Research Article

Analytical Properties of the Generalized Heat Matrix Polynomials Associated with Fractional Calculus

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In this paper, we introduce a matrix version of the generalized heat polynomials. Some analytic properties of the generalized heat matrix polynomials are obtained including generating matrix functions, finite sums, and Laplace integral transforms. In addition, further properties are investigated using fractional calculus operators.

1. Overture

In the past few decades, matrix versions of the orthogonal polynomials have attracted a lot of research interest due to their close relations and various applications in many areas of mathematics, statistics, physics, and engineering, for example, see [1–11].

The recent advances of fractional order calculus (FOC) are dominated by its multidisciplinary applications. Moreover, special functions of fractional order calculus have many applications in various areas of mathematical analysis, probability theory, control systems, and engineering (see, for example, [12–15]).

Moreover, the development of fractional calculus associated with special matrix functions and polynomials has been investigated by many researchers, for example, the recent works [16–22].

Among these classical polynomials are the heat polynomials (also designated as Temperature polynomials) that are polynomial solutions of the heat equation and also are particularly useful in solving the Cauchy problem (see [23–26]). Special functions, such as the confluent hypergeometric function, integral error functions, and Laguerre polynomials, have a close link with the generalized heat polynomials intro-

duced [27–29]. Further, the generalized heat polynomials are mainly used to construct an approximate solution of a given problem as a linear combination of the polynomials. Such solution satisfies the governing equation and other equations (cf., e.g., [30–35]).

In our investigation here, we define a generalized heat matrix polynomial $\mathbb{HP}_m(T; \xi, \nu)$. We then establish certain generating matrix functions, finite sum formulas, Laplace transforms, and fractional calculus operators for these polynomials in Sections 3, 4, 5, and 6, respectively. Further, some interesting special cases and concluding remarks of our main results are pointed out in Section 7.

2. Preliminaries

In this section, we give some basic definitions and terminologies; for more details, we can be referred to [36, 37].

Here and through the work, let $\mathbb{C}^{d \times d}$ be the vector space of all the square matrices of order $d \in \mathbb{N}$, ($\mathbb{N}$ is the set of all positive integers) whose entries are in the set of complex numbers $\mathbb{C}$. For a $E \in \mathbb{C}^{d \times d}$, let $\sigma(E)$ be the set of all eigenvalues of E which is called the spectrum of E . We have

$$\mu(E) := \max \{ \Re(\xi) : \xi \in \sigma(E) \} \text{ and } \tilde{\mu}(E) := \min \{ \Re(z) : z \in \sigma(E) \}, \quad (1)$$

which imply $\tilde{\mu}(E) = -\mu(-E)$. Here, $\mu(E)$ is called the spectral abscissa of E , and the matrix E is said to be positive stable if $\tilde{\mu}(E) > 0$. Further, let I and 0 denote the identity and zero matrices corresponding to a square matrix of any order, respectively.

If E is a positive stable matrix in $\mathbb{C}^{d \times d}$, then the gamma matrix function $\Gamma(E)$ is well defined as follows (cf., e.g., [11, 38, 39]):

$$\Gamma(E) = \int_0^\infty e^{-z} z^{E-I} dz, \quad z^{E-I} := \exp((E-I) \ln z). \quad (2)$$

Moreover, if E is a matrix in $\mathbb{C}^{d \times d}$ which gratifies

$$E + nI \text{ is invertible for every integer } n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \quad (3)$$

then $\Gamma(E)$ is invertible, its inverse coincides with $\Gamma^{-1}(E)$. Under condition (3), we can write the following Pochhammer matrix symbol

$$(E)_n = E(E+I) \cdots (E+(n-1)I) = \Gamma(E+nI) \Gamma^{-1}(E) (n \in \mathbb{N}). \quad (4)$$

