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Research Article

A Note on q-Fubini-Appell Polynomials and Related Properties

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The present article is aimed at introducing and investigating a new class of q-hybrid special polynomials, namely, q-Fubini-Appell polynomials. The generating functions, series representations, and certain other significant relations and identities of this class are established. Some members of q-Fubini-Appell polynomial family are investigated, and some properties of these members are obtained. Further, the class of 3-variable q-Fubini-Appell polynomials is also introduced, and some formulae related to this class are obtained. In addition, the determinant representations for these classes are established.

1. Introduction

The q-calculus subject has gained prominence and numerous popularity during the last three decades or so (see [1–4]). The contemporaneous interest in this subject is due to the fact that q-series has popped in such diverse fields as quantum groups, statistical mechanics, and transcendental number theory. The notations and definitions related to q-calculus used in this article are taken from [2] (see also [5, 6]).

The q-analogues of a number $\ell \in \mathbb{C}$ and the factorial function are, respectively, specified by

$$[\ell]_q = \frac{1-q^{\ell}}{1-q}, \quad (q \in \mathbb{C} \setminus \{1\}), \tag{1}$$

and

$$[\kappa]_q! = \prod_{l=1}^{\kappa} [l]_q = [1]_q[2]_q[3]_q \cdots [\kappa]_q, \; [0]_q! = 1, \quad \kappa \in \mathbb{N}, q \in \mathbb{C} \setminus \{0, 1\}. \tag{2}$$

The q-binomial coefficient $\begin{bmatrix} \kappa \\ l \end{bmatrix}_q$ is specified by

$$\begin{bmatrix} \kappa \\ l \end{bmatrix}_q = \frac{[\kappa]_q!}{[l]_q![\kappa-l]_q!}, \quad l = 0, 1, 2, \cdots, \kappa; \kappa \in \mathbb{N}_0. \tag{3}$$

The q-analogue of $(u \oplus v)^{\kappa}$ is specified as

$$(u \oplus v)_q^{\kappa} = \sum_{l=0}^{\kappa} \begin{bmatrix} \kappa \\ l \end{bmatrix}_q q^{\binom{\kappa-l}{2}} u^l v^{\kappa-l}. \tag{4}$$

The q-derivative of a function f at a point $\tau \in \mathbb{C} \setminus \{0\}$ is given as

$$D_q f(\tau) = \frac{f(\tau) - f(q\tau)}{\tau - q\tau}, \quad 0 < |q| < 1. \tag{5}$$

The functions

$$e_q(\tau)=\sum_{\kappa=0}^{\infty}\frac{\tau^\kappa}{[\kappa]_q!},\quad 0<|q|<1, |\tau|<|1-q|^{-1}, \tag{6}$$

$$E_q(\tau)=\sum_{\kappa=0}^{\infty}q^{\binom{\kappa}{2}}\frac{\tau^\kappa}{[\kappa]_q!},\quad 0<|q|<1, \tau\in\mathbb{C}, \tag{7}$$

are called q-exponential functions and satisfy the following identities:

$$\begin{gathered}D_q e_q(\tau)=e_q(\tau),\quad D_q E_q(\tau)=E_q(q\tau),\\ e_q(\tau)E_q(-\tau)=E_q(\tau)e_q(-\tau)=1.\end{gathered} \tag{8}$$

The Fubini polynomials (FP) $\mathscr{F}_\kappa(w)$ [7] (also known as geometric polynomials) are defined as

$$\frac{1}{1-w(e^\tau-1)}=\sum_{\kappa=0}^{\infty}\mathscr{F}_\kappa(w)\frac{\tau^\kappa}{\kappa!}, \tag{9}$$

together with the geometric series

$$\frac{1}{1-w}\mathscr{F}_m\left(\frac{w}{1-w}\right)=\sum_{l=0}^{\infty}l^m w^l,\quad |w|<1. \tag{10}$$

Recently, Duran et al. [8] introduced the q-analogue of the FP $\mathscr{F}_\kappa(w)$, denoted by $\mathscr{F}_{\kappa,q}(w)$ and defined by means of the generating function

$$\frac{1}{1-w\left(e_q(\tau)-1\right)}=\sum_{\kappa=0}^{\infty}\mathscr{F}_{\kappa,q}(w)\frac{\tau^\kappa}{[\kappa]_q!}. \tag{11}$$

For $w=1$, the q-Fubini polynomials (q-FP) $\mathscr{F}_{\kappa,q}(w)$ reduce to the q-Fubini numbers $\mathscr{F}_{\kappa,q}(1):=\mathscr{F}_{\kappa,q}$, that is

$$\frac{1}{2-e_q(\tau)}=\sum_{\kappa=0}^{\infty}\mathscr{F}_{\kappa,q}\frac{\tau^\kappa}{[\kappa]_q!}. \tag{12}$$

Further, we recall the 3-variable q-Fubini polynomials (3Vq-FP) $\mathscr{F}_{\kappa,q}(u,v,w)$ [8] which are given as

$$\frac{1}{1-w\left(e_q(\tau)-1\right)}e_q(u\tau)E_q(v\tau)=\sum_{\kappa=0}^{\infty}\mathscr{F}_{\kappa,q}(u,v,w)\frac{\tau^\kappa}{[\kappa]_q!}. \tag{13}$$

Substantial properties of Fubini numbers and polynomials and their q-analogue have been studied and investigated by many researchers (see [7–9] and the references cited therein). Further, these numbers and polynomials have enormous applications in analytic number theory, physics, and the other related areas.

The class of the q-special polynomials such as q-Fubini polynomials, q-Appell polynomials, and certain members belonging to the family of q-Appell polynomials such as q-Bernoulli polynomials and q-Euler polynomials is an expanding field in mathematics [3, 7, 8, 10, 11].

The class of q-Appell polynomial sequences $\{\mathscr{A}_{\kappa,q}(w)\}_{\kappa=0}^{\infty}$ was established and investigated by Al-Salam [1]. These polynomials are defined by means of the generating function

$$\mathscr{A}_q(\tau)e_q(w\tau)=\sum_{\kappa=0}^{\infty}\mathscr{A}_{\kappa,q}(w)\frac{\tau^\kappa}{[\kappa]_q!}, \tag{14}$$

where

$$\mathscr{A}_q(\tau)=\sum_{\kappa=0}^{\infty}\mathscr{A}_{\kappa,q}\frac{\tau^\kappa}{[\kappa]_q!},\quad \mathscr{A}_q(\tau)\neq 0\,;\mathscr{A}_{0,q}=1, \tag{15}$$

is an analytic function at $\tau=0$ and $\mathscr{A}_{\kappa,q}:=\mathscr{A}_{\kappa,q}(0)$ denotes the q-Appell numbers.

Certain significant members belonging to q-Appell polynomials class are obtained based on suitable selection for the function $\mathscr{A}_q(\tau)$ as

(1) If $\mathscr{A}_q(\tau)=\tau/(e_q(\tau)-1)$, the q-AP $\mathscr{A}_{\kappa,q}(w)$ reduce to the q-Bernoulli polynomials (q-BP) $\mathfrak{B}_{\kappa,q}(w)$ (see [12, 13]), that is

$$\mathscr{A}_{\kappa,q}(w):=\mathfrak{B}_{\kappa,q}(w), \tag{16}$$

where $\mathfrak{B}_{\kappa,q}(w)$ are defined by

$$\frac{\tau}{e_q(\tau)-1}e_q(w\tau)=\sum_{\kappa=0}^{\infty}\mathfrak{B}_{\kappa,q}(w)\frac{\tau^\kappa}{[\kappa]_q!}, \tag{17}$$

and $\mathfrak{B}_{\kappa,q}$ given by

$$\mathfrak{B}_{\kappa,q}:=\mathfrak{B}_{\kappa,q}(0), \tag{18}$$

denotes the q-Bernoulli numbers.

