

## Research Article

# Functional Coupled Systems with Generalized Impulsive Conditions and Application to a SIRS-Type Model

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In this paper, we consider a first-order coupled impulsive system of equations with functional boundary conditions, subject to the generalized impulsive effects. It is pointed out that this problem generalizes the classical boundary assumptions, allowing two-point or multipoint conditions, nonlocal and integrodifferential ones, or global arguments, as maxima or minima, among others. Our method is based on lower and upper solution technique together with the fixed point theory. The main theorem is applied to a SIRS model where to the best of our knowledge, for the first time, it includes impulsive effects combined with global, local, and the asymptotic behavior of the unknown functions.

## 1. Introduction

The study of impulsive boundary value problems is richer than the related differential equation theory without impulses and has strategic importance in multiple current scientific fields, from sociology and medical sciences to generalized industry production or in any other real-world phenomena where sudden variations occur.

The classic impulsive theory can be seen in [1, 2]. In the last two decades, a vast literature on impulsive differential problems has been produced, such as [3–17], only to mention a few.

Functional problems composed by differential equations and conditions with global dependence on the unknown variable generalize the usual boundary value problems and can include equations and/or conditions with deviating arguments, delays or advances, nonlinear, or nonlocal, increasing in this way the range of applications. The readers interested in results in this direction, on bounded or unbounded domains, may look for in [18–28] and the references therein.

Recently, coupled systems have been studied by many authors, not only from a theoretical point of view but also

due to the huge applications in many sciences and fields, with several methods and approaches. We recommend to the interested readers, for instance, [29–39].

Motivated by the results contained in some of the above references, in this paper, we consider the first-order coupled impulsive system of equations

$$\begin{cases} y_1'(t) = g_1(t, y_1(t), y_2(t), y_3(t)), \\ y_2'(t) = g_2(t, y_1(t), y_2(t), y_3(t)), \\ y_3'(t) = g_3(t, y_1(t), y_2(t), y_3(t)), \end{cases} \quad (1)$$

$a.e. t \in [a, b] \setminus \{t_j\}$ , where  $t_j$  is fixed points, and  $j = 1, 2, \dots, n$  and  $g_i : [a, b] \times \mathbb{R}^3 \rightarrow \mathbb{R}$  are  $L^1$ -Carathéodory functions, for  $i = 1, 2, 3$ , with the functional boundary conditions

$$\begin{aligned} B_1(y_1, y_2, y_3) &= 0, \\ B_2(y_1, y_2, y_3) &= 0, \\ B_3(y_1, y_2, y_3) &= 0, \end{aligned} \quad (2)$$

where  $B_i : (C[a, b])^3 \rightarrow \mathbb{R}$  and  $i = 1, 2, 3$  are continuous functions linearly independent and verifying the generalized impulsive conditions

$$\begin{cases} \Delta y_1(t_j) = H_{1j}(t_j, y_1(t_j), y_2(t_j), y_3(t_j)), \\ \Delta y_2(t_j) = H_{2j}(t_j, y_1(t_j), y_2(t_j), y_3(t_j)), \\ \Delta y_3(t_j) = H_{3j}(t_j, y_1(t_j), y_2(t_j), y_3(t_j)), \end{cases} \quad (3)$$

where  $H_{ij} : [a, b] \times \mathbb{R}^3 \rightarrow \mathbb{R}$  is continuous functions for  $i = 1, 2, 3, j = 1, 2, \dots, n$ , with  $\Delta y_i(t_j) = y_i(t_j^+) - y_i(t_j^-)$ , and  $t_j$  fixed points such that  $a := t_0 < t_1 < t_2 < \dots < t_n < t_{n+1} := b$ .

As far as we know, it is the first time where those three features are taken together to have a coupled impulsive system with functional boundary conditions and generalized impulsive effects, which one including, eventually, impulses on the three unknown functions. We underline two novelties of this paper:

- (1) Condition (2) generalizes the classical boundary assumptions, allowing two-point or multipoint conditions, nonlocal and/or integrodifferential ones, or global arguments, as maxima or minima, among others. In this way, new types of problems and applications could be considered, enabling greater and wider information on the problems studied
- (2) The main theorem is applied to a SIRS model where to the best of our knowledge, for the first time, it includes impulsive effects combined with global, local, and asymptotic behavior of the unknown functions

Our method is based on lower and upper solution technique together with the fixed point theory. In short, the main result is obtained studying a perturbed and truncated system, with modified boundary and impulsive conditions and applying Schauder's fixed point theorem to a completely continuous vectorial operator. Moreover, the paper contains a method to overcome the nonlinearities monotony through a combination with adequate changes in the definition of lower and upper solutions.

The paper is structured in the following way: Section 2 contains the functional framework, definitions, and other known properties. The main result is in Section 3, where the proof is divided into steps, for the reader's convenience. In Section 4, it is shown a method where the definition of coupled lower and upper functions can be used to obtain different versions of the main theorem, with different monotone characteristics on the nonlinearities. The last section contains an application to a vital dynamic SIRS-type model, representing the dynamic epidemiological evolution of susceptible (S), infected (I), Recovered (R), and newly infected individuals in a population on a normalized period, subject to impulsive effects and global restrictions.

## 2. Definitions and Auxiliary Results

Define  $y_i(t_k^\pm) := \lim_{t \rightarrow t_k^\pm} y_i(t)$ , for  $i = 1, 2, 3$ , and consider the sets

$$PC^k([a, b]) = \left\{ \begin{array}{l} y : y \in C^k([a, b], \mathbb{R}^3) \text{ continuous for } t \neq t_j, \\ y^{(k)}(t_j) = y^{(k)}(t_j^-), \\ y^{(k)}(t_j^+) \text{ exists for } j = 1, 2, \dots, n \end{array} \right\}, \quad (4)$$

for  $k = 0, 1$  and the space  $X^3 := (PC([a, b]))^3$  equipped with the norm

$$\|(y_1, y_2, y_3)\|_{X^3} = \max \{\|y_1\|, \|y_2\|, \|y_3\|\}, \quad (5)$$

where

$$\|y\| := \sup_{t \in [a, b]} |y(t)|. \quad (6)$$

It is clear that  $(X^3, \|\cdot\|_{X^3})$  is a Banach space.