Let $r, k \in \mathbb{N}_0$. Also let $(S)_r$ and $(Q)_k$ be arrays of r commutative matrices $S_1, S_2, \dots, S_r$ and k commutative matrices $Q_1, Q_2, \dots, Q_k$ in $\mathbb{C}^{d \times d}$, respectively, such that $Q_k + nI$ are invertible for $1 \leq d \leq k$ and all $n \in \mathbb{N}_0$. Then, the generalized hypergeometric matrix function ${}_rF_k((S)_r; (Q)_k; z) (z \in \mathbb{C})$ can be defined by (see, e.g., [11, 39])

$${}_rF_k((S)_r; (Q)_k; z) = \sum_{s=0}^{\infty} \prod_{j=1}^r (S_j)_s \prod_{i=1}^k [(Q_i)_s]^{-1} \frac{z^s}{s!}. \quad (5)$$

In particular, the hypergeometric matrix function ${}_2F_1(S, P; C; z) \equiv F(S, P; C; z)$ is defined by

$$F(R, P; C; z) = \sum_{s=0}^{\infty} (R)_s (P)_s (C)_s^{-1} \frac{z^s}{s!}, \quad (6)$$

for matrices R, P, C in $\mathbb{C}^{d \times d}$ such that $C + nI$ is invertible for all $n \in \mathbb{N}_0$. Also, note that for $r = 1, k = 0$ in (9), we have the Binomial type matrix function ${}_1F_0(R; -; z)$ as follows:

$${}_1F_0(R; -; z) = (1-z)^{-R} = I + Rz + \frac{R(R+I)z^2}{2!} + \cdots + \frac{(R)_n z^n}{n!} + \cdots, |z| < 1. \quad (7)$$

Let E be a positive stable invertible matrix in $\mathbb{C}^{d \times d}$. Then, the n^{th} Laguerre matrix polynomial is defined in the form (see, e.g., [11, 40])

$$\begin{aligned} \mathbf{L}_n^E(z) &= \sum_{m=0}^n \frac{(-1)^m z^m}{m!(n-m)!} (S+I)_n (E+I)_m^{-1} \\ &= \frac{(E+I)_n}{n!} {}_1F_1(-nI; E+I; z), \quad n \in \mathbb{N}_0. \end{aligned} \quad (8)$$

The Laplace transform of $f(\xi)$ is defined by [7].

$$\mathcal{L}\{f(\xi); \lambda\} = \int_0^\infty e^{-\lambda\xi} f(\xi) d\xi, \quad \Re(\lambda) > 0, \quad (9)$$

provided that the improper integral exists.

Lemma 1. (see [7]). Let S be a positive stable invertible matrix in $\mathbb{C}^{d \times d}$. Then, we have

$$\mathcal{L}\{\xi^S; \lambda\} = \lambda^{-(S+I)} \Gamma(S+I), \quad \Re(\lambda) > 0. \quad (10)$$

3. Generalized Heat Matrix Polynomial and Generating Matrix Functions

A generalized heat matrix polynomial is defined in (11) below; then, a family of generating matrix functions are proposed, see Theorem 4 and Theorem 8 of this section.

Definition 2. Let T be a positive stable matrix in the complex space $\mathbb{C}^{d \times d}$ satisfying the spectral condition (3). Then, we define a generalized heat matrix polynomial of degree $m \in \mathbb{N}_0$ in the following explicit form:

$$\begin{aligned} \mathbb{HP}_m(T; \xi, \nu) &= \sum_{s=0}^m 2^{2s} \binom{m}{s} \Gamma\left(T + \left(m + \frac{1}{2}\right)I\right) \Gamma^{-1} \\ &\quad \cdot \left(T + \left(m - s + \frac{1}{2}\right)I\right) \xi^{2m-2s} \nu^s \\ &= m!(4\nu)^m \mathbf{L}_m^{T-\frac{1}{2}I}\left(\frac{-\xi^2}{4\nu}\right); \quad \nu > 0, \end{aligned} \quad (11)$$

where $\mathbf{L}_m^T(\xi)$ is the Laguerre matrix polynomial in (8).