(2) If $\mathscr{A}_q(\tau)=2/(e_q(\tau)+1)$, the q-AP $\mathscr{A}_{\kappa,q}(w)$ reduce to the q-Euler polynomials (q-EP) $\mathscr{E}_{\kappa,q}(w)$ (see [13, 14]), that is

$$\mathscr{A}_{\kappa,q}(w):=\mathscr{E}_{\kappa,q}(w), \tag{19}$$

where $\mathscr{E}_{\kappa,q}(w)$ are defined by

$$\frac{2}{e_q(\tau)+1}e_q(w\tau)=\sum_{\kappa=0}^{\infty}\mathscr{E}_{\kappa,q}(w)\frac{\tau^\kappa}{[\kappa]_q!}, \tag{20}$$

and $\mathscr{E}_{\kappa,q}$ given by

$$\mathscr{E}_{\kappa,q} := \mathscr{E}_{\kappa,q}(0), \tag{21}$$

denotes the q-Euler numbers.

Also, we recall the family of the numbers denoted by $\mathcal{S}_{2,q}(\kappa, l)$ and defined by

$$\frac{\left(e_q(\tau)-1\right)^l}{[l]_q!} = \sum_{\kappa=l}^{\infty} \mathcal{S}_{2,q}(\kappa, l)\frac{\tau^\kappa}{[\kappa]_q!}. \tag{22}$$

In recent years, many authors have shown their interest to introduce and study new families of q-special polynomials, especially the hybrid type (see [15–17] and the references therein).

The work in this article is summarized as follows: in Section 2, the replacement technique is used to introduce the class of q-Fubini-Appell polynomials by combining the polynomials, q-Fubini polynomials and q-Appell polynomials. In Section 3, the 3-variable q-Fubini-Appell polynomials are introduced which are considered as a generalization of the q-Fubini-Appell polynomials. The generating relations, series representations, and some other useful properties related to these polynomials are established. In Section 4, the determinant representations of these two classes are defined. Further, certain members belonging to these polynomial families are considered, and the corresponding results are also derived.

2. q-Fubini-Appell Polynomials

The q-Fubini-Appell polynomials are established by means of the generating function and series representation. To achieve this, we prove the following results:

Theorem 1. *The q -Fubini-Appell polynomials (q-FAP) ${}_{\mathscr{F}}\mathscr{A}_{\kappa,q}(w)$ are defined by means of the following generating function:*

$$\frac{\mathscr{A}_q(\tau)}{1-w\left(e_q(\tau)-1\right)} = \sum_{\kappa=0}^{\infty} {}_{\mathscr{F}}\mathscr{A}_{\kappa,q}(w)\frac{\tau^\kappa}{[\kappa]_q!}. \tag{23}$$

Proof. Utilizing equation (14), based on expanding the function $e_q(w\tau)$, then replacing the powers of w, i.e., $w^0, w, w^2, \cdots, w^\kappa$ by the corresponding polynomials $\mathscr{F}_{0,q}(w), \mathscr{F}_{1,q}(w), \cdots, \mathscr{F}_{\kappa,q}(w)$ and thereafter summing up the terms in the left-hand side of the resulting equation, we obtain that

$$\mathscr{A}_q(\tau)\sum_{\kappa=0}^{\infty}\mathscr{F}_{\kappa,q}(w)\frac{\tau^\kappa}{[\kappa]_q!} = \sum_{\kappa=0}^{\infty}\mathscr{A}_{\kappa,q}\left(\mathscr{F}_{1,q}(w)\right)\frac{\tau^\kappa}{[\kappa]_q!}. \tag{24}$$

Now, denoting the resultant q-FAP in the right hand side of the above equation by ${}_{\mathscr{F}}\mathscr{A}_{\kappa,q}(w)$ and utilizing equation (11) yield the assertion in equation (23).

Remark 2. Taking $w=1$, the q-FAP ${}_{\mathscr{F}}\mathscr{A}_{\kappa,q}(w)$ reduce to q -Fubini-Appell numbers (q-FAN) ${}_{\mathscr{F}}\mathscr{A}_{\kappa,q}$. Therefore, in view of equation (23), we have

$$\frac{\mathscr{A}_q(\tau)}{2-e_q(\tau)} = \sum_{\kappa=0}^{\infty} {}_{\mathscr{F}}\mathscr{A}_{\kappa,q}\frac{\tau^\kappa}{[\kappa]_q!}. \tag{25}$$

Corollary 3. *Taking $\mathscr{A}_q(\tau)=\tau/(e_q(\tau)-1)$ in equation (23), we get the following generating function of the q -Fubini-Bernoulli polynomials (q-FBP) ${}_{\mathscr{F}}\mathfrak{B}_{\kappa,q}(w)$.*

$$\frac{\tau}{\left(e_q(\tau)-1\right)\left(1-w\left(e_q(\tau)-1\right)\right)} = \sum_{\kappa=0}^{\infty} {}_{\mathscr{F}}\mathfrak{B}_{\kappa,q}(w)\frac{\tau^\kappa}{[\kappa]_q!}. \tag{26}$$

Corollary 4. *Taking $\mathscr{A}_q(\tau)=2/(e_q(\tau)+1)$ in equation (23), we get the following generating function of the q -Fubini-Euler polynomials (q-FEP) ${}_{\mathscr{F}}\mathscr{E}_{\kappa,q}(w)$*

$$\frac{2}{\left(e_q(\tau)+1\right)\left(1-w\left(e_q(\tau)-1\right)\right)} = \sum_{\kappa=0}^{\infty} {}_{\mathscr{F}}\mathscr{E}_{\kappa,q}(w)\frac{\tau^\kappa}{[\kappa]_q!}. \tag{27}$$

Theorem 5. *The following series representation for the q -FAP ${}_{\mathscr{F}}\mathscr{A}_{\kappa,q}(w)$ holds true:*

$${}_{\mathscr{F}}\mathscr{A}_{\kappa,q}(w) = \sum_{l=0}^{\kappa}\begin{bmatrix}\kappa\\ l\end{bmatrix}_q \mathscr{A}_{l,q}\mathscr{F}_{\kappa-l,q}(w). \tag{28}$$

Proof. In view of equations (11) and (15) and equation (23), we have

$$\begin{aligned}\sum_{\kappa=0}^{\infty} {}_{\mathscr{F}}\mathscr{A}_{\kappa,q}(w)\frac{\tau^\kappa}{[\kappa]_q!} &= \frac{\mathscr{A}_q(\tau)}{1-w\left(e_q(\tau)-1\right)}\\ &= \left(\sum_{\kappa=0}^{\infty}\mathscr{A}_{\kappa,q}\frac{\tau^\kappa}{[\kappa]_q!}\right)\left(\sum_{\kappa=0}^{\infty}\mathscr{F}_{\kappa,q}(w)\frac{\tau^\kappa}{[\kappa]_q!}\right)\\ &= \sum_{\kappa=0}^{\infty}\left(\sum_{l=0}^{\kappa}\begin{bmatrix}\kappa\\ l\end{bmatrix}_q \mathscr{A}_{l,q}\mathscr{F}_{\kappa-l,q}(w)\right)\frac{\tau^\kappa}{[\kappa]_q!},\end{aligned} \tag{29}$$

which on comparing the coefficients of $\tau^\kappa/[\kappa]_q!$ yield assertion in equation (28).

