned} \quad (26)$$

So for $0 \le s < \xi_1$, $0 \le t \le 1$, it concludes that

$$0 \le G(t,s) < \frac{1}{\Gamma(q+2)}\left[1-(1-s)^{q+1}-\frac{1-(1-s)^{q-1}}{p}\right]. \quad (27)$$

For $\xi_j \le s \le \xi_{j+1}\,(j=1,2,\cdots,m-3)$, we have

$$\begin{aligned} &\min\{G(t,s) : t \in [0,1]\} \\ &\quad = G(0,s) = \frac{1}{\Gamma(q+2)}\left[\frac{1}{p}\left((1-s)^{q-1}\right.\right. \\ &\qquad \left.\left. -(q-1)\sum_{i=j+1}^{m-2}\alpha_i(\xi_i-s)^{q-2}\right)-(1-s)^{q+1}\right], \\ &\max\{G(t,s) : t \in [0,1]\} = G(1,s) = 0. \end{aligned} \quad (28)$$

For $\xi_{m-2} \le s \le 1$, we have

$$\begin{aligned} &\min\{G(t,s) : t \in [0,1]\} \\ &\quad = G(0,s) = \frac{1}{\Gamma(q+2)}\left[\frac{(1-s)^{q-1}}{p}-(1-s)^{q+1}\right], \\ &\max\{G(t,s) : t \in [0,1]\} = G(1,s) = 0. \end{aligned} \quad (29)$$

□

Let

$$\begin{aligned} G^1 &= \max_{s\in[0,\xi_1]}\left\{\frac{1}{\Gamma(q+2)}\left[1-(1-s)^{q+1}-\frac{1-(1-s)^{q-1}}{p}\right]\right\} \\ &= \frac{1}{\Gamma(q+2)}\left[1-(1-\xi_1)^{q+1}-\frac{1-(1-\xi_1)^{q-1}}{p}\right], \\ G^2 &= \max_{s\in[\xi_{m-2},1]}\left\{\frac{1}{\Gamma(q+2)}\left|\frac{(1-s)^{q-1}}{p}-(1-s)^{q+1}\right|\right\} \\ &= \frac{1}{\Gamma(q+2)}\left[(1-\xi_{m-2})^{q+1}-\frac{(1-\xi_{m-2})^{q-1}}{p}\right], \\ G^3 &= \max_{1\le j\le m-3}\left\{G_j^3\right\}, \end{aligned} \quad (30)$$

where

$$\begin{aligned} G_j^3 = \max_{s\in[\xi_j,\xi_{j+1}]}&\left\{\frac{1}{\Gamma(q+2)}\left|\frac{1}{p}\left[(1-s)^{q-1}\right.\right.\right. \\ &\left.\left.\left. -(q-1)\sum_{i=j+1}^{m-2}\alpha_i(\xi_i-s)^{q-2}\right]-(1-s)^{q+1}\right|\right\}. \end{aligned} \quad (31)$$

From Lemma 5, it is clear that $|G(t,s)| \le \bar{G}$, where $\bar{G} = \max\{G^1, G^2, G^3\}$.

Lemma 6. *Assume that (H1) holds and $\xi_1 > 1-(1/2)^{1/q-1}$. Then, $\bar{G} = G^1$.*

Proof. Let

$$\begin{aligned} G_j(s) = \frac{1}{\Gamma(q+2)}&\left|\frac{1}{p}\left[(1-s)^{q-1}-(q-1)\sum_{i=j+1}^{m-2}\alpha_i(\xi_i-s)^{q-2}\right]\right. \\ &\left. -(1-s)^{q+1}\right|, s\in[\xi_j,\xi_{j+1}], j=1,2,\cdots,m-3. \end{aligned} \quad (32)$$

From (H1) and $\xi_1 > 1-(1/2)^{1/q-1}$, we can verify that

$$\begin{aligned} &G^1 - \left|G_j(s)\right| \\ &\quad = \frac{1}{\Gamma(q+2)}\left[1-(1-\xi_1)^{q+1}-\frac{1}{p}\left(1-(1-\xi_1)^{q-1}\right)\right. \\ &\qquad -(1-s)^{q+1}+\frac{1}{p}\left((1-s)^{q-1}\right. \\ &\qquad \left.\left. -(q-1)\sum_{i=j+1}^{m-2}\alpha_i(\xi_i-s)^{q-2}\right)\right] \end{aligned}$$

