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Research Article

An Attractive Approach Associated with Transform Functions for Solving Certain Fractional Swift-Hohenberg Equation

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Many phenomena in physics and engineering can be built by linear and nonlinear fractional partial differential equations which are considered an accurate instrument to interpret these phenomena. In the current manuscript, the approximate analytical solutions for linear and nonlinear time-fractional Swift-Hohenberg equations are created and studied by means of a recent superb technique, named the Laplace residual power series (LRPS) technique under the time-Caputo fractional derivatives. The proposed technique is a combination of the generalized Taylor's formula and the Laplace transform operator, which depends mainly on the concept of limit at infinity to find the unknown functions for the fractional series expansions in the Laplace space with fewer computations and more accuracy comparing with the classical RPS that depends on the Caputo fractional derivative for each step in obtaining the coefficient expansion. To test the simplicity, performance, and applicability of the present method, three numerical problems of the time-fractional Swift-Hohenberg initial value problems are considered. The impact of the fractional order β on the behavior of the approximate solutions at fixed bifurcation parameter is shown graphically and numerically. Obtained results emphasized that the LRPS technique is an easy, efficient, and speed approach for the exact description of the linear and nonlinear time-fractional models that arise in natural sciences.

1. Introduction

Partial fractional models are a natural generalization of classical multivariable models with arbitrary order derivate that have received great interest in the scientific community due to their diverse applications in engineering, physics, pharmacology, astronomy, and medicine. From the classical point of view, the following leading equation has been proposed in 1977 by Swift and Hohenberg [1] as a global model for describing fluid velocity and temperature dynamics of thermal convection:

$$\frac{\partial w}{\partial t} = rw - \left(1 + \nabla^2\right)^2 w - w^3, \tag{1}$$

where $x \in \mathbb{R}, t > 0$, and r is bifurcation parameter. w is a scaler function of x and t defined on the line or the plane. The Swift-Hohenberg (S-H) equation is a mathematical model that has a great role in modeling the pattern formulation theory which includes the chosen of pattern, the impacts of noise on bifurcations, the dynamics of defects, and spatio-temporal chaos [2–4]. Also, it describes numerous models in engineering and thermal physics including the pattern formulation theory in fluid layers, hydrodynamics, lasers, flame dynamics, and statistical mechanics [5–7].

In the last decades, considerable attention has been paid to the topic of fractional differential equations (FDEs) and fractional partial differential equations (FPDEs) due to the fast-growing and widespread of their applications in various science and engineering areas like medicine, chemistry, biology, electrical engineering, and viscoelasticity, and for more details about these applications and others, we refer to [8–15]. Fractional derivative is a significant instrument that gives ideal assistance to characterize the memory and hereditary features of different processes and materials. With this, numerous mathematicians presented and developed various differential operators that allow analyzing the dynamical behaviors of solutions for FDEs and FPDEs [16–20].

The investigation of the closed-form solutions for the linear and nonlinear FDEs and FPDEs is a difficult task. However, the numerical and analytical techniques have been proposed to solve the nonlinear FPDEs. The most common of these methods to determine approximate analytical solutions for FPDEs are the Adomian decomposition method, variational iteration method, homotopy perturbation method, and reproducing kernel method [21–25]. In this work, we consider the nonlinear time-fractional S-H equation with bifurcation [4].

$$D_t^{\beta} w + \frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^2 w}{\partial x^2} + (1-r)w + w^3 = 0, 0 < \beta \le 1, \quad (2)$$

subject to initial condition $w(x,0) = w_0(x)$, where D_t^{β} indicates to the fractional derivative in the Caputo meaning. Particularly, if $\beta = 1$, the fractional sense (2) reduces to the standard case S-H Equation (1). Equation (2) has been solved by numerous scholars, remarkably, Atangana and Kılıçman [3], Li and Pang [4], Prakasha et al. [5], and others. Solving nonlinear FDEs and nonlinear FPDEs is not an easy matter. However, several mathematical techniques have been employed to investigate their solutions. For example, homotopy analysis technique, variational iteration technique, Adomian decomposition technique, residual power series technique, and other techniques are some of these methods, and for more details, see [21, 26–33].

Unfortunately, the aforesaid standard traditional techniques and others need more computational time and computer memory to determine the closed-form solutions for the nonlinear problems. To beat these defects, some scholars coupled the standard existing approaches and the Laplace transform approach like Kumar in [34] investigated the solution for the time-fractional Cauchy-reaction diffusion equation by using the homotopy perturbation transform approach. In [35], a coupled fractional system of PDEs was studied by the generalized and reduced differential transform approaches. The authors in [36] applied the fractional reduced differential transform approach to provide the solutions of the nonlinear fractional Klein-Gordon equation. In [37, 38], the explicit solutions of some FPDEs are constructed by combining the homotopy perturbation approach and fractional Sumudu transform approach.

The motivation of this work is to construct an approximate analytical solution for the nonlinear time-fractional S-H Equation (2) by directly applying the Laplace residual power series (LRPS) technique. This technique has been presented and proved in [39] to investigate the exact solitary solutions for a certain class of time-FPDEs. It is an efficient analytic-numeric technique that is constructed by coupling the RPS technique with Laplace transform operator. The basic idea of the present approach depends upon transferring the target problem into the Laplace space and creating the solutions for the new algebraic equation, and then finally, by utilizing the inverse Laplace transform of the obtained results, one can get the solution of the target problem. The suggested approach determines the unknown functions of the suggested fractional expansion series based on the limit concept, unlike the RPS approach that based on the concepts of the fractional derivative. So, the unknown functions can be found by fewer time computations, and hence, the approximate analytical solution can be constructed as a convergent multiple fractional power series [40, 41].

The structure of this analysis is as follows: In Section 2, some of the necessary definitions and basic theories of fractional calculus and Laplace power series representation are reviewed. The methodology of the present technique in finding the series solution for (2) is introduced, in Section 3. Three different initial value problems of the time-fractional Swift-Hohenberg problem are studied to confirm the simplicity and applicability of the proposed approach for constructing the series solutions, in Section 4. Lastly, a conclusion is drawn in Section 5.

2. Fundamental Concepts

In this section, essential definitions of fractional and Laplace transform operators and the fundamental properties related to them are revisited. In addition, we review basic theories and primary results related to fractional Taylor's formula in the Laplace space.

Definition 1 (see [10]). For $\beta \in \mathbb{R}^+$, the Riemann-Liouville fractional integral operator for a real-valued function $w(x, t)$ is denoted by $\mathcal{J}_t^{\beta}$ and is defined as follows:

$$\mathcal{J}_t^{\beta} w(x,t) = \begin{cases} \frac{1}{\Gamma(\beta)} \int_0^t \frac{w(x,\eta)}{(t-\eta)^{1-\beta}} d\eta, & 0 < \eta < t, \beta > 0, \\ w(x,t) & \beta = 0. \end{cases} \quad (3)$$

Definition 2 (see [10]). The time fractional derivative of order $\beta > 0$, for the function $w(x,t)$ in the Caputo case, is denoted by D_t^{β}, and it is defined as follows:

$$D_t^{\beta} w(x,t) = \begin{cases} \mathcal{J}_t^{n-\beta}(\mathcal{D}_x^n w(x,t)), & 0 < n-1 < \beta \le n, \\ \mathcal{D}_x^n w(x,t), & \beta = n, \end{cases} \quad (4)$$

where $\mathcal{D}_x^n = \partial^n / \partial x^n$, and $n \in \mathbb{N}$.

Definition 3 (see [39]). The Laplace transform of the piecewise continuous function $w(x,t)$ on $T \times [0,\infty)$ is defined as follows:

$$\mathscr{L}\{w(x,t)\} = W(x,s) = \int_0^\infty w(x,t)\, e^{-st} dt, s > \sigma, \tag{5}$$

where σ is the exponential order of $w(x, t)$.

Definition 4 (see [39]). The inverse Laplace transform of the function $W(x, s)$ is defined as follows:

$$w(x,t) = \mathscr{L}^{-1}\{W(x,s)\} = \int_{a-i\infty}^{a+i\infty} W(x,s)\, e^{st} ds, a = \mathrm{Re}\,(s) > b, \tag{6}$$

where $W(x, s)$ is analytic transform function on the right half plane of the absolute convergence of the Laplace integral.