Theorem 6. *For $n\in\mathbb{N}_0$, the following series representation for the q-FAP ${}_{\mathscr{F}}\mathscr{A}_{\kappa,q}(w)$ holds true:*

$${}_{\mathscr{F}}\mathscr{A}_{\kappa,q}(w) = \sum_{l=0}^{\kappa}\sum_{\sigma=0}^{l}\begin{bmatrix}\kappa\\ l\end{bmatrix}_q [\sigma]_q! w^\sigma \mathscr{A}_{\kappa-l,q}\mathcal{S}_{2,q}(l,\sigma). \tag{30}$$

Proof. In view of equations (15), (22), and (23), we can write

$$\begin{aligned}\sum_{\kappa=0}^{\infty} {}_{\mathscr{F}}\mathscr{A}_{\kappa,q}(w)\frac{\tau^\kappa}{[\kappa]_q!} &= \frac{\mathscr{A}_q(\tau)}{1-w(e_q(\tau)-1)}\\ &= \sum_{\kappa=0}^{\infty}\mathscr{A}_{\kappa,q}\frac{\tau^\kappa}{[\kappa]_q!}\sum_{\sigma=0}^{\infty}w^\sigma(e_q(\tau)-1)^\sigma\\ &= \sum_{\kappa=0}^{\infty}\mathscr{A}_{\kappa,q}\frac{\tau^\kappa}{[\kappa]_q!}\sum_{\sigma=0}^{\infty}w^\sigma[\sigma]_q!\sum_{l=\sigma}^{\infty}\mathscr{S}_{2,q}(l,\sigma)\frac{\tau^l}{[l]_q!}\\ &= \sum_{\kappa=0}^{\infty}\mathscr{A}_{\kappa,q}\frac{\tau^\kappa}{[\kappa]_q!}\sum_{l=0}^{\infty}\left(\sum_{\sigma=0}^{l}w^\sigma[\sigma]_q!\mathscr{S}_{2,q}(l,\sigma)\right)\frac{\tau^l}{[l]_q!}\\ &= \sum_{\kappa=0}^{\infty}\left(\sum_{l=0}^{\kappa}\begin{bmatrix}\kappa\\ l\end{bmatrix}_q\mathscr{A}_{\kappa-l,q}\sum_{\sigma=0}^{l}w^\sigma[\sigma]_q!\mathscr{S}_{2,q}(l,\sigma)\right)\frac{\tau^\kappa}{[\kappa]_q!},\end{aligned} \tag{31}$$

which on comparing the coefficients of $\tau^\kappa/[\kappa]_q!$ yield assertion in equation (30).

Corollary 7. *Taking $\mathscr{A}_q(\tau)=\tau/(e_q(\tau)-1)$ in equations (28) and (30), we get the following series representations of the q-FBP ${}_{\mathscr{F}}\mathfrak{B}_{\kappa,q}(w)$*

$$\begin{aligned}{}_{\mathscr{F}}\mathfrak{B}_{\kappa,q}(w) &= \sum_{l=0}^{\kappa}\begin{bmatrix}\kappa\\ l\end{bmatrix}_q\mathfrak{B}_{l,q}\mathscr{F}_{\kappa-l,q}(w),\\ {}_{\mathscr{F}}\mathfrak{B}_{\kappa,q}(w) &= \sum_{l=0}^{\kappa}\sum_{\sigma=0}^{l}\begin{bmatrix}\kappa\\ l\end{bmatrix}_q[\sigma]_q!w^\sigma\mathfrak{B}_{\kappa-l,q}\mathscr{S}_{2,q}(l,\sigma), \quad n\in\mathbb{N}_0.\end{aligned} \tag{32}$$

Corollary 8. *Taking $\mathscr{A}_q(\tau)=2/(e_q(\tau)+1)$ in equations (28) and (30), we get the following series representations of the q-FEP ${}_{\mathscr{F}}\mathscr{E}_{\kappa,q}(w)$:*

$$\begin{aligned}{}_{\mathscr{F}}\mathscr{E}_{\kappa,q}(w) &= \sum_{l=0}^{\kappa}\begin{bmatrix}\kappa\\ l\end{bmatrix}_q\mathscr{E}_{l,q}\mathscr{F}_{\kappa-l,q}(w),\\ {}_{\mathscr{F}}\mathscr{E}_{\kappa,q}(w) &= \sum_{l=0}^{\kappa}\sum_{\sigma=0}^{l}\begin{bmatrix}\kappa\\ l\end{bmatrix}_q[\sigma]_q!w^\sigma\mathscr{E}_{\kappa-l,q}\mathscr{S}_{2,q}(l,\sigma), \quad n\in\mathbb{N}_0.\end{aligned} \tag{33}$$

Theorem 9. *The following formula for the q-FAP ${}_{\mathscr{F}}\mathscr{A}_{\kappa,q}(w)$ holds true:*

$$\frac{d}{dw}{}_{\mathscr{F}}\mathscr{A}_{\kappa,q}(w)=\sum_{l=0}^{\kappa}\begin{bmatrix}\kappa\\ l\end{bmatrix}_q\left({}_{\mathscr{F}}\mathscr{A}_{\kappa-l,q}(w)\mathscr{F}_{l,q}(1,0,w)-{}_{\mathscr{F}}\mathscr{A}_{l,q}(w)\mathscr{F}_{\kappa-l,q}(w)\right). \tag{34}$$

Proof. Utilizing equation (23), we have

$$\begin{aligned}\frac{d}{dw}\left(\sum_{\kappa=0}^{\infty}{}_{\mathscr{F}}\mathscr{A}_{\kappa,q}(w)\frac{\tau^\kappa}{[\kappa]_q!}\right) &= \frac{d}{dw}\left(\frac{\mathscr{A}_q(\tau)}{1-w(e_q(\tau)-1)}\frac{\mathscr{A}_q(\tau)}{1-w(e_q(\tau)-1)}\right)\\ &= \frac{\mathscr{A}_q(\tau)(e_q(\tau)-1)}{(1-w(e_q(\tau)-1))^2}\\ &= \frac{\mathscr{A}_q(\tau)e_q(\tau)}{(1-w(e_q(\tau)-1))^2}-\frac{\mathscr{A}_q(\tau)}{(1-w(e_q(\tau)-1))^2}\\ &= \sum_{\kappa=0}^{\infty}\sum_{l=0}^{\kappa}\begin{bmatrix}\kappa\\ l\end{bmatrix}_{q\mathscr{F}}\mathscr{A}_{\kappa-l,q}(w)\mathscr{F}_{l,q}(1,0,w)\frac{\tau^\kappa}{[\kappa]_q!}\\ &\quad-\sum_{\kappa=0}^{\infty}\sum_{l=0}^{\kappa}\begin{bmatrix}\kappa\\ l\end{bmatrix}_{q\mathscr{F}}\mathscr{A}_{l,q}(w)\mathscr{F}_{\kappa-l,q}(w)\frac{\tau^h}{[h]_q!},\end{aligned} \tag{35}$$

which on equating the coefficients of the like powers of τ yields the assertion in equation (34).

Corollary 10. *Taking $\mathscr{A}_q(\tau)=\tau/(e_q(\tau)-1)$ in equations (34), we get the formula satisfied by the q-FBP ${}_{\mathscr{F}}\mathfrak{B}_{\kappa,q}(w)$ as*

$$\frac{d}{dw}{}_{\mathscr{F}}\mathfrak{B}_{\kappa,q}(w)=\sum_{l=0}^{\kappa}\begin{bmatrix}\kappa\\ l\end{bmatrix}_q\left({}_{\mathscr{F}}\mathfrak{B}_{\kappa-l,q}(w)\mathscr{F}_{l,q}(1,0,w)-{}_{\mathscr{F}}\mathfrak{B}_{l,q}(w)\mathscr{F}_{\kappa-l,q}(w)\right). \tag{36}$$

Corollary 11. *Taking $\mathscr{A}_q(\tau)=2/e_q(\tau)+1$ in equations (34), we get the formula satisfied by the q-FEP ${}_{\mathscr{F}}\mathscr{E}_{\kappa,q}(w)$ as*

$$\frac{d}{dw}{}_{\mathscr{F}}\mathscr{E}_{\kappa,q}(w)=\sum_{l=0}^{\kappa}\begin{bmatrix}\kappa\\ l\end{bmatrix}_q\left({}_{\mathscr{F}}\mathscr{E}_{\kappa-l,q}(w)\mathscr{F}_{l,q}(1,0,w)-{}_{\mathscr{F}}\mathscr{E}_{l,q}(w)\mathscr{F}_{\kappa-l,q}(w)\right). \tag{37}$$

3. 3-Variable q-Fubini-Appell Polynomials

In this section, the class of 3-variable q-Fubini-Appell polynomials is established, which is a generalization of the class introduced in the previous section. The generating function, series representations, and other formulae for these polynomials are obtained.