$$\geq \frac{1}{\Gamma(q+2)}\left[1-(1-\xi_1)^{q+1}-(1-\xi_j)^{q+1}\right.$$
$$-\frac{1}{p}\left(1-(1-\xi_1)^{q-1}-(1-\xi_1)^{q-1}\right)$$
$$\left.-\frac{q-1}{p}\sum_{i=j+1}^{m-2}\alpha_i(\xi_i-\xi_j)^{q-2}\right]$$
$$\geq \frac{1}{\Gamma(q+2)}\left[1-2(1-\xi_1)^{q+1}-\frac{1}{p}\left(1-2(1-\xi_1)^{q-1}\right)\right.$$
$$\left.-\frac{q-1}{p}\sum_{i=j+1}^{m-2}\alpha_i(\xi_i-\xi_j)^{q-2}\right]$$
$$\geq \frac{1}{\Gamma(q+2)}\left[1-\frac{1}{2^{2/q-1}}-\frac{q-1}{p}\sum_{i=j+1}^{m-2}\alpha_i(\xi_i-\xi_j)\right]>0. \tag{33}$$

Also, we can verify that

$$G^1-G^2=\frac{1}{\Gamma(q+2)}\left[1-(1-\xi_1)^{q+1}-\frac{1}{p}\left(1-(1-\xi_1)^{q-1}\right)\right.$$
$$\left.-(1-\xi_{m-2})^{q+1}+\frac{1}{p}(1-\xi_{m-2})^{q-1}\right]$$
$$\geq \frac{1}{\Gamma(q+2)}\left[1-2(1-\xi_1)^{q+1}-\frac{1}{p}\left(1-2(1-\xi_1)^{q-1}\right)\right]$$
$$\geq \frac{1}{\Gamma(q+2)}\left(1-\frac{1}{2^{2/q-1}}\right)>0. \tag{34}$$

So, it concludes that $G^1>G^2$, $G^1>G^3$, namely, $\bar{G}=G^1$. □

3. Main Results

3.1. Existence Results

Theorem 7. *Assume that (H1)-(H3) hold. In addition, there exists a positive constant $L>0$ such that*

$$|f(u)-f(v)|\leq L|u-v|,\forall u,v\in\mathbb{R}. \tag{35}$$

Then, problem (1) has a unique solution if $L\bar{G}|h|_{L^1}<1$.

Proof. Let $C[0,1]=\{x(t)\colon x(t)\text{ is continuous on }[0,1]\}$ is a Banach space with the norm $\|x\|=\max_{0\leq t\leq 1}|x(t)|$. From Lemma 4, it is clear that solutions of (1) can be rewritten as fixed points of operator T, which is defined by

$$Tu(t)=\int_0^1 G(t,s)h(s)f(u(s))ds. \tag{36}$$

Now, we show that $T:B_r\longrightarrow B_r$ and T is a contraction map, where $B_r=\{u\in E:\|u\|<r\}$ with

$$r>\frac{\bar{G}|h|_{L^1}|f(0)|}{1-\bar{G}L|h|_{L^1}}. \tag{37}$$

On one hand, for any $u\in B_r$, we have

$$\begin{aligned}\|T(u)(t)\|&=\left\|\int_0^1 G(t,s)h(s)f(u(s))ds\right\|\\&\leq\bar{G}\int_0^1|h(s)f(u(s))|ds\\&\leq\bar{G}\int_0^1|h(s)|[|f(u)-f(0)|+|f(0)|]ds\\&\leq\bar{G}\int_0^1|h(s)|[Lr+|f(0)|]ds\\&\leq\bar{G}|h|_{L^1}[Lr+|f(0)|]\leq r,\end{aligned} \tag{38}$$

which implies that $T(B_r)\subset B_r$.

On the other hand, for any $u,v\in E$, we have

$$\begin{aligned}\|T(u)-T(v)\|&=\left\|\int_0^1 G(t,s)h(s)(f(u(s)))ds\right\|\\&\leq\bar{G}\int_0^1|h(s)||f(u(s))-f(v(s))|ds\\&\leq\bar{G}L\int_0^1|h(s)||u(s)-v(s)|ds\\&\leq L\bar{G}|h|_{L^1}\|u(t)-v(t)\|,\end{aligned} \tag{39}$$

which implies that T is a contraction map. □

Therefore, by the Banach contraction mapping principle, it follows that the operator T has a unique fixed point, which is the unique solution for problem (1).

Theorem 8. *Assume that (H1)-(H3) hold. In addition, there exists a positive constant K such that $|f(u)|\leq K$ for $u\in R$. Then, problem (1) has at least one solution.*

The proof is based on the following fixed-point theorem.