Remark 3. Note that

$$\mathbb{HP}_m(T; \xi, \nu) = (4\nu)^m \left(T + \frac{1}{2}I\right)_m {}_1F_1\left(-mI, T + \frac{1}{2}I; \frac{-\xi^2}{4\nu}\right), \quad (12)$$

and that for the scalar case $d = 1$, taking $T = a$ and $a > 0$, the m^{th} polynomial $\mathbb{HP}_m(a; \xi, \nu)$ coincides with the classical scalar generalized heat polynomial, see [24, 26, 33]. Further, the ordinary heat polynomial defined in [23], when $T = 0$; $\mathbb{HP}_m(0; \xi, \nu) = v_{2m}(\xi, \nu)$.

Theorem 4. Let $\xi \in \mathbb{C}$, $\nu > 0$, $m \in \mathbb{N}_0$, $|\xi^2 t| < |1 - 4\nu t|$ and T and R be positive stable matrices in $\mathbb{C}^{d \times d}$ such that $T + nI$ is

invertible for all $n \in \mathbb{N}_0$. A generating matrix function of $\mathbb{H}$ $\mathbb{P}_m(T; \xi, \nu)$ is

$$\begin{aligned} & \sum_{m=0}^{\infty} \left[\left(T + \frac{1}{2}I \right)_m \right]^{-1} (R)_m \mathbb{HP}_m(T; \xi, \nu) \frac{t^m}{m!} \\ &= (1 - 4\nu t)^{-R} {}_1F_1 \left(R, T + \frac{1}{2}I; \frac{\xi^2 t}{1 - 4\nu t} \right). \end{aligned} \quad (13)$$

Proof. For convenience, suppose that the left-hand side of (13) is denoted by $\mathbf{J}$. According to the series expression of (11) and (7) to $\mathbf{J}$, we find that

$$\begin{aligned} \mathbf{J} &= \sum_{m=0}^{\infty} \left[\left(T + \frac{1}{2}I \right)_m \right]^{-1} (R)_m \mathbb{HP}_m(T; \xi, \nu) \frac{t^m}{m!} \\ &= \sum_{m=0}^{\infty} \sum_{s=0}^m \frac{(R)_m [(T + 1/2I)_{m-s}]^{-1}}{s!(m-k)!} (\xi)^{2m-2s} (4\nu)^s t^m \\ &= \sum_{m=0}^{\infty} (R)_m \left[\left(T + \frac{1}{2}I \right)_m \right]^{-1} \frac{(\xi^2 t)^m}{m!} \cdot \sum_{s=0}^{\infty} (R + mI)_s \frac{(\xi \nu t)^s}{s!} \\ &= (1 - 4\nu t)^{-R} \sum_{m=0}^{\infty} (R)_m \left[\left(T + \frac{1}{2}I \right)_m \right]^{-1} \frac{(\xi^2 t/1 - 4\nu t)^m}{m!}. \end{aligned} \quad (14)$$

Upon using the relation (6), the last equality evidently leads us to the required result.

Corollary 5. For $\mathbb{HP}_m(T; \xi, \nu)$, the following generating matrix function holds true

$$\begin{aligned} & \sum_{m=0}^{\infty} \left[\left(T + \frac{1}{2}I \right)_m \right]^{-1} \mathbb{HP}_m(T; \xi, \nu) \frac{t^m}{m!} \\ &= \exp(4\nu t) {}_0F_1 \left(-; T + \frac{1}{2}I; \xi^2 t \right). \end{aligned} \quad (15)$$

Remark 6. The Bessel matrix function $J_R(z)$, for a positive stable matrix $R \in C^{d \times d}$, is expressible in terms of hypergeometric matrix function as follows (see, e.g., [11, 41])

$$J_R(w) = \left(\frac{w}{2} \right)^R \Gamma^{-1}(R+I) {}_0F_1 \left(-; R+I, \frac{-w^2}{4} \right). \quad (16)$$

Thus, by applying the relation (16) to (15) in Corollary 5, we can deduce the following Corollary.