For $\mathscr{L}\{w(x,t)\} = W(x,s)$, $\mathscr{L}\{z(x,t)\} = Z(x,s)$, and λ, $\mu \in \mathbb{R}$, where $w(x, t)$ and $z(x, t)$, are two piecewise continuous functions defined on $T \times [0,\infty)$, of exponential order ϑ_1, ϑ_2, respectively, such that $\vartheta_1 < \vartheta_2$. Following, some of the useful characteristics of the Laplace transform operator and its inverse operator are listed below, which will be essential utilized in this work as follows:

(1) $\mathscr{L}\{\lambda w(x,t) + \mu z(x,t)\} = \lambda W(x,s) + \mu Z(x,s)$, for $\in T$, $s > \vartheta_1$

(2) $\mathscr{L}^{-1}\{\lambda W(x,s) + \mu Z(x,s)\} = \lambda w(x,t) + \mu z(x,t)$, for $\in T$, $t \in [0,\infty)$

(3) $\lim_{s\longrightarrow\infty} sW(x,s) = w(x,0)$, for $x \in T$

(4) $\mathscr{L}\{t^{m\beta}\} = \Gamma(m\beta+1)/s^{m\beta+1}, m\beta > -1$, $s > \vartheta_1$, and $m \in \mathbb{N}$

$$\mathscr{L}\left\{D_t^{\beta} w(x,t)\right\} = s^{\beta} W(x,\mathrm{s}) - \sum_{j=0}^{n-1} s^{\beta-j-1} w_t^{(j)}(x,0), n-1 < \beta \le n, n \in \mathbb{N},$$

$$\mathscr{L}\left\{D_t^{m\beta} w(x,t)\right\} = \mathrm{s}^{m\beta} W(\mathrm{x},\mathrm{s}) - \sum_{j=0}^{m-1} s^{(m-j)\beta-1} D_t^{j\beta} w(x,0), 0 < \beta \le 1, \mathrm{m} \in \mathbb{N}. \tag{7}$$

Next, El-Ajou [39] has been introduced and proved new results related to the generalized Taylor series formula in the Laplace space to identify the series solution of linear and nonlinear FPDEs. Further, the requirements for convergence of the new series expansion will be clarified and proved as follows:

Theorem 5 (see [39]). *For piecewise continuous function of exponential order ϑ, $w(x,t)$ on $T \times [0,\infty)$. Assume that the fractional expansion series of the transform function $W(x,s) = \mathscr{L}\{w(x,t)\}$ has the following shape:*

$$W(x,s) = \sum_{k=0}^{\infty} \frac{w_k(x)}{s^{k\beta+1}}, x \in T, s > \vartheta, \beta \in (0,1]. \tag{8}$$

Then, the unknown functions $w_k(x)$ will be in the form $w_k(x) = D_t^{k\beta} w(x,0)$, where $D_t^{k\beta} = D_t^{\beta} \bullet D_t^{\beta} \bullet\bullet\bullet D_t^{\beta}$ (k-times).

Remark 6. The inverse Laplace transform of the series expansion in Theorem 5 has the following multiple fractional power series (MFPS) shape:

$$w(x,t) = \sum_{k=0}^{\infty} \frac{D_t^{k\beta} w(x,0)}{\Gamma(k\beta+1)} t^{k\beta}, t \ge 0, \beta \in (0,1]. \tag{9}$$

Theorem 7 (see [40]). *$\beta \in (0,1]$. If $|\, s\,\mathscr{L}\{D_t^{(n+1)\beta} w(x,t)\}| \le M(x)$, on $T \times (\delta, d]$ where $\mathscr{L}\{w(x,t)\} = W(x,s)$, can be expanded as a fractional expansion series in Theorem 5, then the reminder $R_m(x,s)$ of (8) satisfies the following inequality:*

$$|R_m(x,s)| \le \frac{M(x)}{s^{1+(m+1)\beta}}, x \in T, \delta < s \le d. \tag{10}$$

3. Methodology of the LRPS Technique

The current section is devoted to illustrating the main principle of the LRPS technique to create the approximate solution for the nonlinear S-H Equation (2). The basic principle of our approach depends at the beginning by converting the target problem to Laplace space and then solving the new Laplace equation algebraically utilizing the limit concept and as a final stage; one converts the obtained Laplace solution into the main space to get the approximate solution for the target problem. For more exciting works on the residual error method and its various applications in physics and engineering, see [42–45]. For more information about the different fractional operators and their applications, we refer to [46–51] and references therein.

However, we apply firstly the Laplace transform to both sides of the main problem (2) as stated in [39] and utilize the fact $\mathscr{L}\{D_t^{\beta} w(x,t)\} = s^{\beta} W(x,s) - s^{1-\beta} w(x,0)$ to get the following algebraic equation:

$$W(x,s) = \frac{w(x,0)}{s} - \frac{1}{s^{\beta}} \mathscr{D}_x^4 W(x,s) - \frac{2}{s^{\beta}} \mathscr{D}_x^2 W(x,s) - (1-r)\frac{1}{s^{\beta}} W(x,s) - \frac{1}{s^{\beta}} \mathscr{L}\left\{\left[\mathscr{L}^{-1} W(x,s)\right]^3\right\}, \tag{11}$$

where $\mathscr{L}\{w(x,t)\} = W(x,s)$.

Since $w(x,0) = w_0(x)$, then we can rewrite (11) as follows:

$$W(x,s) = \frac{w_0(x)}{s} - \frac{1}{s^{\beta}} \mathscr{D}_x^4 W(x,s) - \frac{2}{s^{\beta}} \mathscr{D}_x^2 W(x,s) - (1-r)\frac{1}{s^{\beta}} W(x,s) - \frac{1}{s^{\beta}} \mathscr{L}\left\{\left[\mathscr{L}^{-1} W(x,s)\right]^3\right\}. \tag{12}$$

According to Theorem 5, the proposed solution $W(x,s)$, for the algebraic Equation (12), has the following transform function:

$$W(x,s)=\sum_{k=0}^{\infty}\frac{w_k(x)}{s^{k\beta+1}}\ x\in T,\ s>\vartheta. \tag{13}$$

And the m-th series solution $W_m(x,s)$, of the fractional expansion (13), will be written as the shape:

$$W_m(x,s)=\sum_{k=0}^{m}\frac{w_k(x)}{s^{k\beta+1}}\ x\in T,\ s>\vartheta. \tag{14}$$

Clearly, $w_0(x)=\lim_{s\longrightarrow\infty} sW(x,s)$. So, the fractional expansion (14) becomes the following:

$$W_m(x,s)=\frac{w_0(x)}{s}+\sum_{k=1}^{m}\frac{w_k(x)}{s^{k\beta+1}}\ x\in T,\ s>\vartheta, \tag{15}$$

in which one can find the unknown functions $w_k(x)$, for $k=1,2,3,\cdots,m$, via solving $\lim_{s\longrightarrow\infty} s^{m\beta+1}\mathscr{L}\mathrm{Res}_W^m(x,s)=0$, for $m=1,2,3,\cdots$, $0<\beta\le 1$, where $\mathscr{L}\mathrm{Res}_W^m(x,s)$ is called the m-th Laplace residual function of (11) and which is defined as follows:

$$\begin{aligned}\mathscr{L}\mathrm{Res}_W^m(x,s)&=W_m(x,s)-\frac{w_0(x)}{s}+\frac{1}{s^\beta}\mathscr{D}_x^4W_m(x,s)\\&\quad+\frac{2}{s^\beta}\mathscr{D}_x^2W_m(x,s)+(1-r)\frac{1}{s^\beta}W_m(x,s)\\&\quad+\frac{1}{s^\beta}\mathscr{L}\left\{\left[\mathscr{L}^{-1}W_m(x,s)\right]^3\right\}.\end{aligned} \tag{16}$$

To determine $w_1(x)$, in the fractional expansion (15), we write the first transform function $W_1(x,s)=w_0(x)/s+w_1(x)/s^{\beta+1}$, into the first Laplace residual function as follows:

$$\begin{aligned}\mathscr{L}\mathrm{Res}_W^1(x,s)&=W_1(x,s)-\frac{w_0(x)}{s}+\frac{1}{s^\beta}\mathscr{D}_x^4W_1(x,s)\\&\quad+\frac{2}{s^\beta}\mathscr{D}_x^2W_1(x,s)+(1-r)\frac{1}{s^\beta}W_1(x,s)\\&\quad+\frac{1}{s^\beta}\mathscr{L}\left\{\left[\mathscr{L}^{-1}W_1(x,s)\right]^3\right\}\\&=\frac{w_1(x)}{s^{\beta+1}}+\frac{w_0^{(4)}(x)}{s^{\beta+1}}+\frac{w_1^{(4)}(x)}{s^{2\beta+1}}+\frac{2w_0'(x)}{s^{\beta+1}}\\&\quad+\frac{2w_1'(x)}{s^{2\beta+1}}+(1-r)\left(\frac{w_0(x)}{s^{\beta+1}}+\frac{w_1(x)}{s^{2\beta+1}}\right)\\&\quad+\frac{1}{s^\beta}\mathscr{L}\left\{\left[\mathscr{L}^{-1}\left(\frac{w_0(x)}{s}+\frac{w_1(x)}{s^{\beta+1}}\right)\right]^3\right\}\\&=\frac{1}{s^{\beta+1}}\left(w_1(x)+w_0^{(4)}(x)+2w_0'(x)+(1-r)w_0(x)\right)\\&\quad+\frac{1}{s^{2\beta+1}}\left((1-r)w_1(x)+w_1^{(4)}(x)+2w_1'(x)\right)\\&\quad+\frac{1}{s^\beta}\mathscr{L}\left\{\left[w_0(x)+w_1(x)\frac{t^\beta}{\Gamma(\beta+1)}\right]^3\right\}.\end{aligned} \tag{17}$$