Theorem 12. *The 3-variable q -Fubini-Appell polynomials (3Vq-FAP) ${}_{\mathscr{F}}\mathscr{A}_{\kappa,q}(u,v,w)$ are defined by means of the following generating function:*

$$\frac{\mathscr{A}_q(\tau)}{1-w(e_q(\tau)-1)}e_q(u\tau)E_q(v\tau)=\sum_{\kappa=0}^{\infty}{}_{\mathscr{F}}\mathscr{A}_{\kappa,q}(u,v,w)\frac{\tau^\kappa}{[\kappa]_q!}. \tag{38}$$

Proof. Utilizing equations (13) and (14) and following the same method as in the proof of Theorem 1, we can get the assertion in equation (38).

Remark 13. Setting $w=0$ in equation (38) gives the generating function of the 2-variable q-Appell polynomials (2Vq-AP) $\mathscr{A}_{\kappa,q}(u,v)$ [18], that is

$$\mathscr{A}_q(\tau)e_q(u\tau)E_q(v\tau) = \sum_{\kappa=0}^{\infty} \mathscr{A}_{\kappa,q}(u,v)\frac{\tau^\kappa}{[\kappa]_q!}. \tag{39}$$

Corollary 14. *Taking $\mathscr{A}_q(\tau) = \tau/(e_q(\tau)-1)$ in equation (38), we get the generating function of the 3-variable q-Fubini-Bernoulli polynomials (3Vq-FBP) ${}_{\mathscr{F}}\mathfrak{B}_{\kappa,q}(u,v,w)$ as*

$$\frac{\tau}{(e_q(\tau)-1)(1-w(e_q(\tau)-1))}e_q(u\tau)E_q(v\tau) = \sum_{\kappa=0}^{\infty} {}_{\mathscr{F}}\mathfrak{B}_{\kappa,q}(u,v,w)\frac{\tau^\kappa}{[\kappa]_q!}. \tag{40}$$

Corollary 15. *Taking $\mathscr{A}_q(\tau) = 2/(e_q(\tau)+1)$ in equation (38), we get the generating function of the 3-variable q-Fubini-Euler polynomials (3Vq-FEP) ${}_{\mathscr{F}}\mathscr{E}_{\kappa,q}(u,v,w)$ as*

$$\frac{2}{(e_q(\tau)+1)(1-w(e_q(\tau)-1))}e_q(u\tau)E_q(v\tau) = \sum_{\kappa=0}^{\infty} {}_{\mathscr{F}}\mathscr{E}_{\kappa,q}(u,v,w)\frac{\tau^\kappa}{[\kappa]_q!}. \tag{41}$$

Theorem 16. *The 3Vq-FAP ${}_{\mathscr{F}}\mathscr{A}_{\kappa,q}(u,v,w)$ are defined by the series*

$${}_{\mathscr{F}}\mathscr{A}_{\kappa,q}(u,v,w) = \sum_{l=0}^{\kappa} \begin{bmatrix} \kappa \\ l \end{bmatrix}_q \mathscr{A}_{l,q}\mathscr{F}_{\kappa-l,q}(u,v,w). \tag{42}$$

Proof. In view of equations (13), (15), and (38), we have

$$\begin{aligned}\sum_{\kappa=0}^{\infty} {}_{\mathscr{F}}\mathscr{A}_{\kappa,q}(u,v,w)\frac{\tau^\kappa}{[\kappa]_q!} &= \frac{\mathscr{A}_q(\tau)}{1-w(e_q(\tau)-1)}e_q(u\tau)E_q(v\tau)\\ &= \left(\sum_{\kappa=0}^{\infty}\mathscr{A}_{\kappa,q}\frac{\tau^\kappa}{[\kappa]_q!}\right)\left(\sum_{\kappa=0}^{\infty}\mathscr{F}_{\kappa,q}(u,v,w)\frac{\tau^\kappa}{[\kappa]_q!}\right)\\ &= \sum_{\kappa=0}^{\infty}\left(\sum_{l=0}^{\kappa}\begin{bmatrix} \kappa \\ l \end{bmatrix}_q \mathscr{A}_{l,q}\mathscr{F}_{\kappa-l,q}(u,v,w)\right)\frac{\tau^\kappa}{[\kappa]_q!},\end{aligned} \tag{43}$$

which on comparing the coefficients of $\tau^\kappa/[\kappa]_q!$ yield assertion in equation (42).

Corollary 17. *Taking $\mathscr{A}_q(\tau) = \tau/(e_q(\tau)-1)$ in equation (42), we get the series representation of the 33Vq-FBP ${}_{\mathscr{F}}\mathfrak{B}_{\kappa,q}(u,v,w)$ as*

$${}_{\mathscr{F}}\mathfrak{B}_{\kappa,q}(u,v,w) = \sum_{l=0}^{\kappa} \begin{bmatrix} \kappa \\ l \end{bmatrix}_q \mathfrak{B}_{l,q}\mathscr{F}_{\kappa-l,q}(u,v,w). \tag{44}$$

Corollary 18. *Taking $\mathscr{A}_q(\tau) = 2/(e_q(\tau)+1)$ in equation (42), we get the series representation of the 33Vq-FEP ${}_{\mathscr{F}}\mathscr{E}_{\kappa,q}(u,v,w)$ as*

$${}_{\mathscr{F}}\mathscr{E}_{\kappa,q}(u,v,w) = \sum_{l=0}^{\kappa} \begin{bmatrix} \kappa \\ l \end{bmatrix}_q \mathscr{E}_{l,q}\mathscr{F}_{\kappa-l,q}(u,v,w). \tag{45}$$

Suitably using equations (4), (6), (7), (11), and (23) in generating relation (38) and then making use of the Cauchy product rule in the resultant relations and thereafter comparing the identical powers of τ in both sides of the resultant expressions, we get the formulae given in the following theorem.

Theorem 19. *The 3Vq-FAP ${}_{\mathscr{F}}\mathscr{A}_{\kappa,q}(u,v,w)$ satisfy the following formulae*

$$\begin{aligned}{}_{\mathscr{F}}\mathscr{A}_{\kappa,q}(u,v,w) &= \sum_{l=0}^{\kappa}\begin{bmatrix} \kappa \\ l \end{bmatrix}_{q}{}_{\mathscr{F}}\mathscr{A}_{\kappa-l,q}(w)(u\oplus v)_q^l,\\ {}_{\mathscr{F}}\mathscr{A}_{\kappa,q}(u,v,w) &= \sum_{l=0}^{\kappa}\begin{bmatrix} \kappa \\ l \end{bmatrix}_q q^{\binom{\kappa-l}{2}}{}_{\mathscr{F}}\mathscr{A}_{l,q}(u,0,w)v^{\kappa-l},\\ {}_{\mathscr{F}}\mathscr{A}_{\kappa,q}(u,v,w) &= \sum_{l=0}^{\kappa}\begin{bmatrix} \kappa \\ l \end{bmatrix}_q \mathscr{F}_{l,q}(w){}_{\mathscr{F}}\mathscr{A}_{\kappa-l,q}(u,v,0),\\ {}_{\mathscr{F}}\mathscr{A}_{\kappa,q}(u,v,w) &= \sum_{l=0}^{\kappa}\begin{bmatrix} \kappa \\ l \end{bmatrix}_{q}{}_{\mathscr{F}}\mathscr{A}_{l,q}(0,v,w)u^{\kappa-l}.\end{aligned} \tag{46}$$

Applying the q-derivatives w.r.t. u and v to generating relation (38), we get the results given in the following theorem.