The triple  $(y_1, y_2, y_3)$  is a solution of problem (1)-(3) if  $(y_1, y_2, y_3) \in X^3$  and verifies conditions (1), (2), and (3).

*Definition 1.* A function  $w : [a, b] \times \mathbb{R}^3 \rightarrow \mathbb{R}$  for  $i = 1, 2, 3$  is  $L^1$ -Carathéodory if

- (i) for each  $(x, y, z) \in \mathbb{R}^3$ ,  $t \mapsto w(t, x, y, z)$  is measurable on  $[a, b]$
- (ii) for a.e.  $t \in [a, b]$ ,  $(x, y, z) \mapsto w(t, x, y, z)$  is continuous on  $\mathbb{R}^3$
- (iii) for each  $\rho > 0$ , there exists a positive function  $\psi_\rho \in L^1([a, b])$  and for  $(x, y, z) \in \mathbb{R}^3$  such that

$$\max \{|x|, |y|, |z|\} < \rho, \quad (7)$$

one has

$$|w(t, x, y, z)| \leq \psi_\rho(t), \text{ a.e. } t \in [a, b]. \quad (8)$$

In this paper, the definition of lower and upper solutions plays a key role in our method.

Next definition will be used in the main theorem:

*Definition 2.* Consider the  $PC^1$ -functions  $\alpha_i, \beta_i : [a, b] \rightarrow \mathbb{R}, i = 1, 2, 3$ .

The triple  $(\alpha_1, \alpha_2, \alpha_3) \in X^3$  is a lower solution of the problem (1)-(3) if

$$\begin{aligned} \alpha'_i(t) &\leq g_i(t, \alpha_1(t), \alpha_2(t), \alpha_3(t)), \text{ for } i = 1, 2, 3, \\ B_1(\alpha_1, \alpha_2, \alpha_3) &\geq 0, \\ B_2(\alpha_1, \alpha_2, \alpha_3) &\geq 0, \\ B_3(\alpha_1, \alpha_2, \alpha_3) &\geq 0, \end{aligned} \tag{9}$$

and, for  $j = 1, 2, \dots, n$ ,

$$\begin{aligned} \Delta\alpha_1(t_j) &\leq H_{1j}(t_j, \alpha_1(t_j), \alpha_2(t_j), \alpha_3(t_j)), \\ \Delta\alpha_2(t_j) &\leq H_{2j}(t_j, \alpha_1(t_j), \alpha_2(t_j), \alpha_3(t_j)), \\ \Delta\alpha_3(t_j) &\leq H_{3j}(t_j, \alpha_1(t_j), \alpha_2(t_j), \alpha_3(t_j)). \end{aligned} \tag{10}$$

The triple  $(\beta_1, \beta_2, \beta_3) \in X^3$  is an upper solution of the problems (1)-(3) if the reversed inequalities hold.

### 3. Main Result

The main result will provide the existence of, at least, a solution for the problems (1)-(3).

**Theorem 3.** Assume that there are  $\alpha$  and  $\beta$  lower and upper solutions of problem (1)-(3), according Definition 2, such that

$$\alpha_i(t) \leq \beta_i(t), \forall t \in [a, b], \text{ for } i = 1, 2, 3. \tag{11}$$

Let  $g_i : [a, b] \times \mathbb{R}^3 \rightarrow \mathbb{R}, i = 1, 2, 3$ , be  $L^1$  - Carathéodory functions, not identically null, on the set

$$\{(t, y_i) \in [a, b] \times \mathbb{R}^3 : \alpha_i(t) \leq y_i \leq \beta_i(t), i = 1, 2, 3\}, \tag{12}$$

$$g_1(t, y_1, \alpha_2(t), \alpha_3(t)) \leq g_1(t, y_1, y_2, y_3) \leq g_1(t, y_1, \beta_2(t), \beta_3(t)), \tag{13}$$

for  $t \in [a, b] \setminus \{t_j\}, j \in \{1, 2, \dots, n\}, y_1 \in \mathbb{R}, \alpha_i(t) \leq y_i \leq \beta_i(t)$ , and  $i = 2, 3$ .

$$g_2(t, \alpha_1(t), y_2, \alpha_3(t)) \leq g_2(t, y_1, y_2, y_3) \leq g_2(t, \beta_1(t), y_2, \beta_3(t)), \tag{14}$$

for  $t \in [a, b] \setminus \{t_j\}, j \in \{1, 2, \dots, n\}, y_2 \in \mathbb{R}, \alpha_i(t) \leq y_i \leq \beta_i(t)$ , and  $i = 1, 3$ .

$$g_3(t, \alpha_1(t), \alpha_2(t), y_3) \leq g_3(t, y_1, y_2, y_3) \leq g_3(t, \beta_1(t), \beta_2(t), y_3), \tag{15}$$

for  $t \in [a, b] \setminus \{t_j\}, j \in \{1, 2, \dots, n\}, y_3 \in \mathbb{R}, \alpha_i(t) \leq y_i \leq \beta_i(t)$ , and  $i = 1, 2$ , with

$$B_i(\alpha_1, \alpha_2, \alpha_3) \leq B_i(y_1, y_2, y_3) \leq B_i(\beta_1, \beta_2, \beta_3), \tag{16}$$

for  $\alpha_i \leq y_i \leq \beta_i, i = 1, 2, 3$ .