Multiply both sides of (17) by $s^{\beta+1}$ to get the following:

$$\begin{aligned}s^{\beta+1}\mathscr{L}\mathrm{Res}_W^1(x,s)&=\left(w_1(x)+w_0^{(4)}(x)+2w_0'(x)+(1-r)w_0(x)\right)\\&\quad+\frac{1}{s^\beta}\left((1-r)w_1(x)+w_1^{(4)}(x)+2w_1'(x)\right)\\&\quad+\left(w_0^3(x)+\frac{3w_0^2(x)w_1(x)}{s^\beta}+\frac{3w_0(x)w_1^2(x)}{s^{2\beta}\Gamma(2\beta+1)}\right.\\&\quad\left.+\frac{w_1^3(x)}{s^{3\beta}\Gamma^2(\beta+1)}\right).\end{aligned} \tag{18}$$

Next, solving the limit of the resulting algebraic equation as $s\longrightarrow\infty$, for the unknown function $w_1(x)$, gives the following: $w_1(x)=(r-1)w_0(x)-w_0^3(x)-w_0^{(4)}(x)-2w_0'(x)$.

Likewise, to find $w_2(x)$, let $m=2$, in the fractional expansion (15) and substitute the second transform function $W_2(x,s)$ of (15) into the second Laplace residual function of (16), that is,

$$\begin{aligned}\mathscr{L}\mathrm{Res}_W^2(x,s)&=W_2(x,s)-\frac{w_0(x)}{s}+\frac{1}{s^\beta}\mathscr{D}_x^4W_2(x,s)\\&\quad+\frac{2}{s^\beta}\mathscr{D}_x^2W_2(x,s)+(1-r)\frac{1}{s^\beta}W_2(x,s)\\&\quad+\frac{1}{s^\beta}(W_2(x,s))^3=\frac{w_1(x)}{s^{\beta+1}}+\frac{w_2(x)}{s^{2\beta+1}}\\&\quad+\frac{w_0^{(4)}(x)}{s^{\beta+1}}+\frac{w_1^{(4)}(x)}{s^{2\beta+1}}+\frac{w_2^{(4)}(x)}{s^{3\beta+1}}+\frac{2w_0'(x)}{s^{\beta+1}}\\&\quad+\frac{2w_1'(x)}{s^{2\beta+1}}+\frac{2w_2'(x)}{s^{3\beta+1}}+(1-r)\left(\frac{w_0(x)}{s^{\beta+1}}+\frac{w_1(x)}{s^{2\beta+1}}+\frac{w_2(x)}{s^{3\beta+1}}\right)\\&\quad+\frac{1}{s^\beta}\mathscr{L}\left\{\left[\mathscr{L}^{-1}\left(\frac{w_0(x)}{s}+\frac{w_1(x)}{s^{\beta+1}}+\frac{w_2(x)}{s^{2\beta+1}}\right)\right]^3\right\}\\&=\frac{1}{s^{\beta+1}}\left(w_1(x)+w_0^{(4)}(x)+2w_0'(x)+(1-r)w_0(x)\right)\\&\quad+\frac{1}{s^{2\beta+1}}\left(w_2(x)+(1-r)w_1(x)+w_1^{(4)}(x)+2w_1'(x)\right)\\&\quad+\frac{1}{s^\beta}\mathscr{L}\left\{\left[w_0(x)+w_1(x)\frac{t^\beta}{\Gamma(\beta+1)}+w_2(x)\frac{t^{2\beta}}{\Gamma(2\beta+1)}\right]^3\right\}.\end{aligned} \tag{19}$$

Then, we multiply the resulting equation by the factor $s^{2\beta+1}$ and based on the fact $\lim_{s\longrightarrow\infty} s^{2\beta+1}\mathscr{L}\mathrm{Res}_W^2(x,s)=0$, and this yields the following: $w_2(x)=(r-1)w_1(x)-3w_0^2(x)w_1(x)-w_1^{(4)}(x)-2w_1'(x)$.

To obtain $w_3(x)$, we consider $m=3$ in Equation (9) so that

$$\begin{aligned}\mathscr{L}\mathrm{Res}_W^3(x,s)&=W_3(x,s)-\frac{w_0(x)}{s}+\frac{1}{s^\beta}\mathscr{D}_x^4W_3(x,s)\\&\quad+\frac{2}{s^\beta}\mathscr{D}_x^2W_3(x,s)+(1-r)\frac{1}{s^\beta}W_3(x,s)\\&\quad+\frac{1}{s^\beta}\mathscr{L}\left\{\left[\mathscr{L}^{-1}W_3(x,s)\right]^3\right\},\end{aligned} \tag{20}$$

where

$$W_3(x,s)=\frac{w_0(x)}{s}+\sum_{k=1}^{3}\frac{w_k(x)}{s^{k\beta+1}}. \tag{21}$$

Thus,

$$\begin{aligned}\mathscr{L}\mathrm{Res}_W^3(x,s)&=\sum_{k=1}^{3}\frac{w_k(x)}{s^{k\beta+1}}+\sum_{k=0}^{3}\frac{w_k^{(4)}(x)}{s^{(k+1)\beta+1}}+2\sum_{k=0}^{3}\frac{w_k'(x)}{s^{(k+1)\beta+1}}\\&\quad+(1-r)\sum_{k=0}^{3}\frac{w_k(x)}{s^{(k+1)\beta+1}}+\mathscr{L}\left\{\left[\mathscr{L}^{-1}\left(\sum_{k=1}^{3}\frac{w_k(x)}{s^{(k+1)\beta+1}}\right)\right]^3\right\}\\&=\sum_{k=1}^{3}\frac{w_k(x)}{s^{k\beta+1}}+\sum_{k=0}^{3}\frac{w_k^{(4)}(x)}{s^{(k+1)\beta+1}}+2\sum_{k=0}^{3}\frac{w_k'(x)}{s^{(k+1)\beta+1}}\\&\quad+(1-r)\sum_{k=0}^{3}\frac{w_k(x)}{s^{(k+1)\beta+1}}+\mathscr{L}\left\{\left[\sum_{k=0}^{3}w_k(x)\frac{t^{k\beta}}{\Gamma(k\beta+1)}\right]^3\right\}.\end{aligned} \tag{22}$$

As a last step, after multiplying both sides of (22) by $s^{3\beta+1}$, solve $\lim_{s\longrightarrow\infty}s^{3\beta+1}\mathscr{L}\mathrm{Res}_W^3(x,s)=0$ for the required unknown function to get $w_3(x)=(r-1)w_2(x)-3w_0^2(x)w_2(x)-w_2^{(4)}(x)-2w_2'(x)-(3\Gamma(2\beta+1)/\Gamma^2(\beta+1))w_0(x)w_1^2(x)$.

In the same argument, the fourth unknown function $w_4(x)$ can be obtained via writing $W_4(x,s)=w_0(x)/s+\sum_{k=1}^{3}w_k(x)/s^{k\beta+1}$, into the fourth Laplace residual function, $\mathscr{L}\mathrm{Res}_W^4(x,s)$, of (9), then by multiplying the obtained algebraic equation by $s^{4\beta+1}$, and finally by solving $\lim_{s\longrightarrow\infty}s^{4\beta+1}\mathscr{L}\mathrm{Res}_W^4(x,s)=0$, for $w_4(x)$ to conclude that

$$\begin{aligned}w_4(x)&=(r-1)w_3(x)-3w_0^2(x)w_3(x)-w_3^{(4)}(x)-2w_3'(x)\\&\quad-\frac{6\Gamma(3\beta+1)}{\Gamma(\beta+1)\Gamma(2\beta+1)}w_0(x)w_1(x)w_2(x)-\frac{\Gamma(3\beta+1)}{\Gamma^2(\beta+1)}w_1^3(x).\end{aligned} \tag{23}$$

Continuing with this procedure, the m-th unknown function, $w_m(x)$, can be determined for arbitrary m. So, based on the previous results of $w_m(x)$, the transform function $W(x,s)$ of the Laplace Equation (12) has the following form:

$$\begin{aligned}W(x,s)&=\frac{w_0(x)}{s}+\frac{\left((r-1)w_0(x)-w_0^3(x)-w_0^{(4)}(x)-2w_0'(x)\right)}{s^{\beta+1}}\\&\quad+\frac{\left((r-1)w_1(x)-3w_0^2(x)w_1(x)-w_1^{(4)}(x)-2w_1'(x)\right)}{s^{2\beta+1}}+\cdots.\end{aligned} \tag{24}$$

Lastly, we convert the fractional expansion (24) into the original space by operating the inverse Laplace transform into (24), to get the MFPS approximate solution for the time-fractional S-H Equation (2) as follows:

$$\begin{aligned}w(x,t)&=w_0(x)+\left((r-1)w_0(x)-w_0^3(x)-w_0^{(4)}(x)-2w_0'(x)\right)\frac{t^{\beta}}{\Gamma(1+\beta)}\\&\quad+\left((r-1)w_1(x)-3w_0^2(x)w_1(x)-w_1^{(4)}(x)-2w_1'(x)\right)\\&\quad\cdot\frac{t^{2\beta}}{\Gamma(1+2\beta)}+\cdots.\end{aligned} \tag{25}$$

4. Numerical Examples

This section is aimed at using the proposed approach to construct the Laplace MFPS approximate solutions for three applications of S-H Equation (2) and show the applicability and the notability of our approach as reliable scheme to handle various FPDEs. Notice that all the computations and symbolic were done by using MATHEMATICA 12 software package.