Theorem 20. *The following identities for the 3Vq-FAP ${}_{\mathscr{F}}\mathscr{A}_{\kappa,q}(u,v,w)$ hold true:*

$$\begin{aligned}D_{q,u}{}_{\mathscr{F}}\mathscr{A}_{\kappa,q}(u,v,w) &= [\kappa]_q{}_{\mathscr{F}}\mathscr{A}_{\kappa-1,q}(u,v,w),\\ D_{q,u}^{\xi}{}_{\mathscr{F}}\mathscr{A}_{\kappa,q}(u,v,w) &= \frac{[\kappa]_q!}{[\kappa-\xi]_q!}{}_{\mathscr{F}}\mathscr{A}_{\kappa-\xi,q}(u,v,w),\\ D_{q,v}{}_{\mathscr{F}}\mathscr{A}_{\kappa,q}(u,v,w) &= [\kappa]_q{}_{\mathscr{F}}\mathscr{A}_{\kappa-1,q}(u,qv,w),\\ D_{q,v}^{\xi}{}_{\mathscr{F}}\mathscr{A}_{\kappa,q}(u,v,w) &= \frac{[\kappa]_q!}{[\kappa-\xi]_q!}q^{\binom{\xi}{2}}{}_{\mathscr{F}}\mathscr{A}_{\kappa-\xi,q}\left(u,q^{\xi}v,w\right).\end{aligned} \tag{47}$$

Theorem 21. *The following relation for the 3Vq-FAP ${}_{\mathscr{F}}\mathscr{A}_{\kappa,q}(u,v,w)$ holds true:*

$$\sum_{l=0}^{\kappa}\begin{bmatrix} \kappa \\ l \end{bmatrix}_{q}{}_{\mathscr{F}}\mathscr{A}_{\kappa-l,q}(u,v,w) = \frac{1}{w}\left((w+1){}_{\mathscr{F}}\mathscr{A}_{\kappa,q}(u,v,w) - \mathscr{A}_{\kappa,q}(u,v)\right). \tag{48}$$

Proof. Consider the identity

$$w\frac{e_q(u\tau)E_q(v\tau)}{1-w(e_q(\tau)-1)}e_q(\tau) = (1+w)\frac{e_q(u\tau)E_q(v\tau)}{1-w(e_q(\tau)-1)} - e_q(u\tau)E_q(v\tau). \tag{49}$$

Now, multiplying both sides of the above identity by $\mathscr{A}_{\kappa,q}(\tau)$ and using equations (6), (38), and (39), we get

$$w\sum_{\kappa=0}^{\infty}\left(\sum_{l=0}^{\kappa}\begin{bmatrix}\kappa\\ l\end{bmatrix}_{q\mathcal{F}}\mathscr{A}_{\kappa-l,q}(u,v,w)\right)\frac{\tau^{\kappa}}{[\kappa]_q!} = \sum_{\kappa=0}^{\infty}\left((1+w)_{\mathcal{F}}\mathscr{A}_{\kappa,q}(u,v,w)-\mathscr{A}_{\kappa,q}(u,v)\right)\frac{\tau^{\kappa}}{[\kappa]_q!}, \tag{50}$$

which on equating the coefficients of the like powers of τ yields the assertion in equation (48).

Now, let us recall the generating function of the 2-variable q-generalized tangent polynomials (2Vq-GTP) $\mathscr{C}_{\kappa,\alpha,q}(u,v)$ [19] given as

$$\frac{2}{e_q(\alpha\tau)+1}e_q(u\tau)E_q(v\tau)=\sum_{\kappa=0}^{\infty}\mathscr{C}_{\kappa,\alpha,q}(u,v)\frac{\tau^{\kappa}}{[\kappa]_q!},\ |\alpha\tau|<\pi,\ \alpha\in\mathbb{R}^{+}, \tag{51}$$

and $\mathscr{C}_{\kappa,\alpha,q}:=\mathscr{C}_{\kappa,\alpha,q}(0,0)$ denotes the q-generalized tangent numbers (q-GTN).

Theorem 22. *The following relationships between the 3Vq-FAP $_{\mathcal{F}}\mathscr{A}_{\kappa,q}(u,v,w)$ and 2Vq-GTP $\mathscr{C}_{\kappa,\alpha,q}(u,v)$ holds true:*

$$_{\mathcal{F}}\mathscr{A}_{\kappa,q}(u,v,w)=\frac{1}{2}\sum_{l=0}^{\kappa}\begin{bmatrix}\kappa\\ l\end{bmatrix}_q \cdot\left(\sum_{\sigma=0}^{l}\begin{bmatrix}l\\ \sigma\end{bmatrix}_q\alpha^{\sigma}\mathscr{C}_{\kappa-l,\alpha,q}(u,v)\mathscr{F}_{l-\sigma,q}(w)+{}_{\mathcal{F}}\mathscr{A}_{\kappa-l,q}(u,v,w)\mathscr{C}_{l,\alpha,q}\right). \tag{52}$$

Proof. Utilizing equations (23), (38), and (51), we have

$$\begin{aligned}\sum_{\kappa=0}^{\infty}{}_{\mathcal{F}}\mathscr{A}_{\kappa,q}(u,v,w)\frac{\tau^{\kappa}}{[\kappa]_q!}&=\frac{\mathscr{A}_q(\tau)}{1-w(e_q(\tau)-1)}e_q(u\tau)E_q(v\tau)\\&=\frac{\mathscr{A}_q(\tau)}{1-w(e_q(\tau)-1)}\left(\frac{2}{e_q(\alpha\tau)+1}\right)\left(\frac{e_q(\alpha\tau)+1}{2}\right)e_q(u\tau)E_q(v\tau)\\&=\frac{1}{2}\left[\left(\sum_{\kappa=0}^{\infty}\left(\sum_{\sigma=0}^{\kappa}\begin{bmatrix}\kappa\\ \sigma\end{bmatrix}_q\alpha^{\sigma}\mathscr{F}_{\kappa-\sigma,q}(w)\right)\frac{\tau^{\kappa}}{[\kappa]_q!}\right)\right.\\&\quad\cdot\left(\sum_{\kappa=0}^{\infty}\mathscr{C}_{\kappa,\alpha,q}(u,v)\frac{\tau^{\kappa}}{[\kappa]_q!}\right)+\left(\sum_{\hat{e}=0}^{\infty}{}_{\mathcal{F}}\mathscr{A}_{\kappa,q}(u,v,w)\frac{\tau^{\kappa}}{[\kappa]_q!}\right)\\&\quad\left.\cdot\left(\sum_{\kappa=0}^{\infty}\mathscr{C}_{\kappa,\alpha,q}\frac{\tau^{\kappa}}{[\kappa]_q!}\right)\right]\\&=\frac{1}{2}\sum_{\kappa=0}^{\infty}\left[\sum_{l=0}^{\kappa}\begin{bmatrix}\kappa\\ l\end{bmatrix}_q\sum_{\sigma=0}^{l}\begin{bmatrix}\kappa\\ \sigma\end{bmatrix}_q\alpha^{\sigma}\mathscr{F}_{l-\sigma,q}(w)\mathscr{C}_{\kappa-l,\alpha,q}(u,v)+\sum_{l=0}^{\kappa}\right.\\&\quad\left.\cdot\begin{bmatrix}\kappa\\ l\end{bmatrix}_{q\mathcal{F}}\mathscr{A}_{\kappa-1,q}(u,v,w)\mathscr{C}_{l,\alpha,q}\right]\frac{\tau^{\kappa}}{[\kappa]_q!},\end{aligned} \tag{53}$$

which on comparing the coefficients of $\tau^{\kappa}/[\kappa]_q!$ yield assertion in equation (52).