Assume that  $H_{ij} : [a, b] \times \mathbb{R}^3 \rightarrow \mathbb{R}$  satisfies

$$\begin{aligned} H_{ij}(t_j, \alpha_1(t_j), \alpha_2(t_j), \alpha_3(t_j)) &\leq H_{ij}(t_j, w_1, w_2, w_3) \\ &\leq H_{ij}(t_j, \beta_1(t_j), \beta_2(t_j), \beta_3(t_j)), \end{aligned} \tag{17}$$

for  $\alpha_i(t_j) \leq w_i \leq \beta_i(t_j), i = 1, 2, 3$ , and  $j \in \{1, 2, \dots, n\}$ .  
If there is  $\rho > 0$  such that

$$\max_{i=1,2,3} \left[ \max \{ \|\alpha_i\|, \|\beta_i\| \} + \sum_{j=1}^n |H_{ij}(t_j, w_1, w_2, w_3)| + \int_a^b (\psi_{i,\rho}(s) + 1) ds \right] < \rho, \tag{18}$$

then there exists at least a triple  $(y_1, y_2, y_3) \in X^3$ , solution of (1)-(3), such that

$$\alpha_i(t) \leq y_i(t) \leq \beta_i(t), \forall t \in [a, b], \text{ for } i = 1, 2, 3. \tag{19}$$

*Proof.* Let  $(\alpha_1, \alpha_2, \alpha_3), (\beta_1, \beta_2, \beta_3) \in X^3$  be, respectively, lower and upper solutions of (1)-(3), as in Definition 2, verifying (11).

Consider the continuous truncatures  $\delta_i : [a, b] \times \mathbb{R} \rightarrow \mathbb{R}, i = 1, 2, 3$ , denoted, for short, as  $\delta_i(t)$ , defined by

$$\delta_i(t) := \delta_i(t, y_i(t)) = \begin{cases} \beta_i(t) & , y_i(t) > \beta_i(t), \\ y_i(t) & , \alpha_i(t) \leq y_i(t) \leq \beta_i(t), \\ \alpha_i(t) & , y_i(t) < \alpha_i(t), \end{cases} \tag{20}$$

and consider the modified and perturbed problem composed by the differential system

$$y'_1(t) = g_1(t, \delta_1(t), \delta_2(t), \delta_3(t)) + \frac{y_1(t) - \delta_1(t)}{1 + |y_1(t) - \delta_1(t)|}, \tag{21}$$

$$y'_2(t) = g_2(t, \delta_1(t), \delta_2(t), \delta_3(t)) + \frac{y_2(t) - \delta_2(t)}{1 + |y_2(t) - \delta_2(t)|}, \tag{22}$$

$$y'_3(t) = g_3(t, \delta_1(t), \delta_2(t), \delta_3(t)) + \frac{y_3(t) - \delta_3(t)}{1 + |y_3(t) - \delta_3(t)|}, \tag{23}$$

for  $t \in [a, b] \setminus \{t_j\}, j \in \{1, 2, \dots, n\}$ , together with the truncated boundary conditions

$$y_i(a) = \delta_i(a, y_i(a) + B_i(\delta_1^*(y_1), \delta_2^*(y_2), \delta_3^*(y_3))), \tag{24}$$

for  $i = 1, 2, 3$ , with the functional truncations

$$\delta_i^*(w) = \begin{cases} \beta_i & , w > \beta_i, \\ w & , \alpha_i \leq w \leq \beta_i, \\ \alpha_i & , w < \alpha_i, \end{cases} \tag{25}$$

and the truncated impulsive conditions

$$\Delta y_i(t_j) = H_{ij}(t_j, \delta_1(t_j), \delta_2(t_j), \delta_3(t_j)), \quad (26)$$

for  $j \in \{1, 2, \dots, n\}$ .  $\square$

*Claim 4.* The problems (21), (24), and (26) have at least a solution.

This claim will be proved by the fixed point theory, applied to the vectorial operator

$$\mathcal{T} : X^3 \longrightarrow X^3, \quad (27)$$

given by

$$\mathcal{T}(y_1, y_2, y_3) = (\mathcal{T}_1(y_1, y_2, y_3), \mathcal{T}_2(y_1, y_2, y_3), \mathcal{T}_3(y_1, y_2, y_3)), \quad (28)$$

where for  $i = 1, 2, 3$ ,

$$\mathcal{T}_i : X^3 \longrightarrow X \quad (29)$$

defined as

$$\begin{aligned} \mathcal{T}_i(y_1, y_2, y_3)(t) = & \delta_i(a, y_i(a) + B_i(\delta_1^*(y_1), \delta_2^*(y_2), \delta_3^*(y_3))) \\ & + \sum_{j:t_j < t} H_{ij}(t_j, \delta_1(t_j), \delta_2(t_j), \delta_3(t_j)) + \int_a^t \\ & \cdot \left( g_i(s, \delta_1(s), \delta_2(s), \delta_3(s)) + \frac{y_i(s) - \delta_i(s)}{1 + |y_i(s) - \delta_i(s)|} \right) ds. \end{aligned} \quad (30)$$

Integrating (21) for  $t \in [a, t_1]$ , by (24), we have

$$\begin{aligned} y_i(t) = & \delta_i(a, y_i(a) + B_i(\delta_1^*(y_1), \delta_2^*(y_2), \delta_3^*(y_3))) + \int_a^t \\ & \cdot \left( g_i(s, \delta_1(s), \delta_2(s), \delta_3(s)) + \frac{y_i(s) - \delta_i(s)}{1 + |y_i(s) - \delta_i(s)|} \right) ds. \end{aligned} \quad (31)$$