Example 1. Consider the linear time-fractional S-H equation of the form [4].

$$D_t^{\beta}w+(1-r)w+\frac{\partial^4 w}{\partial x^4}+2\frac{\partial^2 w}{\partial x^2}=0, \tag{26}$$

subject to initial condition

$$w(x,0)=e^x, \tag{27}$$

where $(x,t)\in\mathbb{R}\times[0,1]$, $0<\beta\le 1$. The exact solution of fractional IVPs (26) and (27) at $\beta=1$ is $w(x,t)=e^{x+(r-4)t}$.

Before applying the LRPS method, we convert the time-fractional S-H Equation (26) to the Laplace space by taking the Laplace transform to both sides of (26) and utilizing the initial data (27), and we get the following Laplace algebraic equation:

$$W(x,s)=\frac{e^x}{s}-\frac{1}{s^{\beta}}\mathscr{D}_x^4W(x,s)-\frac{2}{s^{\beta}}\mathscr{D}_x^2W(x,s)-(1-r)\frac{1}{s^{\beta}}W(x,s). \tag{28}$$

According to layout of the proposed approach, we can solve (28) by identifying the following m-th Laplace residual function:

$$\begin{aligned}\mathscr{L}\mathrm{Res}_W^m(x,s)&=W_m(x,s)-\frac{e^x}{s}+\frac{1}{s^{\beta}}\mathscr{D}_x^4W_m(x,s)\\&\quad+\frac{2}{s^{\beta}}\mathscr{D}_x^2W_m(x,s)+(1-r)\frac{1}{s^{\beta}}W_m(x,s),\end{aligned} \tag{29}$$

where $W_m(x,s)$ represents the m-th transform function as follows:

$$W_m(x,s)=\frac{e^x}{s}+\sum_{k=1}^{m}\frac{w_k(x)}{s^{k\beta+1}}\ x\in\mathbb{R},s>\vartheta. \tag{30}$$

To determine $w_1(x)$, consider $m=1$ in the series expansion (30) and substitute it into the first Laplace residual function as follows:

$$\mathscr{L}\mathrm{Res}_W^1(x,s) = \frac{w_1(x)}{s^{\beta+1}} + \frac{1}{s^\beta}\left(\frac{e^x}{s} + \frac{w_1^{(4)}(x)}{s^{\beta+1}}\right) + \frac{2}{s^\beta}\left(\frac{e^x}{s} + \frac{w_1'(x)}{s^{\beta+1}}\right) + (1-r)\frac{1}{s^\beta}\left(\frac{e^x}{s} + \frac{w_1(x)}{s^{\beta+1}}\right). \tag{31}$$

Next, multiply both sides of (31) by $s^{\beta+1}$ to get the following:

$$\begin{aligned} s^{\beta+1}\mathscr{L}\mathrm{Res}_W^1(x,s) &= s^{\beta+1}\left(\frac{w_1(x)}{s^{\beta+1}} + \frac{1}{s^\beta}\left(\frac{e^x}{s} + \frac{w_1^{(4)}(x)}{s^{\beta+1}}\right) + \frac{2}{s^\beta}\left(\frac{e^x}{s} + \frac{w_1'(x)}{s^{\beta+1}}\right) + (1-r)\frac{1}{s^\beta}\left(\frac{e^x}{s} + \frac{w_1(x)}{s^{\beta+1}}\right)\right) \\ &= e^x - e^x r + 2e^x + e^x + w_1(x) + \frac{w_1(x)}{s^\beta} - r\frac{w_1(x)}{s^\beta} + \frac{2w_1'(x)}{s^\beta} + \frac{w_1^{(4)}(x)}{s^\beta}. \end{aligned} \tag{32}$$

Then, solving $\lim_{s\longrightarrow\infty} s^{\beta+1}\mathscr{L}\mathrm{Res}_W^1(x,s)=0$, for $w_1(x)$ gives $w_1(x)=(r-4)e^x$. So, the 1^{st}-transform function for the Laplace Equation (28) has the following expansion: $W_1(x,s) = e^x/s(e^x/s) + (r-4)e^x/s^{\beta+1}$.

Similarly, for constructing the 2^{nd}-Laplace residual function, we substitute $W_2(x,s) = e^x/s + (r-4)e^x/s^{\beta+1} + w_2(x)/s^{2\beta+1}$ into the second Laplace residual function $\mathscr{L}\mathrm{Res}_W^2(x,s)$ of (29) such that

$$\mathscr{L}\mathrm{Res}_W^2(x,s) = \frac{(r-4)e^x}{s^{\beta+1}} + \frac{w_2(x)}{s^{2\beta+1}} + \frac{1}{s^\beta}\left(\frac{e^x}{s} + \frac{(r-4)e^x}{s^{\beta+1}} + \frac{w_2(x)}{s^{2\beta+1}}\right) + \frac{2}{s^\beta}\left(\frac{e^x}{s} + \frac{(r-4)e^x}{s^{\beta+1}} + \frac{w_2(x)}{s^{2\beta+1}}\right) + (1-r)\frac{1}{s^\beta}\cdot\left(\frac{e^x}{s} + \frac{(r-4)e^x}{s^{\beta+1}} + \frac{w_2(x)}{s^{2\beta+1}}\right). \tag{33}$$

Thereafter, we multiply (33) by the factor $s^{2\beta+1}$ to give the following:

$$\begin{aligned} s^{2\beta+1}\mathscr{L}\mathrm{Res}_W^2(x,s) &= s^{\beta+1}\left(\frac{(r-4)e^x}{s^{\beta+1}} + \frac{w_2(x)}{s^{2\beta+1}} + \frac{1}{s^\beta}\left(\frac{e^x}{s} + \frac{(r-4)e^x}{s^{\beta+1}} + \frac{w_2(x)}{s^{2\beta+1}}\right) + \frac{2}{s^\beta}\left(\frac{e^x}{s} + \frac{(r-4)e^x}{s^{\beta+1}} + \frac{w_2(x)}{s^{2\beta+1}}\right) + (1-r)\frac{1}{s^\beta}\cdot\left(\frac{e^x}{s} + \frac{(r-4)e^x}{s^{\beta+1}} + \frac{w_2(x)}{s^{2\beta+1}}\right)\right) \\ &= -4e^x + 5e^x r - e^x r^2 - 3e^x s^\beta - 8e^x + 2e^x r + 2e^x s^\beta - 4e^x + e^x r + e^x s^\beta + w_2(x) + \frac{w_2(x)}{s^\beta} - \frac{rw_2(x)}{s^\beta} + \frac{2w_2'(x)}{s^\beta} + \frac{w_2^{(4)}(x)}{s^\beta}. \end{aligned} \tag{34}$$

Depending on the result $\lim_{s\longrightarrow\infty} s^{m\beta+1}\mathscr{L}\mathrm{Res}_W^m(x,s)=0$, for $m=2$, we conclude that $w_2(x)=(r-4)^2e^x$. Thus, the second Laplace function for the Laplace Equation (28) will be $W_2(x,s) = e^x/s + (r-4)e^x/s^{\beta+1} + (r-4)^2e^x/s^{2\beta+1}$.