Corollary 7. For $\mathbb{HP}_m(T; \xi, \nu)$, the following holds true

$$\begin{aligned} & \sum_{m=0}^{\infty} \left[\left(T + \frac{1}{2}I \right)_m \right]^{-1} \mathbb{HP}_m(T; \xi, \nu) \frac{t^m}{m!} = \Gamma \left(\left(T + \frac{1}{2}I \right) \right) \\ & \cdot \left(i\xi\sqrt{t} \right)^{-(T-\frac{1}{2}I)} \exp(4\nu t) J_{T-\frac{1}{2}I} \left(2i\xi\sqrt{t} \right). \end{aligned} \quad (17)$$

Theorem 8. Let $\xi \in \mathbb{C}$, $\nu > 0$, $m, l \in \mathbb{N}_0$ and T be a positive stable matrix in $C^{d \times d}$ such that $T + nI$ is invertible for all $n \in \mathbb{N}_0$. The following relation holds true

$$\begin{aligned} & \sum_{m=0}^{\infty} \mathbb{HP}_{m+l}(T; \xi, \nu) \frac{t^m}{m!} = (1 - 4\nu t)^{-(T+(l+\frac{1}{2})I)} \exp \\ & \cdot \left(\frac{\xi^2 t}{1 - 4\nu t} \right) \mathbb{HP}_{m+l} \left(T; \frac{\xi}{\sqrt{1 - 4\nu t}}, \nu \right). \end{aligned} \quad (18)$$

Proof. Follows by induction or by the successive application of Theorem 4 when $R = (T + 1/2I)$. The details are omitted.

Corollary 9. For $l = 0$ in Theorem 8, the following holds true

$$\sum_{m=0}^{\infty} \mathbb{HP}_m(T; \xi, \nu) \frac{t^m}{m!} = (1 - 4\nu t)^{-(T+\frac{1}{2}I)} \exp \left(\frac{\xi^2 t}{1 - 4\nu t} \right). \quad (19)$$

Remark 10. The special cases of (18) and (19) when $d = 1$ are seen to yield the classical generating functions of the generalized Heat polynomials (see [24, 33]).

4. Finite Sums

Here, various finite sums of $\mathbb{HP}_m(T; \xi, \nu)$ can be obtained in the following results.

Theorem 11. Let $\xi, z \in \mathbb{C}$, $\nu > 0$, $m \in \mathbb{N}_0$, and T be a positive stable matrix in $C^{d \times d}$ such that $T + nI$ is invertible for all $n \in \mathbb{N}_0$. Then, we have

$$\begin{aligned} \mathbb{HP}_m(T; \xi z, \nu) &= \sum_{s=0}^m \binom{m}{s} \left(T + \frac{1}{2}I \right)_m \left[\left(T + \frac{1}{2}I \right)_s \right]^{-1} \\ & \cdot \{ 4\nu(1 - z^2) \}^{m-s} z^{2s} \mathbb{HP}_s(T; \xi z, \nu). \end{aligned} \quad (20)$$

Proof. From (15) and the following fact

$$e^{4\nu t} {}_0F_1 \left(; T + \frac{1}{2}I; \xi^2 z^2 t \right) = e^{4\nu t(1-z^2)+4\nu z^2 t} {}_0F_1 \left(; T + \frac{1}{2}I; \xi^2 z^2 t \right). \quad (21)$$

We thus find that

$$\begin{aligned} & \sum_{m=0}^{\infty} \sum_{s=0}^{\infty} \mathbb{HP}_m(T; \xi, \nu) \left[\left(T + \frac{1}{2}I \right)_s \right]^{-1} \frac{\{ 4\nu(1 - z^2) \}^m (z^2 t)^s}{m!s!} \\ &= \sum_{m=0}^{\infty} \sum_{s=0}^m \frac{\{ 4\nu(1 - z^2) \}^{m-s} (z^2 t)^s}{(m-s)!s!} \left[\left(T + \frac{1}{2}I \right)_s \right]^{-1} \mathbb{HP}_m(T; \xi, \nu). \end{aligned} \quad (22)$$

Comparing the coefficient of t^m on both side, we thus get the required a finite sum formula (20).