By integration of (21) for  $t \in ]t_1, t_2]$ , by (26) and (31), we get

$$\begin{aligned} y_i(t) = & y_i(t_1^+) + \int_a^t \left( g_i(s, \delta_1(s), \delta_2(s), \delta_3(s)) + \frac{y_i(s) - \delta_i(s)}{1 + |y_i(s) - \delta_i(s)|} \right) ds \\ = & y_i(t_1^-) + H_{i1}(t_1, \delta_1(t_1), \delta_2(t_1), \delta_3(t_1)) + \int_{t_1}^t \\ & \cdot \left( g_i(s, \delta_1(s), \delta_2(s), \delta_3(s)) + \frac{y_i(s) - \delta_i(s)}{1 + |y_i(s) - \delta_i(s)|} \right) ds \\ = & \delta_i(a, y_i(a) + B_i(\delta_1^*(y_1), \delta_2^*(y_2), \delta_3^*(y_3))) \\ & + H_{i1}(t_1, \delta_1(t_1), \delta_2(t_1), \delta_3(t_1)) + \int_a^t \\ & \cdot \left( g_i(s, \delta_1(s), \delta_2(s), \delta_3(s)) + \frac{y_i(s) - \delta_i(s)}{1 + |y_i(s) - \delta_i(s)|} \right) ds. \end{aligned} \quad (32)$$

Iterating these arguments, it is clear that the fixed points of  $\mathcal{T}$ , that is, the set of the fixed points of  $\mathcal{T}_i$ , for  $i = 1, 2, 3$ , are solutions of the problems (21), (24), and (26).

As  $g_i$  is a  $L^1$ -Carathéodory function,  $H_{ij}$  and the truncatures  $\delta_i, \delta_i^*$  are continuous; therefore,  $\mathcal{T}_i$  are well defined and continuous. Therefore,  $\mathcal{T}$  is well defined and continuous.

Consider a bounded set  $D \subset X^3$ . So, there is  $k > 0$  such that  $\|(x, y, z)\|_{X^3} < k$ , for  $(x, y, z) \in D$ .

$\mathcal{T}_i D$  is uniformly bounded, as, for  $i = 1, 2, 3$ ,

$$\begin{aligned} \|\mathcal{T}_i(y_1, y_2, y_3)\| = & \sup_{t \in [a, b]} |\mathcal{T}_i(y_1, y_2, y_3)(t)| \\ \leq & \sup_{t \in [a, b]} |\delta_i(a, y_i(a) + B_i(\delta_1^*(y_1), \delta_2^*(y_2), \delta_3^*(y_3)))| \\ & + \sum_{j:t_j < t} |H_{ij}(t_j, \delta_1(t_j), \delta_2(t_j), \delta_3(t_j))| \\ & + \int_a^t \left| g_i(s, \delta_1(s), \delta_2(s), \delta_3(s)) + \frac{y_i(s) - \delta_i(s)}{1 + |y_i(s) - \delta_i(s)|} \right| ds \\ \leq & \max_{i=1,2,3} \left[ \max \{ \|\alpha_i\|, \|\beta_i\| \} + \sum_{j=1}^n |H_{ij}(t_j, \delta_1(t_j), \delta_2(t_j), \delta_3(t_j))| \right] \\ & + \max_{i=1,2,3} \int_a^b (\psi_{i,k}(s) + 1) ds < \infty, \end{aligned} \quad (33)$$

where  $\psi_{i,k}$  is the positive function given by Definition 1.

$\mathcal{T}_i D$  is equicontinuous because, for  $i = 1, 2, 3$ , and  $t_1, t_2 \in [a, b]$  with  $t_1 < t_2$  (without loss of generality),

$$\begin{aligned} & |\mathcal{T}_i(y_1, y_2, y_3)(t_1) - \mathcal{T}_i(y_1, y_2, y_3)(t_2)| \\ \leq & \sum_{j:t_1 < t_j < t_2} |H_{ij}(t_j, \delta_1(t_j), \delta_2(t_j), \delta_3(t_j))| \\ & + \int_{t_1}^{t_2} (\psi_{i,k}(s) + 1) ds \longrightarrow 0, \end{aligned} \quad (34)$$

as  $t_1 \longrightarrow t_2$ .

$\mathcal{T}_i D$  is equiconvergent on the impulsive moments, as  $|\mathcal{T}_i(y_1, y_2, y_3)(t) - \lim_{t \rightarrow t_j^+} \mathcal{T}_i(y_1, y_2, y_3)(t)| \leq |\sum_{j:t_j < t} H_{ij}(t_j,$

$\delta_1(t_j), \delta_2(t_j), \delta_3(t_j)) - \sum_{j:t_j < t_j^+} |H_{ij}(t_j, \delta_1(t_j), \delta_2(t_j), \delta_3(t_j))| + |\int_a^t (\psi_{i,k}(s) + 1) ds - \int_a^{t_j^+} (\psi_{i,k}(s) + 1) ds| \longrightarrow 0$ , (42) when  $t \longrightarrow t_j^+$ . Therefore,  $\mathcal{T}_i$  and  $\mathcal{T}$  are compact operators.

Consider now the closed, bounded, and convex set  $\Omega \subset X^3$ , defined by

$$\Omega = \{w \in X^3 : \|\mathcal{T}w\|_{X^3} \leq R\} \quad (35)$$

with  $R > 0$  such that

$$\begin{aligned} R > & \max_{i=1,2,3} \left[ \max \{ \|\alpha_i\|, \|\beta_i\| \} + \sum_{j=1}^n |H_{ij}(t_j, \delta_1(t_j), \delta_2(t_j), \delta_3(t_j))| \right] \\ & + \max_{i=1,2,3} \int_a^b (\psi_{i,R}(s) + 1) ds. \end{aligned} \quad (36)$$

By the above calculus,  $\mathcal{F}\Omega \subset \Omega$ , and from Schauders' fixed point theorem,  $\mathcal{F}$  has a fixed point  $y^* = (y_1^*, y_2^*, y_3^*)$ , which is solution of the problems (21), (24), and (26).

*Claim 5.* This function  $y^* = (y_1^*, y_2^*, y_3^*)$  is a solution of problems (1)-(3), too.