Since for $\alpha=1$, the 2-variable q-generalized tangent polynomials (2Vq-GTP) $\mathscr{C}_{\kappa,\alpha,q}(u,v)$ reduce to 2-variable q-Euler polynomials $\mathscr{E}_{\kappa,q}(u,v)$ [20]. Therefore, setting $\alpha=1$ in equation (52) gives the following result.

Corollary 23. *The following relationships between the 3Vq-FAP $_{\mathcal{F}}\mathscr{A}_{\kappa,q}(u,v,w)$ and 2Vq-EP $\mathscr{E}_{\kappa,q}(u,v)$ holds true:*

$$_{\mathcal{F}}\mathscr{A}_{\kappa,q}(u,v,w)=\frac{1}{2}\sum_{l=0}^{\kappa}\begin{bmatrix}\kappa\\ l\end{bmatrix}_q\left(\sum_{\sigma=0}^{l}l\sigma_q\mathscr{E}_{\kappa-l,\alpha,q}(u,v)\mathscr{F}_{l-\sigma,q}(w)+{}_{\mathcal{F}}\mathscr{A}_{\kappa-l,q}(u,v,w)\mathscr{E}_{l,\alpha,q}\right). \tag{54}$$

Let us recall the generating function of the 2-variable q-Euler-Bernoulli polynomials (2Vq-EBP) $_{\mathscr{E}}\mathfrak{B}_{\kappa,q}(u,v)$ [16] given by

$$\frac{2\tau}{(e_q(\tau)+1)(e_q(\tau)-1)}e_q(u\tau)E_q(v\tau)=\sum_{\kappa=0}^{\infty}{}_{\mathscr{E}}\mathfrak{B}_{\kappa,q}(u,v)\frac{t^{\kappa}}{[\kappa]_q!}. \tag{55}$$

Theorem 24. *The following relationships between the 3Vq-FAP $_{\mathcal{F}}\mathscr{A}_{\kappa,q}(u,v,w)$ and 2Vq-EBP $_{\mathscr{E}}\mathfrak{B}_{\kappa,q}(u,v)$ holds true:*

$$_{\mathcal{F}}\mathscr{A}_{\kappa-1,q}(u,v,w)=\frac{1}{2[k]_q}\sum_{l=0}^{\kappa}\begin{bmatrix}\kappa\\ l\end{bmatrix}_q\left(\sum_{\sigma=0}^{\kappa}\sum_{h=0}^{l}\begin{bmatrix}\kappa-l\\ \sigma\end{bmatrix}_q\cdot\begin{bmatrix}l\\ h\end{bmatrix}_{q\mathcal{F}}\mathscr{A}_{\kappa-\sigma-l,q}(w)_{\mathscr{E}}\mathfrak{B}_{l-h,q}(u,v)-{}_{\mathcal{F}}\mathscr{A}_{\kappa-l,q}(u,v,w)_{\mathscr{E}}\mathfrak{B}_{l,q}\right). \tag{56}$$

Proof. Utilizing equations (6), (38), and (55), we have

$$\begin{aligned}\sum_{\kappa=0}^{\infty}{}_{\mathcal{F}}\mathscr{A}_{\kappa,q}(u,v,w)\frac{\tau^{\kappa}}{[\kappa]_q!}&=\frac{\mathscr{A}_q(\tau)}{1-w(e_q(\tau)-1)}e_q(u\tau)E_q(v\tau)\\&=\frac{\mathscr{A}_q(\tau)}{1-w(e_q(\tau)-1)}\left(\frac{2t}{(e_q(\tau)+1)(e_q(\tau)-1)}\right)\\&\quad\cdot\left(\frac{(e_q(\tau)+1)(e_q(\tau)-1)}{2t}\right)e_q(u\tau)E_q(v\tau)\\&=\frac{1}{2t}\left[\left(\frac{\mathscr{A}_q(\tau)}{1-w(e_q(\tau)-1)}e_q(\tau)\right)\left(\frac{2te_q(u\tau)E_q(v\tau)}{(e_q(\tau)+1)(e_q(\tau)-1)}e_q(\tau)\right)\right.\\&\quad\left.-\left(\frac{\mathscr{A}_q(\tau)e_q(u\tau)E_q(v\tau)}{1-w(e_q(\tau)-1)}\right)\left(\frac{2t}{(e_q(\tau)+1)(e_q(\tau)-1)}\right)\right]\\&=\frac{1}{2t}\left[\left(\sum_{\kappa=0}^{\infty}\sum_{\sigma=0}^{\kappa}\begin{bmatrix}\kappa\\ \sigma\end{bmatrix}_{q\mathcal{F}}\mathscr{A}_{\kappa-\sigma,q}(w)\frac{\tau^{\kappa}}{[\kappa]_q!}\right)\right.\\&\quad\cdot\left(\sum_{l=0}^{\infty}\sum_{h=0}^{l}\begin{bmatrix}l\\ h\end{bmatrix}_q\varepsilon\mathfrak{B}_{l-h,q}(u,v)\frac{\tau^{l}}{[l]_q!}\right)\\&\quad\left.-\sum_{\kappa=0}^{\infty}\sum_{l=0}^{\kappa}\begin{bmatrix}\kappa\\ l\end{bmatrix}_{q\mathcal{F}}\mathscr{A}_{\kappa-l,q}(u,v,w)_{\mathscr{E}}\mathfrak{B}_{l,q}\frac{\tau^{\kappa}}{[\kappa]_q!}\right]\\&=\frac{1}{2t}\sum_{\kappa=0}^{\infty}\left[\sum_{l=0}^{\kappa}\sum_{\sigma=0}^{\kappa}\sum_{h=0}^{l}\begin{bmatrix}\kappa\\ l\end{bmatrix}_q\begin{bmatrix}\kappa-l\\ \sigma\end{bmatrix}_q\right.\\&\quad\cdot\begin{bmatrix}l\\ h\end{bmatrix}_{q\mathcal{F}}\mathscr{A}_{\kappa-l,q}(w)_{\mathscr{E}}\mathfrak{B}_{l-h,q}(u,v)-\sum_{l=0}^{\kappa}\\&\quad\left.\cdot\begin{bmatrix}\kappa\\ l\end{bmatrix}_{q\mathcal{F}}\mathscr{A}_{\kappa-l,q}(u,v,w)_{\mathscr{E}}\mathfrak{B}_{l,q}\right]\frac{\tau^{\kappa}}{[\kappa]_q!},\end{aligned} \tag{57}$$

which on comparing the coefficients of $\tau^{\kappa}/[\kappa]_q!$ yield assertion in equation (56).