Theorem 12. Let $\xi \in \mathbb{C}$, $\nu > 0$, $m \in \mathbb{N}_0$, also let T and R be positive stable matrices in $C^{d \times d}$ such that $T + nI$ and $R + nI$ are invertible for all $n \in \mathbb{N}_0$. Then, we have

$$\begin{aligned} \mathbb{HP}_m(T; \xi, \nu) &= \left(T + \frac{1}{2}I\right)_m \left[(R)_m\right]^{-1} \sum_{s=0}^m \binom{n}{s} \left(T - R + \frac{1}{2}I\right)_s \\ &\quad \cdot \left[\left(T + \frac{1}{2}I\right)_s\right]^{-1} \times \mathbb{HP}_s(T; i\xi, \nu) \mathbb{HP}_{m-s} \\ &\quad \cdot (2R - T - I; \xi, \nu). \end{aligned} \quad (23)$$

Proof. By using the series (11) and Theorem 4 with applying to Kummer's matrix formula (see [5]), we observe that

$$\begin{aligned} &\sum_{m=0}^{\infty} \left[\left(T + \frac{1}{2}I\right)_m\right]^{-1} (R)_m \mathbb{HP}_m(T; \xi, \nu) \frac{t^m}{m!} \\ &= (1 - 4\nu t)^{-R} \exp\left(\frac{\xi^2 t}{1 - 4\nu t}\right) {}_1F_1\left(T - R + \frac{1}{2}I, T + \frac{1}{2}I; \frac{-\xi^2 t}{1 - 4\nu t}\right) \\ &= (1 - 4\nu t)^{-(2R - T - I) + \frac{1}{2}I} \exp\left(\frac{\xi^2 t}{1 - 4\nu t}\right) (1 - 4\nu t)^{-(T - R + \frac{1}{2}I)} {}_1F_1 \\ &\quad \cdot \left(T - R + \frac{1}{2}I, T + \frac{1}{2}I; \frac{-\xi^2 t}{1 - 4\nu t}\right) \\ &= \sum_{m=0}^{\infty} \frac{\mathbb{HP}_m(2R - T - I; \xi, \nu)}{m!} t^m \sum_{s=0}^{\infty} \left(T - R + \frac{1}{2}I\right)_s \\ &\quad \cdot \left[\left(T + \frac{1}{2}I\right)_s\right]^{-1} \frac{\mathbb{HP}_m(T; i\xi, \nu)}{m!} t^s. \end{aligned} \quad (24)$$

Equating the coefficient of t^m on both sides, we thus arrive at the desired result (23).

5. Laplace Transforms

Here, Laplace integral transforms of the generalized heat matrix polynomials are derived as follows.

Theorem 13. Let $\xi \in \mathbb{C}$, $\nu > 0$, $\Re(\lambda) > 0$, $m \in \mathbb{N}_0$. Also, let T and A be a positive stable matrices in $C^{d \times d}$ such that $T + nI$ is invertible for all $n \in \mathbb{N}_0$. The following Laplace transform formula hold

$$\begin{aligned} \mathcal{L}\{u^A \mathbb{HP}_m(T; \xi\sqrt{u}, \nu): \lambda\} &= (4\nu)^m \left(T + \frac{1}{2}I\right)_m \lambda^{-(A+I)} \Gamma \\ &\quad \cdot (A + I)_2 {}_2F_1\left(-mI, A + I; T + \frac{1}{2}I; \frac{-\xi^2}{4\nu\lambda}\right). \end{aligned} \quad (25)$$

Proof. Making a particular use of (9) with (11), (6) and applying to Lemma 1, yields our desired result (25) in Theorem 13. The details are omitted.