Lemma 9 [39]. *Let E be a Banach space, E_1 is a closed, convex subset of E, Ω an open subset of E_1, and $0\in\Omega$. Suppose that $T:\Omega\longrightarrow E_1$ is completely continuous. Then, either*

(i) T has a fixed point in Ω, or

(ii) there are $u\in\partial\Omega$ (the boundary of Ω in E_1) and $\rho\in(0,1)$ with $u=\rho Tu$

Proof of Theorem 8. First, we show that the operator T is uniformly bounded.

For any $u \in \bar{\Omega}_\delta = \{u \in C[0,1]: \|u\| \le \delta\}$, we have

$$\|T(u)(t)\| = \left\|\int_0^1 G(t,s)h(s)f(u(s))ds\right\| \le \bar{G}K|h|_{L^1}, \quad (40)$$

which implies that $T(\Omega_\delta)$ is uniformly bounded.

Second, for $0 \le s \le \xi_1$, from Lemma 4, we have

(i) if $s \le t \le 1$

$$\begin{aligned}\left|\frac{\partial G(t,s)}{\partial t}\right| &= \frac{1}{\Gamma(q+1)}\left|(t-s)^q - t^q + \frac{t^q\left[1-(1-s)^{q-1}\right]}{p}\right| \\ &\le \frac{1}{\Gamma(q+1)}\left((t-s)^q + t^q - \frac{t^q\left[1-(1-s)^{q-1}\right]}{p}\right) \\ &\le \frac{1}{\Gamma(q+1)}\left(2-\frac{1}{p}\right).\end{aligned} \quad (41)$$

(ii) if $0 \le t < s$

$$\begin{aligned}\left|\frac{\partial G(t,s)}{\partial t}\right| &= \frac{1}{\Gamma(q+1)}\left|-t^q + \frac{t^q\left[1-(1-s)^{q-1}\right]}{p}\right| \\ &\le \frac{1}{\Gamma(q+1)}\left(t^q - \frac{t^q\left[1-(1-s)^{q-1}\right]}{p}\right) \\ &\le \frac{1}{\Gamma(q+1)}\left(1-\frac{1}{p}\right),\end{aligned} \quad (42)$$

which implies that $|\partial G(t,s)/\partial t|$ is bounded for $0 \le s < \xi_1$, $0 \le t \le 1$. In the similar way, we know that there exists a $S > 0$ such that $|\partial G(t,s)/\partial t| \le S$ for $0 \le s, t \le 1$.

Furthermore, for $t_1, t_2 \in [0,1]$, we have

$$\begin{aligned}&|Tu(t_2) - Tu(t_1)| \\ &\quad = \left|\int_0^1 G(t_2,s)h(s)f(u(s))ds - \int_0^1 G(t_1,s)h(s)f(u(s))ds\right| \\ &\quad = \left|\int_0^1 [G(t_2,s) - G(t_1,s)]h(s)f(u(s))ds\right| \\ &\quad \le SK|h|_{L^1}|t_2 - t_1|.\end{aligned} \quad (43)$$

Therefore, applying the Arzela-Ascoli theorem [39], we can find that $T(\Omega_\delta)$ is relatively compact.

Third, we claim that $T : \bar{\Omega}_\delta \longrightarrow \mathbb{R}$ is continuous. Assume that $\{u_n\}_{n=1}^\infty \subset \bar{\Omega}_\delta$, which converges to u_0 uniformly on [0,1]. Since $\{(Tu_n)(t)\}_{n=1}^\infty$ is uniformly bounded and equicontinuous on [0,1], from the Arzela-Ascoli theorem, it follows that there exists a uniformly convergent subsequence in $\{(Tu_n)(t)\}_{n=1}^\infty$. Let $\{(Tu_{n(m)})(t)\}_{m=1}^\infty$ be a subsequence which converges to $v(t)$ uniformly on [0,1]. Observe that

$$Tu_{n(m)}(t) = \int_0^1 G(t,s)h(s)f\left(u_{n(m)}(s)\right)ds. \quad (44)$$

Furthermore, by Lebesgue's dominated convergence theorem and letting $m \longrightarrow \infty$, we have

$$v(t) = \int_0^1 G(t,s)h(s)f(u_0(s))ds, \quad (45)$$

namely, $v(t) = Tu_0(t)$. This shows that each subsequence of $\{(Tu_n)(t)\}_{n=1}^\infty$ uniformly converges to $v(t)$. Therefore, the sequence $\{(Tu_n)(t)\}_{n=1}^\infty$ uniformly converges to $Tu_0(t)$. This means that T is continuous at $u_0 \in \Omega_\delta$. So, T is continuous on $\bar{\Omega}_\delta$. Thus, T is completely continuous.