Theorem 25. *The following relationships between the 3Vq-FAP ${}_{\mathscr{F}}\mathscr{A}_{\kappa,q}(u,v,w)$ and 2Vq-AP $\mathscr{A}_{\kappa,q}(u,v)$ holds true:*

$$\sum_{l=0}^{\kappa}\begin{bmatrix}\kappa\\ l\end{bmatrix}_{q\mathscr{F}}\mathscr{A}_{\kappa-l,q}\left(u,v,\frac{w}{1-w}\right)=\frac{1}{w}\left({}_{\mathscr{F}}\mathscr{A}_{\kappa,q}\left(u,v,\frac{w}{1-w}\right)-(1-w)\mathscr{A}_{\kappa,q}(u,v)\right). \tag{58}$$

Proof. Replacing w by $w/(1-w)$ in generating relation (38), we have

$$\sum_{\kappa=0}^{\infty}{}_{\mathscr{F}}\mathscr{A}_{\kappa,q}\left(u,v,\frac{w}{1-w}\right)\frac{\tau^{\kappa}}{[\kappa]_q!}=\frac{\mathscr{A}_q(\tau)}{1-(w/(1-w))(e_q(\tau)-1)}e_q(u\tau)E_q(v\tau). \tag{59}$$

Rewriting the above equation then using equations (38) and (39), we obtain

$$\sum_{\kappa=0}^{\infty}{}_{\mathscr{F}}\mathscr{A}_{\kappa,q}\left(u,v,\frac{w}{1-w}\right)\frac{\tau^{\kappa}}{[\kappa]_q!}-w\sum_{\kappa=0}^{\infty}\sum_{l=0}^{\kappa}\begin{bmatrix}\kappa\\ l\end{bmatrix}_{q\mathscr{F}}\mathscr{A}_{\kappa-l,q}\left(u,v,\frac{w}{1-w}\right)\frac{\tau^{\kappa}}{[\kappa]_q!}=(1-w)\sum_{\kappa=0}^{\infty}\mathscr{A}_{\kappa,q}(u,v)\frac{\tau^{\kappa}}{[\kappa]_q!}. \tag{60}$$

which on comparing the coefficients of $\tau^{\kappa}/[\kappa]_q!$ yield assertion in equation (58).

4. Determinant Representations

One of the significant representations of the q-special polynomials is the determinant representation due to its importance for the computational and applied purposes. In 2015, Keleshteri and Mahmudov [18] established the determinant representation of the q-Appell polynomials. In the section, the determinant representations of the q-FAP ${}_{\mathscr{F}}\mathscr{A}_{\kappa,q}(w)$ and the 3Vq-FAP ${}_{\mathscr{F}}\mathscr{A}_{\kappa,q}(u,v,w)$ are introduced.

Definition 26. The determinant representation for the q-FAP ${}_{\mathscr{F}}\mathscr{A}_{\kappa,q}(w)$ of degree κ is given as

$${}_{\mathscr{F}}\mathscr{A}_{0,q}(w)=\frac{1}{\mathscr{B}_{0,q}}, \tag{61}$$

$${}_{\mathscr{F}}\mathscr{A}_{\kappa,q}(w)\begin{bmatrix}1 & \mathscr{F}_{1,q} & \mathscr{F}_{2,q}(w) & \cdots & \mathscr{F}_{\kappa-1,q}(w) & \mathscr{F}_{\kappa,q}(w)\\ \mathscr{B}_{0,q} & \mathscr{B}_{1,q} & \mathscr{B}_{2,q} & \cdots & \mathscr{B}_{\kappa-1,q} & \mathscr{B}_{\kappa,q}\\ 0 & \mathscr{B}_{0,q} & \begin{bmatrix}2\\1\end{bmatrix}_q\mathscr{B}_{1,q} & \cdots & \begin{bmatrix}\kappa-1\\1\end{bmatrix}_q\mathscr{B}_{\kappa-2,q} & \begin{bmatrix}\kappa\\1\end{bmatrix}_q\mathscr{B}_{\kappa-1,q}\\ 0 & 0 & \mathscr{B}_{0,q} & \cdots & \begin{bmatrix}\kappa-1\\2\end{bmatrix}_q\mathscr{B}_{\kappa-3,q} & \begin{bmatrix}\kappa\\2\end{bmatrix}_q\mathscr{B}_{\kappa-2,q}\\ . & . & . & \cdots & . & .\\ . & . & . & \cdots & . & .\\ 0 & 0 & 0 & \cdots & \mathscr{B}_{0,q} & \begin{bmatrix}\kappa\\\kappa-1\end{bmatrix}_q\mathscr{B}_{1,q}\end{bmatrix}, \tag{62}$$

$$\mathscr{B}_{\kappa,q}=-\frac{1}{{}_{\mathscr{F}}\mathscr{A}_{0,q}}\left(\sum_{\nu=1}^{\kappa}\begin{bmatrix}\kappa\\ \nu\end{bmatrix}_{\mathscr{F}}\mathscr{A}_{\nu,q}\mathscr{B}_{\kappa-\nu,q}\right),\quad \mathscr{B}_{0,q}\neq 0,\ \kappa=1,2,3,\cdots. \tag{63}$$

Setting $\mathscr{B}_{0,q}=1$ and $\mathscr{B}_{\delta,q}=(1/[\delta+1]_q)(\delta=1,2,\cdots,\kappa)$ in equations (61) and (62) gives the determinant representation of the q-FBP ${}_{\mathscr{F}}\mathfrak{B}_{\kappa,q}(w)$ as:

Definition 27. The determinant representation for the q -FBP ${}_{\mathscr{F}}\mathfrak{B}_{\kappa,q}(w)$ of degree κ is given as

$${}_{\mathscr{F}}\mathfrak{B}_{0,q}(w)=1,$$

$${}_{\mathscr{F}}\mathfrak{B}_{\kappa,q}(w)=(-1)^{\kappa}\cdot\begin{vmatrix}1 & \mathscr{F}_{1,q}(w) & \mathscr{F}_{2,q}(w) & \cdots & \mathscr{F}_{\kappa-1,q}(w) & \mathscr{F}_{\kappa,q}(w)\\ 1 & \frac{1}{[2]_q} & \frac{1}{[3]_q} & \cdots & \frac{1}{[\kappa]_q} & \frac{1}{[\kappa+1]_q}\\ 0 & 1 & \begin{bmatrix}2\\1\end{bmatrix}_q\frac{1}{[2]_q} & \cdots & \begin{bmatrix}\kappa-1\\1\end{bmatrix}_q\frac{1}{[\kappa-1]_q} & \begin{bmatrix}\kappa\\1\end{bmatrix}_q\frac{1}{[\kappa]_q}\\ 0 & 0 & 1 & \cdots & \begin{bmatrix}\kappa-1\\2\end{bmatrix}_q\frac{1}{[\kappa-2]_q} & \begin{bmatrix}\kappa\\2\end{bmatrix}_q\frac{1}{[\kappa-1]_q}\\ . & . & . & \cdots & . & .\\ . & . & . & \cdots & . & .\\ 0 & 0 & 0 & \cdots & 1 & \begin{bmatrix}\kappa\\\kappa-1\end{bmatrix}_q\frac{1}{[2]_q}\end{vmatrix},\quad \kappa=1,2\cdots. \tag{64}$$

Setting $\mathscr{B}_{0,q}=1$ and $\mathscr{B}_{\delta,q}=(1/2)(\delta=1,2,\cdots,\kappa)$ in equations (61) and (62) gives the determinant representation of the q-FEP ${}_{\mathscr{F}}\mathscr{E}_{\kappa,q}(w)$ as:

Definition 28. The determinant representation for the q -FEP ${}_{\mathscr{F}}\mathscr{E}_{\kappa,q}(w)$ of degree κ is given as

$$
\begin{aligned}
&{}_{\mathscr{F}}\varepsilon_{0,q}(w)=1,\\
&{}_{\mathscr{F}}\varepsilon_{\kappa,q}(w)=(-1)^{\kappa}\cdot\begin{vmatrix}
1 & \mathscr{F}_{1,q}(w) & \mathscr{F}_{2,q}(w) & \cdots & \mathscr{F}_{\kappa-1,q}(w) & \mathscr{F}_{\kappa,q}(w)\\
1 & \frac{1}{2} & \frac{1}{2} & \cdots & \frac{1}{2} & \frac{1}{2}\\
0 & 1 & \begin{bmatrix}2\\1\end{bmatrix}_q\frac{1}{2} & \cdots & \begin{bmatrix}\kappa-1\\1\end{bmatrix}_q\frac{1}{2} & \begin{bmatrix}\kappa\\1\end{bmatrix}_q\frac{1}{2}\\
0 & 0 & 1 & \cdots & \begin{bmatrix}\kappa-1\\2\end{bmatrix}_q\frac{1}{2} & \begin{bmatrix}\kappa\\2\end{bmatrix}_q\frac{1}{2}\\
. & . & . & \cdots & . & .\\
. & . & . & \cdots & . & .\\
0 & 0 & 0 & \cdots & 1 & \begin{bmatrix}\kappa\\\kappa-1\end{bmatrix}_q\frac{1}{2}
\end{vmatrix},\quad \kappa=1,2,3,\cdots.
\end{aligned}
\tag{65}
$$