A similar procedure yields the following Laplace integral transforms. So we prefer to omit the proofs.

Theorem 14. Let $\xi \in \mathbb{C}$, $\nu > 0$, $\Re(\lambda) > 0$, $m \in \mathbb{N}_0$. Also, let T and A be a positive stable matrices in $C^{d \times d}$ such that $T + nI$ is invertible for all $n \in \mathbb{N}_0$. The following Laplace transform formula hold

$$\begin{aligned} \mathcal{L}\{u^A \mathbb{HP}_m(T; \xi u, \nu): \lambda\} &= (4\nu)^m \left(T + \frac{1}{2}I\right)_m \lambda^{-(A+I)} \Gamma(A + I)_3 {}_3F_1 \\ &\quad \cdot \left(-mI, \frac{1}{2}(A + I), \frac{1}{2}(A + 2I); T + \frac{1}{2}I; \frac{-\xi^2}{\nu\lambda^2}\right). \end{aligned} \quad (26)$$

Theorem 15. Let $\xi \in \mathbb{C}$, $\nu > 0$, $\Re(\lambda) > 0$, $m \in \mathbb{N}_0$. Also, let T and A be a positive stable matrices in $C^{d \times d}$ such that $T + nI$ is invertible for all $n \in \mathbb{N}_0$. The following Laplace transform formula hold

$$\begin{aligned} \mathcal{L}\{u^A \mathbb{HP}_m(T; u, \nu): \lambda\} &= (4\nu)^m \left(T + \frac{1}{2}I\right)_m \lambda^{-(A+I)} \Gamma(A + I)_3 {}_3F_1 \\ &\quad \cdot \left(-mI, \frac{1}{2}(A + I), \frac{1}{2}(A + 2I); T + \frac{1}{2}I; \frac{-1}{\nu\lambda^2}\right). \end{aligned} \quad (27)$$

The above Theorems lead to the following special cases.

Corollary 16. For T is a positive stable matrix in $C^{d \times d}$, and $\Re(\lambda) > 0$, then we have the following Laplace transforms

$$\begin{aligned} \mathcal{L}\{u^{T-\frac{1}{2}I} \mathbb{HP}_m(T; \xi\sqrt{u}, \nu): \lambda\} &= (4\nu)^m \lambda^{-(A+\frac{1}{2}I)} \Gamma\left(A + \left(m + \frac{1}{2}\right)I\right) \left(1 + \frac{\xi^2}{4\nu\lambda}\right)^m, \end{aligned}$$

$$\begin{aligned} \mathcal{L}\{u^{2T-I} \mathbb{HP}_m(T; \xi u, \nu): \lambda\} &= (4\nu)^m \left(T + \frac{1}{2}I\right)_m \lambda^{-2T} \Gamma(2T)_2 {}_2F_0\left(-mI, T; -; \frac{-\xi^2}{\nu\lambda^2}\right), \end{aligned}$$

$$\begin{aligned} \mathcal{L}\{u^{2T-I} \mathbb{HP}_m(T; u, \nu): \lambda\} &= (4\nu)^m \left(T + \frac{1}{2}I\right)_m \lambda^{-(2T)} \Gamma(2T)_3 {}_3F_1\left(-mI, T; -; \frac{-1}{\nu\lambda^2}\right). \end{aligned} \quad (28)$$

6. Fractional Calculus Approach

Here, we consider the Riemann. Liouville fractional integral and derivative operators I^γ and D_z^γ of order $\gamma \in \mathbb{C}$, $\Re(\gamma) > 0$, respectively (see, for details, [19])

$$(I^\gamma f(\tau))(z) = \frac{1}{\Gamma(\gamma)} \int_0^z f(\tau)(z - \tau)^{\gamma-1} d\tau, \quad (29)$$

where $f(\tau)$ is a function of τ and some square matrices so that this integral converges.