Finally, let $\Omega_\delta = \{u \in C[0,1]: \|u\| < \delta\}$ with $\delta = \bar{G}K|h|_{L^1} + 1$. If u is a solution of problem (1), then, for $\rho \in (0,1)$, $u \in \partial\Omega_\delta$, we have

$$\begin{aligned}\|u\| &= \rho\|Tu(t)\| = \rho\left\|\int_0^1 G(t,s)h(s)f(u(s))ds\right\| \\ &\le \rho\bar{G}\int_0^1 |h(s)f(u(s))|ds \le \bar{G}K|h|_{L^1},\end{aligned} \quad (46)$$

which yields a contradiction. Therefore, by Lemma 9, the operator T has a fixed point in Ω_δ.

Theorem 10. *Assume that (H1)-(H3) hold. In addition, f satisfies the following assumptions:*

(H4) There exists a nondecreasing function $\psi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ such that

$$|f(u)| \le \psi(\|u\|), \forall u \in \mathbb{R}. \quad (47)$$

(H5) There exists a constant $R > 0$ such that $R/\bar{G}|h|_{L^1}\psi(R) > 1$. Then, problem (1) has at least one solution.

Proof. Now we show that (ii) of Lemma 9 does not hold. If u is a solution of problem (1), then for $\rho \in (0,1)$, we obtain

$$\begin{aligned}\|u\| &= \rho\|T(u(t))\| \le \rho\int_0^1 |G(t,s)h(s)f(u(s))|ds \\ &\le \rho\bar{G}\int_0^1 |h(s)f(u(s))|ds \le \bar{G}|h|_{L^1}\psi(\|u\|).\end{aligned} \quad (48)$$

Let $B_R = \{u \in C[0,1]: \|u\| < R\}$. From the above inequality and (H5), it yields a contradiction. Therefore, by Lemma 9, the operator T has a fixed point in B_R. □

3.2. Lyapunov's Inequality

Theorem 11. *Assume that (H1)-(H3) hold. In addition, $f(u)$ is a concave function on $\mathbb{R}$. Then, for any nontrivial solution of problem (1), we have*

$$\int_0^1 |h(t)|dt > \frac{\|u\|}{\bar{G}\max_{u\in[u_*,u^*]}|f(u)|}, \quad (49)$$

where

$$u_* = \min_{t\in[0,1]} u(t),\ u^* = \max_{t\in[0,1]} u(t). \tag{50}$$

Proof. If $u(t)$ is a nontrivial solution of problem (1), then by Lemma 4, we have

$$u(t) = \int_0^1 G(t,s)h(s)f(u(s))ds. \tag{51}$$

Furthermore, by Lemma 6, we have

$$|u(t)| \le \int_0^1 |G(t,s)h(s)f(u(s))|ds. \tag{52}$$

Since f is continuous and concave, then from Jensen's inequality, it follows that

$$\begin{aligned} \|u(t)\| &\le \max_{t\in[0,1]} \int_0^1 |G(t,s)||h(s)f(u(s))|ds \\ &\le \int_0^1 \left[\max_{t\in[0,1]} |G(t,s)|\right] |h(s)f(u(s))|ds \\ &\le \bar{G}|h(t)|_{L^1} \int_0^1 \frac{|h(s)|}{|A(t)|_{L^1}} |f(u(s))|ds \\ &\le \bar{G} \max_{u\in[u_*,u^*]} |f(u)||h|_{L^1}, \end{aligned} \tag{53}$$

namely,

$$\int_0^1 |h(t)dt| > \frac{\|u\|}{\bar{G}\max_{u\in[u_*,u^*]}|f(u)|}. \tag{54}$$

□

3.3. Stability Analysis

Definition 12 [34]. Equation (1) is said to be Ulam-Hyers-Rassias stability with respect to $\Psi \in C[0,1]$ if there exists a nonzero positive real number μ such that for every $\varepsilon > 0$ and each solution $v \in C[0,1]$ of the inequality

$$\left|D_{0^+}^q v''(t) - h(t)f(v(t))\right| \le \varepsilon\Psi(t),\ t \in [0,1], \tag{55}$$

there exists a solution $u \in C[0,1]$ of problem (1) such that $|u(t) - v(t)| \le \mu\varepsilon\Psi(t)$, $t \in [0,1]$.

Theorem 13. *Assume that (H1)-(H3) hold. In addition, there exists a positive constant $L > 0$ such that*

$$|f(u) - f(v)| \le L|u - v|, \forall u, v \in C[0,1]. \tag{56}$$

Then, problem (1) is Ulam-Hyers-Rassias stability if $LG|h|_{L^1} < 1$.