Following the same argument, the 3^{rd}-transform function can be constructed by writing $W_3(x,s)$, of the fractional expansion (30) into the third Laplace residual function $\mathscr{L}\mathrm{Res}_W^3(x,s)$ of (29) such that

$$\begin{aligned} \mathscr{L}\mathrm{Res}_W^3(x,s) = {} & \frac{(r-4)e^x}{s^{\beta+1}} + \frac{(r-4)^2e^x}{s^{2\beta+1}} + \frac{w_3(x)}{s^{3\beta+1}} + \frac{1}{s^\beta}\cdot\left(\frac{e^x}{s} + \frac{(r-4)e^x}{s^{\beta+1}} + \frac{(r-4)^2e^x}{s^{2\beta+1}} + \frac{w_3(x)}{s^{3\beta+1}}\right) \\ & + \frac{2}{s^\beta}\left(\frac{e^x}{s} + \frac{(r-4)e^x}{s^{\beta+1}} + \frac{(r-4)^2e^x}{s^{2\beta+1}} + \frac{w_3(x)}{s^{3\beta+1}}\right) \\ & + (1-r)\frac{1}{s^\beta}\left(\frac{e^x}{s} + \frac{(r-4)e^x}{s^{\beta+1}} + \frac{(r-4)^2e^x}{s^{2\beta+1}} + \frac{w_3(x)}{s^{3\beta+1}}\right). \end{aligned} \tag{35}$$

And by solving $\lim_{s\longrightarrow\infty} s^{3\beta+1}\mathscr{L}\mathrm{Res}_W^3(x,s)=0$, for the desired unknown function $w_3(x)$, then one can get $w_3(x)=(r-4)^3e^x$. Therefore, the 3^{rd}-transform function the Laplace Equation (28) has the following expansion: $W_3(x,s) = e^x/s + (r-4)e^x/s^{\beta+1} + (r-4)^2e^x/s^{2\beta+1} + (r-4)^3e^x/s^{3\beta+1}$.

Continuing in this manner, one can obtain the m-th unknown function $w_m(x)$, for arbitrary m, and hence, the m-th transform function $W_m(x,s)$, for the Laplace Equation (28), is obtained in the following expansion:

$$W_m(x,s) = \frac{e^x}{s} + \frac{(r-4)e^x}{s^{\beta+1}} + \frac{(r-4)^2e^x}{s^{2\beta+1}} + \frac{(r-4)^3e^x}{s^{3\beta+1}} + \cdots + \frac{(r-4)^m e^x}{s^{m\beta+1}}. \tag{36}$$

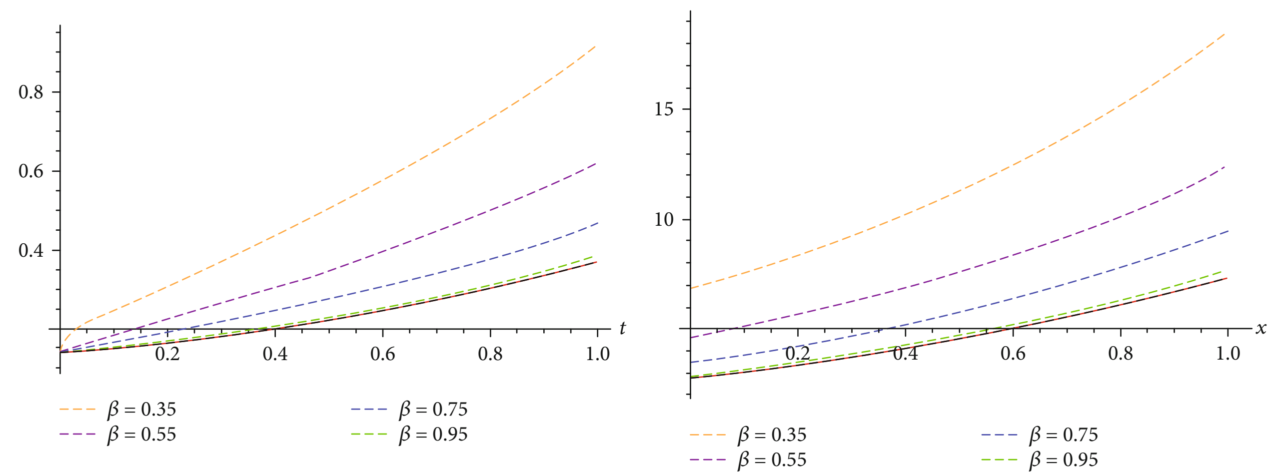
As $m\longrightarrow\infty$, the transform function $W(x,s)$, for the Laplace Equation (28), can be expressed as the following infinite series:

$$\begin{aligned} W(x,s) &= e^x\left(\frac{1}{s} + \frac{(r-4)}{s^{\beta+1}} + \frac{(r-4)^2}{s^{2\beta+1}} + \frac{(r-4)^3}{s^{3\beta+1}} + \cdots + \frac{(r-4)^m}{s^{m\beta+1}} + \cdots\right) \\ &= e^x\sum_{k=0}^{\infty}\frac{(r-4)^k}{s^{k\beta+1}}. \end{aligned} \tag{37}$$

Finally, to get the MFPS approximate solution for the fractional IVPs (26) and (27), we apply the inverse Laplace transform to both sides of (37) as follows:

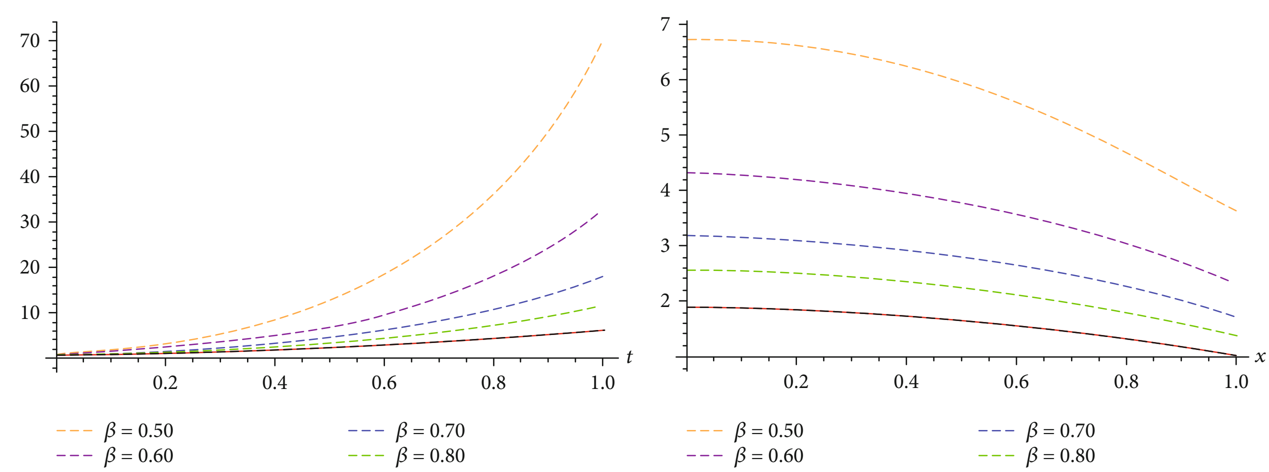
Table 1: Absolute errors for Example1 at $\beta = 1$ and $n = 10$ for different values of r.

t_i	$r = 1$	$r = 3$	$r = 5$
0.15	$1.005617811 \times 10^{-11}$	$4.440892099 \times 10^{-16}$	0.0
0.30	$1.987156617 \times 10^{-8}$	$1.176836406 \times 10^{-13}$	$1.239008895 \times 10^{-13}$
0.45	$1.660194246 \times 10^{-6}$	$1.005617811 \times 10^{-11}$	$1.083932943 \times 10^{-11}$
0.60	$3.800576801 \times 10^{-5}$	$2.352540385 \times 10^{-10}$	$2.600089033 \times 10^{-10}$
0.75	$4.281970881 \times 10^{-4}$	$2.706239455 \times 10^{-9}$	$3.066872978 \times 10^{-9}$
1.00	$3.081913228 \times 10^{-3}$	$1.987156617 \times 10^{-8}$	$2.309139457 \times 10^{-8}$



(a) (b)

Figure 1: (a) Plots of exact solution $w(x, t)$ and 10^{th}-MFPS approximate solution $w_{10}(x, t)$ at various β values and fixed values of $x = -2$ and $r = 5$; (b) plots of exact solution $w(x, t)$ and 10^{th}-MFPS approximate solution $w_{10}(x, t)$ at various β values and $t = 1$ and $r = 5$, for Example 1.



(a) (b)

Figure 2: (a) Plots of exact solution $w(x, t)$ and 10^{th}-MFPS approximate solution $w_{10}(x, t)$ at fixed value of $x = \pi/6$; (b) plots of exact solution $w(x, t)$ and 10^{th}-MFPS approximate solution $w_{10}(x, t)$ at fixed value of $t = 0.32$, with various β values and $r = 2$, for Example 2.