$$D_z^\gamma \{f(z)\} = \begin{cases} \frac{1}{\Gamma(-\gamma)} \int_0^z (z-\tau)^{-\gamma-1} f(\tau) d\tau, & (\Re(\gamma) < 0), \\ \frac{d^n}{dz^n} (D_z^{\gamma-n} \{f(z)\}), & (n-1 \leq \Re(\gamma) < n \ (n \in \mathbb{N})), \end{cases} \quad (30)$$

where

$$D_z^\gamma \{z^\beta\} = \frac{\Gamma(\beta+1)}{\Gamma(\beta-(\gamma-1)I)} z^{\beta-\gamma} \Re(\beta) > -1. \quad (31)$$

It is noted in passing that (29) and (30) are applied in recent works, for example, (see [16–22]).

Theorem 17. For the generalized heat matrix polynomials of degree $m \in \mathbb{N}_0$, the following fractional integral operator holds true:

$$\begin{aligned} \mathbf{I}^\gamma \left\{ \xi^{T-\frac{1}{2}I} \mathbb{HP}_m(T; \sqrt{\xi}, \nu) \right\} &= \Gamma \left(T + \left(m + \frac{1}{2} \right) I \right) \Gamma^{-1} \\ &\cdot \left(T + \left(\gamma + m + \frac{1}{2} \right) I \right) \\ &\times \xi^{T+(\gamma-\frac{1}{2})I} \mathbb{HP}_m(T + \gamma I; \sqrt{\xi}, \nu). \end{aligned} \quad (32)$$

Proof. From (29) and (11), we have

$$\begin{aligned} \mathbf{I}^\gamma \left\{ \xi^{T-\frac{1}{2}I} \mathbb{HP}_m(T; \sqrt{\xi}, \nu) \right\} &= \frac{1}{\Gamma(\gamma)} \int_0^\xi (\xi-\tau)^{\gamma-1} \tau^{T-\frac{1}{2}I} \mathbb{HP}_m \\ &\cdot (T; \sqrt{\tau}, \nu) d\tau. \end{aligned} \quad (33)$$

Setting $\tau = \xi\eta$ and after a simplification, we get

$$\begin{aligned} \mathbf{I}^\gamma \left\{ \xi^{T-\frac{1}{2}I} \mathbb{HP}_m(T; \sqrt{\xi}, \nu) \right\} &= \frac{\xi^{T+(\gamma-1/2)I}}{\Gamma(\gamma)} \int_0^1 \eta^{T-\frac{1}{2}I} (1-\eta)^{\gamma-1} \mathbb{HP}_m(T; \sqrt{\xi\eta}, \nu) d\eta \\ &= \xi^{T+(\gamma-\frac{1}{2})I} (4\nu)^m \left(T + \frac{1}{2}I \right)_m \sum_{s=0}^m (-mI)_s \left[\left(T + \frac{1}{2}I \right)_s \right]^{-1} \Gamma \\ &\cdot \left(T + \left(s + \frac{1}{2} \right) I \right) \times \Gamma^{-1} \left(T + \left(s + \gamma + \frac{1}{2} \right) I \right) \frac{\left(-(\sqrt{\xi})^2 / 4\nu \right)^s}{s!}, \end{aligned} \quad (34)$$

whose last summation, in view of (11), is easily seen to arrive at the expression in (32). This completes the proof of Theorem 17.

Theorem 18. For the generalized heat matrix polynomials $\mathbb{HP}_m(T; \xi, \nu)$, we have the following formula

$$\begin{aligned} \sum_{m=0}^{\infty} \left[\left(T + \frac{1}{2}I \right)_m \right]^{-1} \frac{(\beta)_m}{(\gamma)_m} \mathbb{HP}_m(T; \xi, \nu) \frac{t^m}{m!} {}_1F_1(\gamma - \beta; \gamma + m; 4\nu t) \\ = \exp(4\nu t) {}_1F_1 \left(\beta I; T + \frac{1}{2}I; \xi^2 t \right), \Re(\beta) > 0, \Re(\gamma) > 0. \end{aligned} \quad (35)$$