To prove this claim is enough to show that, for every solution  $(y_1, y_2, y_3) \in X^3$  of problems (21), (24), and (26), the following inequalities hold:

$$\alpha_i(t) \leq y_i(t) \leq \beta_i(t), \text{ for } i = 1, 2, 3, \text{ and } t \in [a, b], \quad (37)$$

$$\alpha_i(a) \leq y_i(a) + B_i(\delta_1^*(a, y_1^*(a)), \delta_2^*(a, y_2^*(a)), \delta_3^*(a, y_3^*(a))) \leq \beta_i(a). \quad (38)$$

Let  $y := (y_1, y_2, y_3) \in X^3$  be a solution of the problems (21), (24), and (26).

To prove the first inequality of (37), for  $i = 1$ , assume that there is  $t \in [a, b]$  such that  $\alpha_1(t) - y_1(t) > 0$  and define

$$\sup_{t \in [a, b]} (\alpha_1(t) - y_1(t)) := \alpha_1(t_*) - y_1(t_*) > 0. \quad (39)$$

Remark that  $t_* \neq a$ , as, by (24) and (20),

$$\alpha_1(a) - y_1(a) = \alpha_1(a) - \delta_1(a, y_1(a) + B_1(\delta_1^*(y_1), \delta_2^*(y_2), \delta_3^*(y_3))) \leq 0. \quad (40)$$

If  $t_*$  is between two consecutive impulses, that is,  $t \in ]t_p, t_{p+1}]$ , for fixed  $p = 0, 1, \dots, n$ , then  $\alpha_1'(t_*) - y_1'(t_*) = 0$ , by (24), (13), and Definition 2, this contradiction is achieved

$$\begin{aligned} 0 \leq \alpha_1'(t_*) - y_1'(t_*) &= \alpha_1'(t_*) - g_1(t_*, \alpha_1(t_*), \delta_2(t_*), \delta_3(t_*)) \\ &\quad - \frac{y_1(t_*) - \alpha_1(t_*)}{1 + |y_1(t_*) - \alpha_1(t_*)|} < \alpha_1'(t_*) - g_1(t_*, \alpha_1(t_*), \delta_2(t_*), \delta_3(t_*)) \\ &\leq \alpha_1'(t_*) - g_1(t_*, \alpha_1(t_*), \alpha_2(t_*), \alpha_3(t_*)) \leq 0. \end{aligned} \quad (41)$$

If  $t_*$  is an impulsive moment, that is, there is  $j \in \{1, 2, \dots, n\}$  such that  $t_* = t_j^+$ ; then, by (26), (17), and Definition 2, we have

$$\begin{aligned} 0 \leq \Delta \alpha_1(t_j) - \Delta y_1(t_j) &= \Delta \alpha_1(t_j) - H_{1j}(t_j, \delta_1(t_j), \delta_2(t_j), \delta_3(t_j)) \\ &\leq \Delta \alpha_1(t_j) - H_{1j}(t_j, \alpha_1(t_j), \alpha_2(t_j), \alpha_3(t_j)) \leq 0. \end{aligned} \quad (42)$$

Therefore,

$$\Delta y_1(t_j) - \Delta \alpha_1(t_j) = 0, \quad (43)$$

that is, there are no jumps at any point  $t_j$ . Then, by (39),

$$0 \leq \alpha_1'(t_j^-) - y_1'(t_j^-), \quad (44)$$

and the contradiction is obtained as in the previous case.

Therefore,  $\alpha_1(t) \leq y_1(t)$ , for  $t \in [a, b]$ . With the same arguments, it can be proved that  $y_1(t) \leq \beta_1(t)$ , for  $t \in [a, b]$ .

A similar technique can be applied for functions  $g_2$  and  $g_3$ , applying conditions (14) and/or (15), respectively.

Suppose now, by contradiction, that

$$\alpha_i(a) > y_i(a) + B_i(\delta_1^*(y_1), \delta_2^*(y_2), \delta_3^*(y_3)). \quad (45)$$

Then, by (24),

$$y_i(a) = \delta_i(a, y_i(a) + B_i(\delta_1^*(y_1), \delta_2^*(y_2), \delta_3^*(y_3))) = \alpha_i(a), \quad (46)$$

which is in contradiction with (45), by (16) and Definition 2,

$$0 = y_i(a) - \alpha_i(a) > B_i(\delta_1^*(y_1), \delta_2^*(y_2), \delta_3^*(y_3)) - B_i(\alpha_1, \alpha_2, \alpha_3) \geq 0, \quad (47)$$

for  $i = 1, 2, 3$ .

The remaining inequalities can be proved with similar arguments.

#### 4. Relation between Monotonies and Lower and Upper Definitions

The monotone assumptions required on the nonlinearities and on the impulsive functions, by conditions (13)-(15) and (17), although local, can seem too restrictive. Indeed, these monotonies can be modified since they are combined with different definitions of coupled lower and upper solutions, following the method described in this section.

*Definition 6.* Consider the  $PC^1$ -functions  $\alpha_i, \beta_i : [a, b] \rightarrow \mathbb{R}, i = 1, 2, 3$ .