Table 2: Absolute errors for Example2 at $\beta = 1$, and $n = 10$, with different values of r.

t_i	$r = 1$	$r = 2$	$r = 3$
0.16	$4.440892099 \times 10^{-16}$	$9.281464486 \times 10^{-14}$	$8.131717522 \times 10^{-12}$
0.32	$9.281464485 \times 10^{-14}$	$1.952191742 \times 10^{-10}$	$1.736975808 \times 10^{-8}$
0.48	$8.131717522 \times 10^{-12}$	$1.736975808 \times 10^{-8}$	$1.569467859 \times 10^{-6}$
0.64	$1.952191742 \times 10^{-10}$	$4.233315316 \times 10^{-7}$	$3.888067467 \times 10^{-5}$
0.80	$2.304785252 \times 10^{-9}$	$5.076409958 \times 10^{-6}$	$4.743977753 \times 10^{-4}$
0.96	$1.736975807 \times 10^{-8}$	$3.888067467 \times 10^{-5}$	$3.701046898 \times 10^{-3}$

$$w(x,t) = e^x\left(1 + \frac{(r-4)t^\beta}{\Gamma(\beta+1)} + \frac{(r-4)^2 t^{2\beta}}{\Gamma(2\beta+1)} + \cdots + \frac{(r-4)^m t^{m\beta}}{\Gamma(m\beta+1)} + \cdots\right) = e^x \sum_{k=0}^{\infty} \frac{(r-4)^k t^{k\beta}}{\Gamma(k\beta+1)}. \tag{38}$$

Based on the obtained results and with no loss of generality, the accuracy and efficiency of our proposed approach have been illustrated; the absolute errors at $x = 1$, with various values of t and parameter r, have been computed and summarized in Table 1. One can notice from Table 1 that the results refer to well-harmony between the exact and MFPS approximate solutions. Also, the behavior of the 10^{th}-MFPS approximate solution for the time-fractional IVPs (26) and (27) has been studied and compared with the exact solution as in Figure 1.

Example 2. Consider the linear time-fractional S-H equation of the form [4]:

$$D_t^\beta w + (1-r)w + \frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^2 w}{\partial x^2} = 0, \tag{39}$$

subject to initial condition

$$w(x,0) = \cos(x), \tag{40}$$

where $(x,t) \in \mathbb{R} \times [0,1], 0 < \beta \le 1$. The exact solution of fractional IVPs (39) and (40) at $\beta = 1$ is $w(x,t) = \cos(x)e^{rt}$.

According to employing the LRPS scheme, we convert the time-fractional S-H Equation (39) into the Laplace space and use the initial condition (40), to get the following Laplace algebraic equation:

$$W(x,s) = \frac{\cos(x)}{s} - \frac{1}{s^\beta}\mathscr{D}_x^4 W(x,s) - \frac{2}{s^\beta}\mathscr{D}_x^2 W(x,s) - (1-r)\frac{1}{s^\beta}W(x,s). \tag{41}$$

Following, let the approximate solution of (41), and it has the following m-th transform function:

$$W_m(x,s) = \frac{\cos(x)}{s} + \sum_{k=1}^{m} \frac{w_k(x)}{s^{k\beta+1}} \; x \in \mathbb{R}, s > \vartheta. \tag{42}$$

Table 3: Numerical results of the approximate solution at $n = 10$ for different values of β and parameter r for Example 2.

r_i	t_i	10^{th}-MFPS approximate solution $\beta = 1$	$\beta = 0.95$	$\beta = 0.85$	$\beta = 0.75$
3	0.25	0.326550322	0.355783973	0.439413482	0.585936283
	0.50	0.691306656	0.793037250	1.112531266	1.772249160
	0.75	1.463461560	1.759067720	2.776148009	3.199715336
	1.00	3.097317467	3.893554312	4.862973030	6.698826476
5	0.25	0.538390412	0.630100786	0.946089559	1.726592675
	0.50	1.879051549	2.464692765	4.996474868	6.001370563
	0.75	6.547505591	7.560995725	9.140575178	9.781842380
	1.00	22.57941994	23.04998298	24.13141857	25.36859332

Table 4: Numerical results of the approximated solution at $n = 5$ and $L = 7$ with different values of β and r for Example 3.

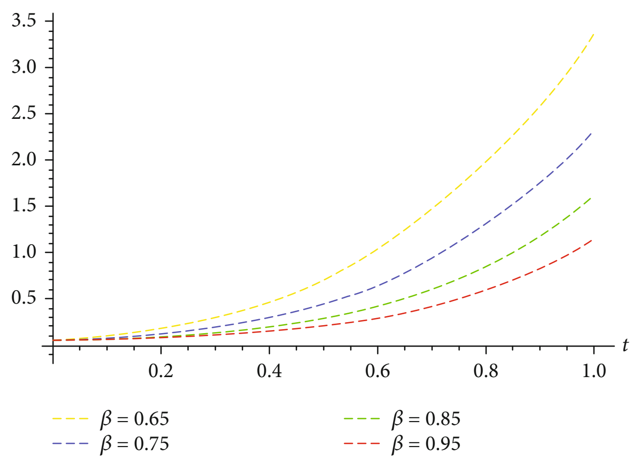
r	t_i	$\beta = 1$	$\beta = 0.95$	$\beta = 0.85$	$\beta = 0.75$
3	0.25	0.130235598	0.138601698	0.163195116	0.208272383
	0.50	0.241985850	0.274987590	0.378120295	0.568190733
	0.75	0.523404754	0.620746749	0.909781606	1.393600940
	1.00	1.190766401	1.408620937	2.014116748	2.936614248
5	0.25	0.249269193	0.301512503	0.498227923	0.970174288
	0.50	1.351750386	1.770999038	3.193745388	4.019381092
	0.75	6.353284329	7.840191289	8.379398245	9.168510709
	1.00	21.86124928	22.09938948	22.49186598	23.27968921

In order to investigate the unknown functions $w_k(x)$, we identify the following the m-th Laplace residual function:

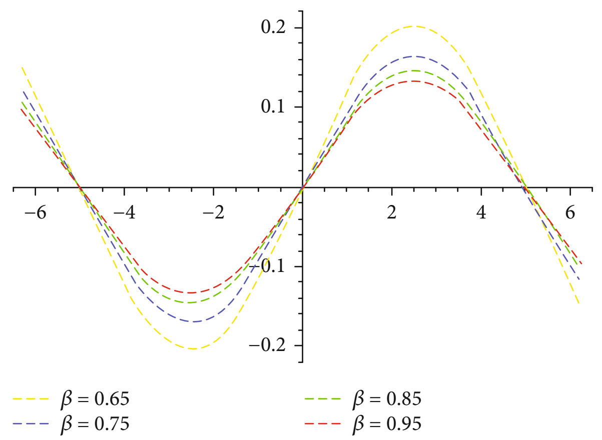
$$\mathscr{L}\mathrm{Res}_W^m(x,s) = \sum_{k=1}^{m}\frac{w_k(x)}{s^{k\beta+1}} + \left(\frac{\cos(x)}{s^{\beta+1}} + \sum_{k=1}^{m}\frac{w_k^{(4)}(x)}{s^{(k+1)\beta+1}}\right) + 2\left(-\frac{\cos(x)}{s^{\beta+1}} + \sum_{k=1}^{m}\frac{w_k'(x)}{s^{(k+1)\beta+1}}\right) + (1-r)\left(\frac{\cos(x)}{s^{\beta+1}} + \sum_{k=1}^{m}\frac{w_k(x)}{s^{(k+1)\beta+1}}\right). \tag{43}$$

Table 5: The recurrence errors $|w_5(x,t) - w_4(x,t)|$ of the fifth approximate solution with different values of β, for Example 3.

t_i	$\beta = 0.45$	$\beta = 0.65$	$\beta = 0.85$	$\beta = 1.00$
0.16	$2.330727639 \times 10^{-9}$	$3.175844934 \times 10^{-10}$	$5.367856853 \times 10^{-11}$	$1.632998306 \times 10^{-11}$
0.32	$1.108687157 \times 10^{-8}$	$3.021389905 \times 10^{-9}$	$1.021358990 \times 10^{-9}$	$5.225594212 \times 10^{-10}$
0.48	$2.760669069 \times 10^{-8}$	$1.128504684 \times 10^{-8}$	$5.722242603 \times 10^{-9}$	$3.968185638 \times 10^{-9}$
0.64	$5.273834620 \times 10^{-8}$	$2.874446701 \times 10^{-8}$	$1.943371804 \times 10^{-8}$	$1.672190145 \times 10^{-8}$
0.80	$8.713126837 \times 10^{-8}$	$5.936244804 \times 10^{-8}$	$5.016761473 \times 10^{-8}$	$5.103119306 \times 10^{-8}$
0.80	$1.313202919 \times 10^{-7}$	$1.073620640 \times 10^{-7}$	$1.088789059 \times 10^{-7}$	$1.269819411 \times 10^{-7}$



(a)



(b)

Figure 3: (a) Plots of $w_5(x,t)$ at various β values and $x = 0$; (b) plots $w_5(x,t)$ at various β values and $t = 0.125$, for Example 3.