Similarly, the determinant representation of the 3Vq-FAP ${}_{\mathscr{F}}\mathscr{A}_{\kappa,q}(u,v,w)$, 3Vq-FBP ${}_{\mathscr{F}}\mathfrak{B}_{\kappa,q}(u,v,w)$, and 3Vq-FEP ${}_{\mathscr{F}}\mathscr{E}_{\kappa,q}(u,v,w)$ are established as:

Definition 29. The determinant representation for the 3Vq-FAP ${}_{\mathscr{F}}\mathscr{A}_{\kappa,q}(u,v,w)$ of degree κ is given as

$$
\begin{aligned}
&{}_{\mathscr{F}}\mathscr{A}_{0,q}(u,v,w)=\frac{1}{\mathscr{B}_{0,q}},\\
&{}_{\mathscr{F}}\mathscr{A}_{\kappa,q}(u,v,w)=\frac{(-1)^{\kappa}}{(\mathscr{B}_{0,q})^{\kappa+1}}\cdot\begin{vmatrix}
1 & \mathscr{F}_{1,q}(u,v,w) & \mathscr{F}_{2,q}(u,v,w) & \cdots & \mathscr{F}_{\kappa-1,q}(u,v,w) & \mathscr{F}_{\kappa,q}(u,v,w)\\
\mathscr{B}_{0,q} & \mathscr{B}_{1,q} & \mathscr{B}_{2,q} & \cdots & \mathscr{B}_{\kappa-1,q} & \mathscr{B}_{\kappa,q}\\
0 & \mathscr{B}_{0,q} & \begin{bmatrix}2\\1\end{bmatrix}_q\mathscr{B}_{1,q} & \cdots & \begin{bmatrix}\kappa-1\\1\end{bmatrix}_q\mathscr{B}_{\kappa-2,q} & \begin{bmatrix}\kappa\\1\end{bmatrix}_q\mathscr{B}_{\kappa-1,q}\\
0 & 0 & \mathscr{B}_{0,q} & \cdots & \begin{bmatrix}\kappa-1\\2\end{bmatrix}_q\mathscr{B}_{\kappa-3,q} & \begin{bmatrix}\kappa\\2\end{bmatrix}_q\mathscr{B}_{\kappa-2,q}\\
. & . & . & \cdots & . & .\\
. & . & . & \cdots & . & .\\
0 & 0 & 0 & \cdots & \mathscr{B}_{0,q} & \begin{bmatrix}\kappa\\\kappa-1\end{bmatrix}\mathscr{B}_{1,q}
\end{vmatrix},\\
&\mathscr{B}_{\kappa,q}=-\frac{1}{{}_{\mathscr{F}}\mathscr{A}_{0,q}}\left(\sum_{v=1}^{\kappa}\begin{bmatrix}\kappa\\v\end{bmatrix}_{q\mathscr{F}}\mathscr{A}_{v,q}\mathscr{B}_{\kappa-v,q}\right),\ \mathscr{B}_{0,q}\neq 0,\ \kappa=1,2,3,\cdots.
\end{aligned}
\tag{66}
$$

Definition 30. The determinant representation for the 3Vq-FBP ${}_{\mathscr{F}}\mathfrak{B}_{\kappa,q}(u,v,w)$ of degree κ is given as

$$
\begin{aligned}
&{}_{\mathscr{F}}\mathfrak{B}_{0,q}(u,v,w)=1,\\
&{}_{\mathscr{F}}\mathfrak{B}_{0,q}(u,v,w)=(-1)^{\kappa}\cdot\begin{vmatrix}
1 & \mathscr{F}_{1,q}(u,v,w) & \mathscr{F}_{2,q}(u,v,w) & \cdots & \mathscr{F}_{\kappa-1,q}(u,v,w) & \mathscr{F}_{\kappa,q}(u,v,w)\\
1 & & & \cdots & \frac{1}{[\kappa]_q} & \frac{1}{[\kappa+1]_q}\\
0 & 1 & \begin{bmatrix}2\\1\end{bmatrix}_q\frac{1}{[2]_q} & \cdots & \begin{bmatrix}\kappa-1\\1\end{bmatrix}_q\frac{1}{[\kappa-1]_q} & \begin{bmatrix}\kappa\\1\end{bmatrix}_q\frac{1}{[\kappa]_q}\\
0 & 0 & 1 & \cdots & \begin{bmatrix}\kappa-1\\2\end{bmatrix}_q\frac{1}{[\kappa-2]_q} & \begin{bmatrix}\kappa\\2\end{bmatrix}_q\frac{1}{[\kappa-2]_q}\\
. & . & . & \cdots & . & .\\
. & . & . & \cdots & . & .\\
0 & 0 & 0 & \cdots & 1 & \begin{bmatrix}\kappa\\\kappa-1\end{bmatrix}_q\frac{1}{[2]_q}
\end{vmatrix},\\
&\kappa=1,2\cdots.
\end{aligned}
\tag{67}
$$

Definition 31. The determinant representation for the 3Vq-FEP ${}_{\mathscr{F}}\mathscr{E}_{\kappa,q}(u,v,w)$ of degree κ is given as

$$
\begin{aligned}
&{}_{\mathscr{F}}\mathscr{E}_{0,q}(u,v,w)=1,\\
&{}_{\mathscr{F}}\varepsilon_{\kappa,q}(u,v,w)=(-1)^{\kappa}\cdot\begin{vmatrix}
1 & \mathscr{F}_{1,q}(u,v,w) & \mathscr{F}_{2,q}(u,v,w) & \cdots & \mathscr{F}_{\kappa-1,q}(u,v,w) & \mathscr{F}_{\kappa,q}(u,v,w)\\
1 & \frac{1}{2} & \frac{1}{2} & \cdots & \frac{1}{2} & \frac{1}{2}\\
0 & 1 & \begin{bmatrix}2\\1\end{bmatrix}_q\frac{1}{2} & \cdots & \begin{bmatrix}\kappa-1\\1\end{bmatrix}_q\frac{1}{2} & \begin{bmatrix}\kappa\\1\end{bmatrix}_q\frac{1}{2}\\
0 & 0 & 1 & \cdots & \begin{bmatrix}\kappa-1\\2\end{bmatrix}_q\frac{1}{2} & \begin{bmatrix}\kappa\\2\end{bmatrix}_q\frac{1}{2}\\
. & . & . & \cdots & . & .\\
. & . & . & \cdots & . & .\\
0 & 0 & 0 & \cdots & 1 & \begin{bmatrix}\kappa\\\kappa-1\end{bmatrix}_q\frac{1}{2}
\end{vmatrix},\\
&\kappa=1,2,3\cdots.
\end{aligned}
\tag{68}
$$

5. Conclusions

Recently, the Fubini polynomials and their q-analogue have been studied and investigated by many researchers. Motivated by various recent studies related to these type of polynomials (see for example [8, 21, 22]), in this article, we introduced two important families of q-hybrid special polynomials, namely, the q-Fubini-Appell polynomials and 3-variable q-Fubini-Appell polynomials. Certain properties related to these families are derived.

Further investigations along the results obtained in this article, which are associated with many recent generalizations and extensions of the q-Appell polynomial family, especially, the parametric types, may be worthy of consideration in future investigations.

Data Availability

There is no data availability in this manuscript.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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