The triples  $(\alpha_1, \alpha_2, \alpha_3), (\beta_1, \beta_2, \beta_3) \in X^3$  are coupled lower and upper solutions of the problems (1)-(3) if

$$\begin{aligned} \alpha_1'(t) &\leq g_1(t, \alpha_1(t), \beta_2(t), \alpha_3(t)), \\ \alpha_i'(t) &\leq g_i(t, \alpha_1(t), \alpha_2(t), \alpha_3(t)), \text{ for } i = 2, 3, \\ \beta_1'(t) &\geq g_1(t, \beta_1(t), \alpha_2(t), \beta_3(t)), \\ \beta_i'(t) &\geq g_i(t, \beta_1(t), \beta_2(t), \beta_3(t)), \text{ for } i = 2, 3, \end{aligned} \quad (48)$$

and, for  $j = 1, 2, \dots, n$ ,

$$\Delta \alpha_1(t_j) \leq H_{1j}(t_j, \beta_1(t_j), \alpha_2(t_j), \alpha_3(t_j)), \quad (49)$$

$$\Delta \alpha_2(t_j) \leq H_{2j}(t_j, \alpha_1(t_j), \alpha_2(t_j), \alpha_3(t_j)), \quad (50)$$

$$\Delta \alpha_3(t_j) \leq H_{3j}(t_j, \alpha_1(t_j), \alpha_2(t_j), \beta_3(t_j)), \quad (51)$$

$$\Delta \beta_1(t_j) \geq H_{1j}(t_j, \alpha_1(t_j), \beta_2(t_j), \beta_3(t_j)), \quad (52)$$

$$\Delta \beta_2(t_j) \geq H_{2j}(t_j, \beta_1(t_j), \beta_2(t_j), \beta_3(t_j)), \quad (53)$$

$$\Delta \beta_3(t_j) \geq H_{3j}(t_j, \beta_1(t_j), \beta_2(t_j), \alpha_3(t_j)). \quad (54)$$

The inequalities for boundary conditions are similar to Definition 2.

With this definition, the assumption on the local monotony of function  $g_1$  and on the impulsive functions  $H_{1j}$  and  $H_{3j}$  can be replaced, as in the following version of Theorem 3:

**Theorem 7.** *Assuming that all the assumptions of Theorem 3, with coupled lower and upper solutions defined as in Definition 6, (13) is replaced by*

$$g_1(t, \beta_1(t), \beta_2(t), \alpha_3(t)) \leq g_1(t, \gamma_1, \gamma_2, \gamma_3) \leq g_1(t, \alpha_1, \alpha_2(t), \beta_3(t)), \quad (55)$$

for  $t \in [a, b] \setminus \{t_j\}, j \in \{1, 2, \dots, n\}, \gamma_1 \in \mathbb{R}, \alpha_i(t) \leq \gamma_i \leq \beta_i(t), i = 2, 3$ , and (12) by

$$\begin{aligned} H_{1j}(t_j, \beta_1(t_j), \alpha_2(t_j), \alpha_3(t_j)) &\leq H_{1j}(t_j, w_1, w_2, w_3) \\ &\leq H_{1j}(t_j, \alpha_1(t_j), \beta_2(t_j), \beta_3(t_j)), \end{aligned}$$

$$\begin{aligned} H_{2j}(t_j, \alpha_1(t_j), \alpha_2(t_j), \alpha_3(t_j)) &\leq H_{2j}(t_j, w_1, w_2, w_3) \\ &\leq H_{2j}(t_j, \beta_1(t_j), \beta_2(t_j), \beta_3(t_j)), \end{aligned}$$

$$\begin{aligned} H_{3j}(t_j, \alpha_1(t_j), \alpha_2(t_j), \beta_3(t_j)) &\leq H_{3j}(t_j, w_1, w_2, w_3) \\ &\leq H_{3j}(t_j, \beta_1(t_j), \beta_2(t_j), \alpha_3(t_j)), \end{aligned} \quad (56)$$

for  $\alpha_i(t_j) \leq w_i \leq \beta_i(t_j), i = 1, 2, 3$ , and  $j \in \{1, 2, \dots, n\}$ . Then, there is at least  $(\gamma_1, \gamma_2, \gamma_3) \in X^3$  solution of (1)-(3) such that

$$\alpha_i(t) \leq \gamma_i(t) \leq \beta_i(t), \forall t \in [a, b], \text{ for } i = 1, 2, 3. \quad (57)$$

*Proof.* The proof of Theorem 3 holds, and it remains to prove that every solution  $(\gamma_1, \gamma_2, \gamma_3) \in X^3$  of problems (21), (24), and (26) verifies

$$\alpha_1(t) \leq \gamma_1(t) \leq \beta_1(t), \text{ for } t \in [a, b]. \quad (58)$$

Assume that there is  $t \in [a, b]$  such that  $\alpha_i(t) - \gamma_i^*(t) > 0$  and define

$$\sup_{t \in [a, b]} (\alpha_i(t) - \gamma_i^*(t)) := \alpha_i(t_*) - \gamma_i^*(t_*) > 0. \quad (59)$$

Consider  $t_*$  between two consecutive impulses, that is,  $t \in ]t_p, t_{p+1}]$ . Then, by (24), (13), and Definition 2, this contradiction is achieved

$$\begin{aligned} 0 \leq \alpha_1'(t) - \gamma_1^{*'}(t) &= \alpha_1'(t) - g_1(t, \alpha_1(t), \delta_2(t), \delta_3(t)) \\ &\quad - \frac{\gamma_1^*(t) - \alpha_1(t)}{1 + |\gamma_1^*(t) - \alpha_1(t)|} < \alpha_1'(t) - g_1(t, \alpha_1(t), \delta_2(t), \delta_3(t)) \\ &\leq \alpha_1'(t) - g_1(t, \alpha_1(t), \beta_2(t), \alpha_3(t)) \leq 0. \end{aligned} \quad (60)$$

In the impulsive points, case where  $t_* = t_j^+$ , by (49),

$$\begin{aligned} 0 \leq \Delta \alpha_1(t_j) - \Delta \gamma_1(t_j) &= \Delta \alpha_1(t_j) - H_{1j}(t_j, \delta_1(t_j), \delta_2(t_j), \delta_3(t_j)) \\ &\leq \Delta \alpha_1(t_j) - H_{1j}(t_j, \beta_1(t_j), \alpha_2(t_j), \alpha_3(t_j)) \leq 0, \end{aligned} \quad (61)$$

analogously, for  $\Delta(\alpha_3 - \gamma_3)(t_j)$ .  $\square$

Following these arguments, we can obtain different versions of Theorem 3, combining adequate definitions of coupled lower and upper solutions, as in Definition 6, and alternative monotone assumptions on  $g_2$  and  $g_3$  and on the impulsive functions  $H_{ij}$ .