Following the procedure of the Laplace RPS algorithm and based on the result $\lim_{s\longrightarrow\infty} s^{m\beta+1}\mathscr{L}\text{Res}_W^m(x,s) = 0$, for $m = 1, 2, 3, \cdots$, then the first few unknown functions $w_k(x)$ are as follows:

$$\begin{aligned} w_1(x) &= r\ \cos(x), \\ w_2(x) &= r^2\ \cos(x), \\ w_3(x) &= r^3\ \cos(x), \\ w_4(x) &= r^4\ \cos(x). \\ &\ \vdots . \end{aligned} \tag{44}$$

So, the m-th transform function $W_m(x,s)$, for the Laplace Equation (41) will be written as follows:

$$W_m(x,s) = \frac{\cos(x)}{s} + \frac{r\ \cos(x)}{s^{\beta+1}} + \frac{r^2\ \cos(x)}{s^{2\beta+1}} + \frac{r^3\ \cos(x)}{s^{3\beta+1}} + \cdots + \frac{r^m\ \cos(x)}{s^{m\beta+1}}. \tag{45}$$

Thus, the transform function $W(x,s)$, for fractional IVPs (39) and (40) can be expressed as the following infinite series:

$$\begin{aligned} W(x,s) &= \cos(x)\left(\frac{1}{s} + \frac{r}{s^{\beta+1}} + \frac{r^2}{s^{2\beta+1}} + \frac{r^3}{s^{3\beta+1}} + \cdots + \frac{r^m}{s^{m\beta+1}} + \cdots\right) \\ &= \cos(x)\sum_{k=0}^{\infty}\frac{r^k}{s^{k\beta+1}}. \end{aligned} \tag{46}$$

Finally, by applying the inverse Laplace transform to both sides of (46), we get the following MFPS solution for the fractional IVPs (39) and (40):

$$\begin{aligned} w(x,t) &= \cos(x)\left(1 + \frac{rt^{\beta}}{\Gamma(\beta+1)} + \frac{r^2t^{2\beta}}{\Gamma(2\beta+1)} + \cdots + \frac{r^m t^{m\beta}}{\Gamma(m\beta+1)} + \cdots\right) \\ &= \cos(x)\sum_{k=0}^{\infty}\frac{r^k t^{k\beta}}{\Gamma(k\beta+1)}. \end{aligned} \tag{47}$$

The convergence of the MFPS approximate solution to the exact solution for the fractional IVPs (39) and (40) has been shown graphically as in Figure 2 and numerically as in Tables 2 and 3. It is evident from the obtained results that the present technique is an effective and convenient algorithm to solve certain classes of FPDEs with fewer calculations and iteration steps.

Example 3. Consider the nonlinear time-fractional S-H equation of the form [5]:

$$D_t^{\beta} w + \frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^2 w}{\partial x^2} + (1-r)w + w^3 = 0, 0 < \beta \le 1, \quad (48)$$

subject to initial condition

$$w(x,0) = \frac{1}{10}\sin\left(\frac{\pi x}{L}\right). \quad (49)$$

As stated previously in Section 3, we operate the Laplace transform to both sides of (48), and using the initial data (49), we get the following Laplace algebraic equation:

$$W(x,s) = \frac{w_0(x)}{s} - \frac{1}{s^{\beta}}\mathscr{D}_x^4 W(x,s) - \frac{2}{s^{\beta}}\mathscr{D}_x^2 W(x,s) - (1-r)\frac{1}{s^{\beta}} W(x,s) - \frac{1}{s^{\beta}}\mathscr{L}\left\{\left[\mathscr{L}^{-1} W_m(x,s)\right]^3\right\}. \quad (50)$$

According to LRPS scheme, the proposed m-th approximate solution of (50) has the following series shape:

$$W_m(x,s) = \frac{1}{10s}\sin\left(\frac{\pi x}{L}\right) + \sum_{k=1}^{m}\frac{w_k(x)}{s^{k\beta+1}} x \in \mathbb{R}, s > \vartheta. \quad (51)$$

To investigate the unknown coefficients $w_k(x)$ via defining the following m-th Laplace residual function,

$$\mathscr{L}\mathrm{Res}_W^m(x,s) = \sum_{k=1}^{m}\frac{w_k(x)}{s^{k\beta+1}} + \left(\frac{\pi^4 \sin(\pi x/L)}{10L^4 s^{\beta+1}} + \sum_{k=1}^{m}\frac{w_k^{(4)}(x)}{s^{(k+1)\beta+1}}\right) - 2\left(\frac{\pi^2\sin(\pi x/L)}{10L^2 s^{\beta+1}} - \sum_{k=1}^{m}\frac{w_k'(x)}{s^{(k+1)\beta+1}}\right) + (1-r)\left(\frac{\sin(\pi x/L)}{10\, s^{\beta+1}} + \sum_{k=1}^{m}\frac{w_k(x)}{s^{(k+1)\beta+1}}\right) + \frac{1}{s^{\beta}}\mathscr{L}\left\{\left[\mathscr{L}^{-1}\left(\frac{\sin(\pi x/L)}{10\, s} + \sum_{k=1}^{m}\frac{w_k(x)}{s^{k\beta+1}}\right)\right]^3\right\}. \quad (52)$$

Now, based on the same methodology that is discussed in the last section to find out the forms of $w_k(x)$, then the first few five unknown functions in the fractional expansion (51) are given as follows:

$$w_1(x) = \frac{1}{2\times 10^3}\left(-201 + 200r + \cos\left(\frac{2\pi x}{L}\right)\right)\sin\left(\frac{\pi x}{L}\right),$$

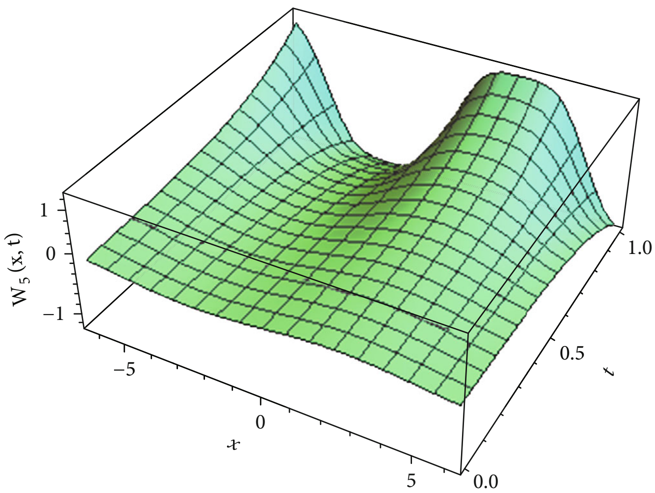
$$w_2(x) = \frac{1}{4\times 10^5}\left(-201 + 200r + \cos\left(\frac{2\pi x}{L}\right)\right) \cdot \left(-203 + 200r + 3\cos\left(\frac{2\pi x}{L}\right)\right)\sin\left(\frac{\pi x}{L}\right),$$

$$w_3(x) = \frac{1}{4\times 10^7}\left(-201 + 200r + \cos\left(\frac{2\pi x}{L}\right)\right)\sin\left(\frac{\pi x}{L}\right) \cdot \left(\left(-203 + 200r + 3\cos\left(\frac{2\pi x}{L}\right)\right)^2 - \frac{6(-201 + 200r + \cos(2\pi x/L))\Gamma(1+3\beta)\sin(\pi x/L)^2}{\Gamma^2(1+\beta)}\right),$$

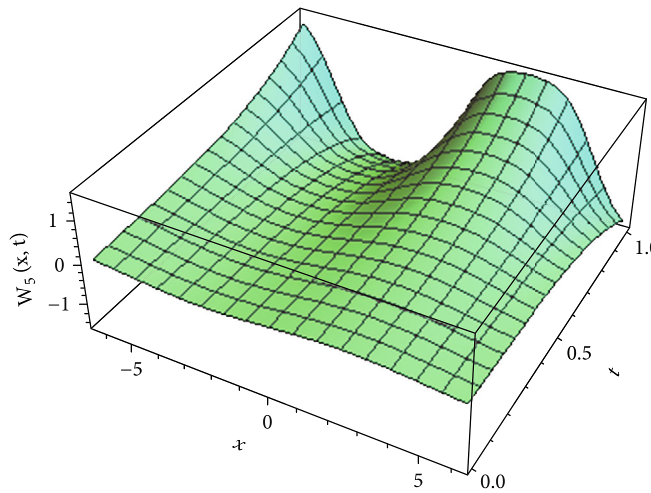
$$w_4(x) = \frac{1}{16\times 10^9}\left(-201 + 200r + \cos\left(\frac{2\pi x}{L}\right)\right)\sin\left(\frac{\pi x}{L}\right) + \left(\left(-203 + 200r + 3\cos\left(\frac{2\pi x}{L}\right)\right)^3 + \frac{1}{\Gamma^2(1+\beta)\Gamma(1+2\beta)} \cdot \left(12\left(-201 + 200r + \cos\left(\frac{2\pi x}{L}\right)\right) \cdot \left(\left(203 - 200r - 3\cos\left(\frac{2\pi x}{L}\right)\right)\Gamma(1+\beta) - 2 \cdot \left(-101 + 100r + \cos\left(\frac{2\pi x}{L}\right)\right)\Gamma(1+2\beta)\right)\right)\Gamma(1+3\beta)\mathrm{Sin} \cdot \left(\frac{\pi x}{L}\right)^2\right)\right),$$