Proof. Let $v \in C[0,1]$ be the solution of the inequality (55); then,

$$\left|D_{0^+}^1 v''(t) - h(t)f(v(t))\right| \le \varepsilon\Psi(t),\ t \in [0,1]. \tag{57}$$

Thus, for $\varepsilon > 0$, we get

$$\left|v(t) - \int_0^1 G(t,s)h(s)f(v(s))ds\right| \le \varepsilon\Psi(t),\ t \in [0,1]. \tag{58}$$

By Theorem 7, problem (1) has a solution $u(t)$ satisfies

$$u(t) = \int_0^1 G(t,s)h(s)f(u(s))ds. \tag{59}$$

Then, for $t \in [0,1]$, we have

$$\begin{aligned} |v(t) - u(t)| &= \left|v(t) - \int_0^1 G(t,s)h(s)f(u(s))ds\right| \\ &\le \left|v(t) - \int_0^1 G(t,s)h(s)f(v(s))ds\right| \\ &\quad + \left|\int_0^1 G(t,s)h(s)(f(u(s)) - f(v(s)))ds\right| \\ &\le \varepsilon\Psi(t) + L\int_0^1 |G(t,s)h(s)(u(s) - v(s))|ds \\ &\le \varepsilon\Psi(t) + L\bar{G}\int_0^1 |h(s)||u(s) - v(s)|ds \\ &\le \varepsilon\Psi(t) + L\bar{G}|h|_{L^1}|u(t) - v(t)|, \end{aligned} \tag{60}$$

which yields

$$|v(t) - u(t)| \le \frac{\varepsilon\Psi(t)}{1 - L\bar{G}|h|_{L^1}} = \mu\varepsilon\Psi(t),\ t\varepsilon[0,1]. \tag{61}$$

Therefore, problem (1) is Ulam-Hyers-Rassias stability. □

4. Examples

Now we give some examples to illustrate our main results.

Example 1. We consider the following problem:

$$\begin{cases} D_{0^+}^{3/2} u''(t) = 6t\ \arctan u,\ t \in [0,1], \\ u'(0) = u''(0) = u(1) = 0,\ u''(1) - 2u'''\left(\frac{4}{5}\right) - \frac{1}{8}u'''\left(\frac{6}{7}\right) = 0, \end{cases} \tag{62}$$

where $h(t) = 12t$ and $f(u) = 1/2 \arctan u$. It is obvious that (H1)-(H3) hold. Via some computations, we have

$$\xi_1 = \frac{4}{5} > 1 - \frac{1}{2^{1/q-1}} = \frac{3}{4},$$
$$p = 1 - \frac{\sqrt{5}}{2} - \frac{\sqrt{7}}{16\sqrt{6}},$$
$$\bar{G} = \frac{8}{105\sqrt{\pi}}\left[1 - \frac{1}{\sqrt{5^5}} - \frac{\sqrt{5}-1}{\sqrt{5 - 5/2 - 35/16\sqrt{6}}}\right] \approx 0.170285. \quad (63)$$

Since

$$f'(u) = \left(\frac{1}{2}\arctan u\right)' = \frac{1}{2(1+u^2)} \le \frac{1}{2} = L, \quad (64)$$

the function f satisfies the condition

$$|f(u) - f(v)| \le L|u - v|, \forall u, v \in C[0,1]. \quad (65)$$

Furthermore, we can verify that $GL|h|_{L^1} \approx 0.510855 < 1$. Therefore, by Theorem 7 and Theorem 13, problem (62) has a solution $u(t)$, which is Ulam-Hyers-Rassias stability.

Example 2. Let us consider the following problem:

$$\begin{cases} D_{0^+}^{3/2} u''(t) = 6t\left(2\arctan u^{1/2} + \sin u\right), t \in [0,1], \\ u'(0) = u''(0) = u(1) = 0, u''(1) - 2u'''\left(\frac{4}{5}\right) - \frac{1}{8}u'''\left(\frac{6}{7}\right) = 0, \end{cases} \quad (66)$$

where $h(t) = 6t$ and

$$|f(u)| = \left|2\arctan u^{1/2} + \sin u\right| \le 2\|u\|^{1/2} + \|u\| = \psi(\|u\|). \quad (67)$$

It is obvious that (H1)-(H4) hold. By computations of Example 2, we have

$$\bar{G} \approx 0.170285. \quad (68)$$

Furthermore, for $R > 4.362954$, the inequality $R/\bar{G}|h|_{L^1}\psi(R) > 1$ holds, Therefore, by Theorem 10, problem (66) has at least one solution.

Data Availability

All data generated or analyzed during this study are included in this article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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