## 5. Application to a Vital Dynamic SIRS Model

The study of epidemiological phenomena via compartmental models is currently a special concern as it simplifies the mathematical modeling of infectious diseases. These types of models try to predict, for instance, how a disease spreads, the duration of an epidemic, the variation of the number of infected people, and other epidemiological parameters. So, they are important tools to help the definition of rules for public health interventions and how they may affect the outcome of the epidemic.

The classic SIR model is a basic compartmental model where the population is divided into three groups: susceptible (S), infected (I), and recovered (R). People may change groups, but the SIR model assumes that the population gains lifelong immunity to some disease upon recovery. This is true for some infectious diseases, such as measles, mumps, or rubella, but it is not the case for some airborne diseases, such as seasonal influenza, where the individual's immunity may wane over time. In this situation, the SIRS model is more adequate as it allows that the recovered individuals can return to a susceptible state and be infected again.

These compartmental models were introduced in the early 20th century, by Kermack and McKendrick in 1927, ([40]), but since then, many authors study these topics, under different and varied features, objectives, and techniques. As examples, we mention only some recent works on the field: [41–48].

Motivated by the papers above, we apply our technique to a vital dynamic SIRS system composed by the differential equations

$$\begin{cases} S'(t) = -\gamma S(t)I(t) + \lambda R(t), \\ I'(t) = \gamma S(t)I(t) - (\mu + d)I(t), \\ R'(t) = \mu I(t) - \lambda R(t), \end{cases} \quad (62)$$

for  $t$  in a normalized interval  $[0, 1]$ ,  $\gamma, \mu$ , representing the infection and recover rates;  $\lambda$  is the rate of recovered individuals becoming susceptible again, and  $d$  is the death number by infection.



Our method allows to consider global and asymptotic data as a particular case of functional boundary conditions:

$$\begin{aligned} \inf_{t \in [0,1]} S(t) &= \lim_{t \rightarrow (\frac{1}{4})^+} S(t), \\ \max_{t \in [0, \frac{1}{4}]} I(t) &= \lim_{t \rightarrow (\frac{1}{4})^+} I(t), \\ \sup_{t \in [0,1]} R(t) &= R(1), \end{aligned} \tag{63}$$

and generalized impulsive functions, with only one impulsive moment, for the sake of clarity,

$$\Delta S\left(\frac{1}{4}\right) = -5\gamma S\left(\frac{1}{4}\right) I\left(\frac{1}{4}\right) + \lambda R\left(\frac{1}{4}\right) I\left(\frac{1}{4}\right), \tag{64}$$

$$\Delta I\left(\frac{1}{4}\right) = 5\gamma S\left(\frac{1}{4}\right) I\left(\frac{1}{4}\right), \tag{65}$$

$$\Delta R\left(\frac{1}{4}\right) = -\lambda R\left(\frac{1}{4}\right) I\left(\frac{1}{4}\right). \tag{66}$$

It is clear that problems (62)-(64) are a particular case of problem (1)-(3), with  $y_1 = S, y_2 = I, y_3 = R, a = 0, b = 1, n = j = 1, t_1 = 1/4$ ,

$$\begin{aligned} g_1(t, S(t), I(t), R(t)) &= -\gamma S(t)I(t) + \lambda R(t), \\ g_2(t, S(t), I(t), R(t)) &= \gamma S(t)I(t) - (\mu + d)I(t), \\ g_3(t, S(t), I(t), R(t)) &= \mu I(t) - \lambda R(t), \end{aligned} \tag{67}$$

and the functional boundary conditions

$$\begin{aligned} B_1(S, I, R) &= \inf_{t \in [0,1]} S(t) - \lim_{t \rightarrow (\frac{1}{4})^+} S(t) = 0, \\ B_2(S, I, R) &= \max_{t \in [0, \frac{1}{4}]} I(t) - \lim_{t \rightarrow (\frac{1}{4})^+} I(t) = 0, \\ B_3(S, I, R) &= \sup_{t \in [0,1]} R(t) - R(1) = 0, \end{aligned} \tag{68}$$

and the impulsive effects

$$\begin{aligned} \Delta S\left(\frac{1}{4}\right) &= H_{11}\left(S\left(\frac{1}{4}\right), I\left(\frac{1}{4}\right), R\left(\frac{1}{4}\right)\right) = -5\gamma S\left(\frac{1}{4}\right) I\left(\frac{1}{4}\right) + \lambda R\left(\frac{1}{4}\right) I\left(\frac{1}{4}\right), \\ \Delta I\left(\frac{1}{4}\right) &= H_{21}\left(S\left(\frac{1}{4}\right), I\left(\frac{1}{4}\right), R\left(\frac{1}{4}\right)\right) = 5\gamma S\left(\frac{1}{4}\right) I\left(\frac{1}{4}\right) - kI\left(\frac{1}{4}\right), \\ \Delta R\left(\frac{1}{4}\right) &= H_{31}\left(S\left(\frac{1}{4}\right), I\left(\frac{1}{4}\right), R\left(\frac{1}{4}\right)\right) = -\lambda R\left(\frac{1}{4}\right) I\left(\frac{1}{4}\right) + kI\left(\frac{1}{4}\right). \end{aligned} \tag{69}$$

As a numeric example we consider the rates  $\gamma_1 = 0.017, \lambda_1 = 0.02, \mu_1 = 0.93$ , and  $d_1 = 0.2$  before the impulsive moment, that is, for  $0 \leq t \leq 1/4$  and  $\gamma_2 = 0.001, \lambda_2 = 0.1, \mu_2 = 0.162$ , and  $d_2 = 0$ , after the impulsive effect, i.e., for  $1/4 < t \leq 1$ .