$$w_5(x) = \frac{1}{32\times 10^{11}\Gamma^3(1+\beta)\Gamma^2(1+2\beta)\Gamma(1+3\beta)} \cdot \left(\left(-201 + 200r + \cos\left(\frac{2\pi x}{L}\right)\right)\sin\left(\frac{\pi x}{L}\right) \cdot (-3(81609 + 1600r(-101 + 50r) + 4(-403 + 400r)\cos \cdot \left(\frac{2\pi x}{L}\right) + 3\cos\left(\frac{4\pi x}{L}\right)\right)\Gamma(1+\beta)\Gamma(1+2\beta)\Gamma(1+3\beta) \cdot \left(4\left(-101 + 100r + \cos\left(\frac{2\pi x}{L}\right)\right)\Gamma(1+2\beta)\Gamma(1+3\beta) + \left(-201 + 200r + \cos\left(\frac{2\pi x}{L}\right)\right)\Gamma(1+4\beta)\right)\sin^2 \cdot \left(\frac{\pi x}{L}\right) - 12\left(-201 + 200r + \cos\left(\frac{2\pi x}{L}\right)\right) \cdot \left(-203 + 200r + 3\cos\left(\frac{2\pi x}{L}\right)\right)^2\Gamma^2(1+\beta)\Gamma(1+2\beta) \cdot \left(\Gamma^2(1+3\beta) + \Gamma(1+2\beta)\Gamma(1+4\beta)\right)\sin^2\left(\frac{\pi x}{L}\right) + 72\left(-201 + 200r + \cos\left(\frac{2\pi x}{L}\right)\right)^2\Gamma^2(1+2\beta)\Gamma \cdot (1+3\beta)\Gamma(1+4\beta)\sin^4\left(\frac{\pi x}{L}\right) + \left(-203 + 200r + 3\cos\left(\frac{2\pi x}{L}\right)\right)^2\Gamma^2(1+\beta)\Gamma(1+3\beta) \cdot \left(\left(-203 + 200r + 3\cos\left(\frac{2\pi x}{L}\right)\right)^2\Gamma^2(1+2\beta) - 6\left(-201 + 200r + \cos\left(\frac{2\pi x}{L}\right)\right)\Gamma(1+4\beta)\sin^2\left(\frac{\pi x}{L}\right)\right)\right). \quad (53)$$

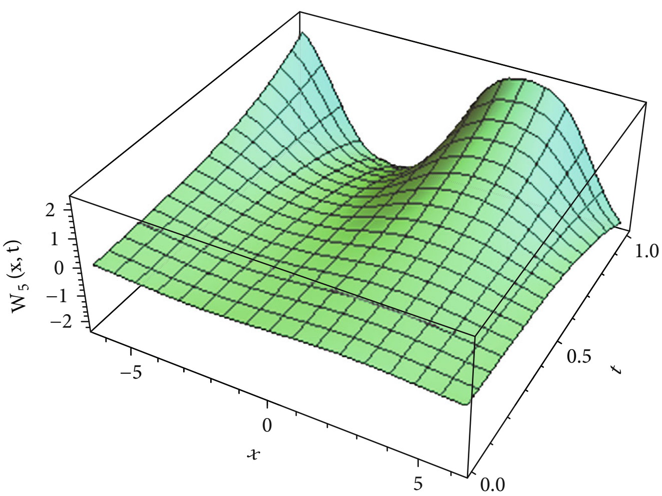
Therefore, the 5th-transform function to the Laplace algebraic equation of (50) can be reformulated:



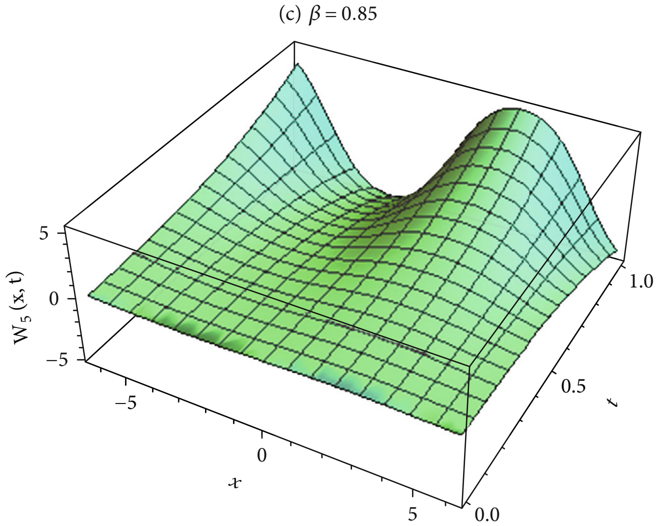
(a) $\beta = 1$



(b) $\beta = 0.95$



(c) $\beta = 0.85$



(d) $\beta = 0.65$

Figure 4: 3D graph representation of the 5^{th}-MFPS approximate solution of Example 3, for all $t \in [0, 1]$, $x \in [-2\pi, 2\pi]$, $L = 5$, and $r = 3$.

$$W_5(x,s)=\frac{1}{10\,s}\sin\left(\frac{\pi x}{L}\right)+\frac{1}{2\times 10^3 s^{\beta+1}}\cdot\left(-201+200r+\cos\left(\frac{2\pi x}{L}\right)\right)\sin\left(\frac{\pi x}{L}\right)+\frac{1}{4\times 10^5\, s^{2\beta+1}}\left(-201+200r+\cos\left(\frac{2\pi x}{L}\right)\right)\cdot\left(-203+200r+3\cos\left(\frac{2\pi x}{L}\right)\right)\sin\left(\frac{\pi x}{L}\right)+\frac{w_3(x)}{s^{3\beta+1}}+\frac{w_4(x)}{s^{4\beta+1}}+\frac{w_5(x)}{s^{5\beta+1}}. \tag{54}$$

As a last step, we take the inverse Laplace transform to both sides of (54), to conclude that the 5th-MFPS approximate solution for the fractional IVPs (48) and (49) will be expressed as follows:

$$w_5(x,t)=\frac{1}{10}\sin\left(\frac{\pi x}{L}\right)+\frac{1}{2\times 10^3}\left(-201+200r+\cos\left(\frac{2\pi x}{L}\right)\right)\sin\cdot\left(\frac{\pi x}{L}\right)\frac{t^{\beta}}{\Gamma(\beta+1)}+\frac{1}{4\times 10^5}\left(-201+200r+\cos\left(\frac{2\pi x}{L}\right)\right)\cdot\left(-203+200r+3\cos\left(\frac{2\pi x}{L}\right)\right)\sin\cdot\left(\frac{\pi x}{L}\right)\frac{t^{2\beta}}{\Gamma(2\beta+1)}+w_3(x)\frac{t^{3\beta}}{\Gamma(3\beta+1)}+w_4(x)\frac{t^{4\beta}}{\Gamma(4\beta+1)}+w_5(x)\frac{t^{5\beta}}{\Gamma(5\beta+1)}. \tag{55}$$

For the numerical simulation of the 5th-MFPS approximate solution (48), some numerical results are calculated for the approximate solution $w_5(x,t)$, at different values of β and r with some selected grid points with step size 0.25 on the interval $[0,1]$ and for $x=\pi/3$, and given in Table 4. Further, numerical comparisons are performed to validate the accuracy of our approach by establishing the recurrence errors $|w_5(x,t)-w_4(x,t)|$ for the obtained approximate solution of fractional IVPs (48) and (49) at various values of β and at fixed values $L=10$, $r=0.9$, as in Table 5. Graphically, we have drawn the profile solution $w_5(x,t)$, for different values of β and at fixed values $L=5$, $r=3$, as in Figure 3. Moreover, the geometric behavior of the 5th-MFPS approximate solution for the fractional IVPs (48) and (49) is studied for different values of β and fixed values $L=5$, $r=3$ by drawing 3D-graphs for the obtained approximate solution as in Figure 4.

5. Conclusion

In this analysis, a recent numeric-analytic iterative technique has been utilized for finding the approximate analytical solutions to both linear and nonlinear time-fractional S-H equations with appropriate initial conditions based on coupling the RPS approach with the Laplace transform operator. The proposed technique has an advantage over other techniques in getting the approximate solution as the MFPS formula reduces the steps of mathematical computations to find out the unknown coefficients for proposed fractional series throughout applying the concept of limit at infinity. Three linear and nonlinear time-fractional S-H equations are solved using the LRPS technique, and its efficiency has been shown via graphical and numerical obtained results which showed the similar and coinciding behavior of the MFPS approximate solution for various values of β and for the classical case $\beta=1$, in terms of the accuracy. Therefore, the current results confirm that the LRPS technique gives notable merits in terms of efficiency, accuracy, and applicability, and it is a convenient method to solve various ranges of linear and nonlinear FPDEs.

Data Availability

No data were used to support this study.

Conflicts of Interest

There is no conflict of interest regarding the publication of this article.

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