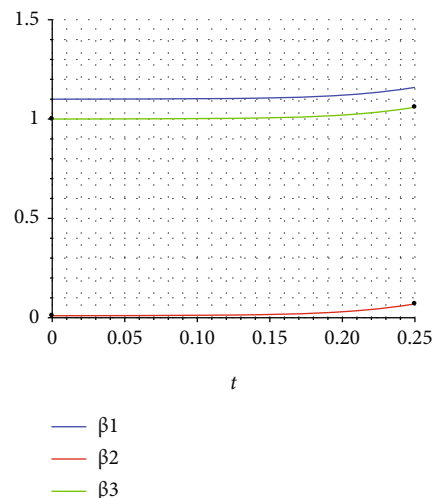


FIGURE 1: Graphs in  $[0, 1/4]$ .

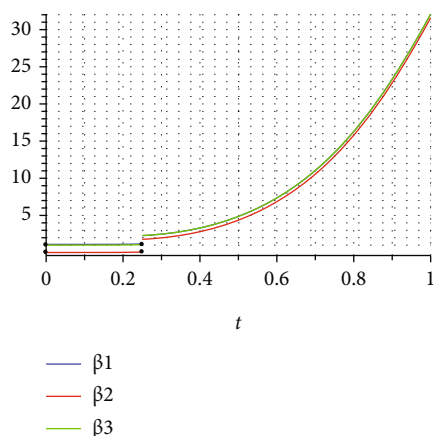


FIGURE 2: Graphs in  $[0, 1]$ .

For these values, the triple null functions  $(\alpha_1, \alpha_2, \alpha_3) = (0, 0, 0)$  and the piecewise one  $(\beta_1, \beta_2, \beta_3)$  given by

$$\begin{aligned} \beta_1(t) &= \begin{cases} 112t^6 + 17.12t^5 + 4.6t^4 - 0.4t^3 - 0.02t^2 + 0.02t + 1.1, & 0 \leq t \leq \frac{1}{4}, \\ -14.5t^6 + 20t^5 + 10t^4 + 14.6t^3 - 0.02t^2 + 0.006t + 2, & \frac{1}{4} < t \leq 1, \end{cases} \\ \beta_2(t) &= \begin{cases} 112t^6 + 17.12t^5 + 4.6t^4 - 0.4t^3 - 0.02t^2 + 0.02t + 0.01, & 0 \leq t \leq \frac{1}{4}, \\ -14.5t^6 + 20t^5 + 10t^4 + 14.6t^3 - 0.02t^2 + 0.006t + 1.5, & \frac{1}{4} < t \leq 1, \end{cases} \\ \beta_3(t) &= \begin{cases} 112t^6 + 17.12t^5 + 4.6t^4 - 0.4t^3 - 0.02t^2 + 0.02t + 1, & 0 \leq t \leq \frac{1}{4}, \\ -14.5t^6 + 20t^5 + 10t^4 + 14.6t^3 - 0.02t^2 + 0.006t + 2, & \frac{1}{4} < t \leq 1, \end{cases} \end{aligned} \tag{70}$$

are, respectively, lower and upper solutions of problems (62)-(64), according to Definition 6, as it can be seen in Figures 1 and 2.

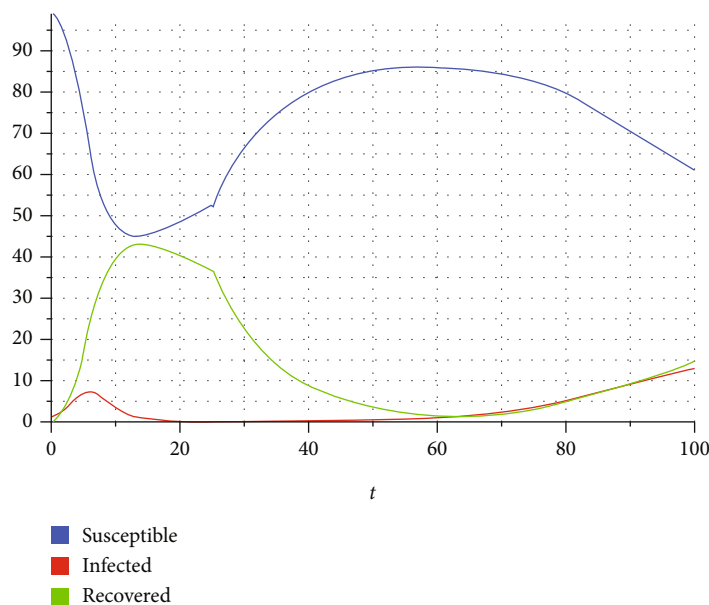


FIGURE 3: Evolution of functions  $S$ ,  $I$ , and  $R$  in the first 100 days.

All the assumptions of Theorem 7 are fulfilled for  $56.22 \leq \rho < 137.23$ , and therefore, there is a solution of problems (62)-(64), such that

$$\begin{aligned} 0 &\leq S(t) \leq \beta_1(t), \\ 0 &\leq I(t) \leq \beta_2(t), \\ 0 &\leq R(t) \leq \beta_3(t), \text{ for } t \in [0, 1]. \end{aligned} \quad (71)$$

Applying an adequate mathematical software, these inequalities can be illustrated by the graph of the correspondent solution, given in Figure 3, considering the population in percentage in the first 100 days.

## 6. Conclusion

The paper's main goal is to present sufficient conditions for the solvability of impulsive coupled systems with functional boundary conditions generalizing the classical boundary ones. In this way, problems may consider restrictions related to the global variation of solutions or their asymptotic behavior near the impulsive moments, as can be seen in Theorem 3.

Moreover, in Section 4, it is shown how we can use the definition of lower and upper solutions to overcome restrictions on the monotone conditions on the nonlinearities.

The application's aim is to illustrate how the theoretical results could be applied to real phenomena.

## Data Availability

No data were used to support the study.

## Conflicts of Interest

The authors declare that they have no conflicts of interest.

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