Proof. According to (15), we find that

$$\begin{aligned} \exp(-4\nu t) \sum_{m=0}^{\infty} \left[\left(T + \frac{1}{2}I \right)_m \right]^{-1} \mathbb{HP}_m(T; \xi, \nu) \frac{t^m}{m!} \\ = {}_0F_1 \left(-; T + \frac{1}{2}I; \xi^2 t \right). \end{aligned} \quad (36)$$

Assume that the left-hand side of (35) be denoted by J. Multiply the left-hand side (35) by $t^{\beta-1}$ and applying the relation (31), we get

$$\begin{aligned} J &= \frac{\Gamma(\beta)}{\Gamma(\gamma)} t^{\gamma-1} \sum_{m=0}^{\infty} \sum_{s=0}^{\infty} \left[\left(T + \frac{1}{2}I \right)_m \right]^{-1} \mathbb{HP}_m \\ &\cdot (T; \xi, \nu) (4\nu)^s \frac{(-1)^s (\beta)_{m+s} t^{m+s}}{(\gamma)_{m+s} s! m!} \\ &= \frac{\Gamma(\beta)}{\Gamma(\gamma)} t^{\gamma-1} \sum_{m=0}^{\infty} \left[\left(T + \frac{1}{2}I \right)_m \right]^{-1} \mathbb{HP}_m(T; \xi, \nu) \frac{(\beta)_m t^m}{(\gamma)_m m!} {}_1F_1 \\ &\cdot \left(\beta + m; \gamma + \frac{1}{2}; -4\nu t \right). \end{aligned} \quad (37)$$

Invoking Kummer's matrix transformation [5] leads to

$$\begin{aligned} J &= \frac{\Gamma(\beta)}{\Gamma(\gamma)} t^{\gamma-1} \exp(-4\nu t) \sum_{m=0}^{\infty} \left[\left(T + \frac{1}{2}I \right)_m \right]^{-1} \mathbb{HP}_m \\ &\cdot (T; \xi, \nu) \frac{(\beta)_m t^m}{(\gamma)_m m!} {}_1F_1(\gamma - \beta; \gamma + m; 4\nu t). \end{aligned} \quad (38)$$

It is easy to multiply right-hand side of (35) by $t^{\beta-1}$ and applying the fractional differentiation operator $D_t^{\beta-\gamma}$ from (31), whereupon this completes the establishment of the Theorem 18.

7. Concluding Remarks

In [23], Rosenbloom and Widder investigated expansions of solutions $u(x, t)$ of the heat equation $u_{xx} = u_t$ in series of polynomial solutions $v_n(x, t)$. Further, Haimo and Markett [33, 34] discussed the generalized heat equation

$$\Delta_x u(x, t) = \frac{\partial}{\partial t} u(x, t), \quad (39)$$

where $\Delta_x h(x) = h''(x) + (2\nu/x)h'(x)$, ν is a fixed positive number and they introduced the generalized heat polynomial solution of (39) as

$$\mathbb{P}_{n,v}(x, t) = \sum_{j=0}^n 2^{2j} \binom{n}{j} \frac{\Gamma(v + 1/2 + n)}{\Gamma(v + 1/2 + n - j)} x^{2n-2j} t^j. \quad (40)$$

Note that $\mathbb{P}_{n,0}(x, t) = v_{2n}(x, t)$ is the ordinary heat polynomials of even order. Also, $\mathbb{P}_{n,0}(x, -1) = H_{2n}(x/2)$ is the Hermite polynomials of even order.

Recently, many studies and extensions of the well-known special matrix polynomials have been in a focus of increasing attention leading to new and interesting problems. In this perspective, we defined the generalized heat matrix polynomials. Also, we have given some of their main properties, namely, the generating matrix functions, finite sum formulas, Laplace integral transforms, and fractional calculus operators. Further, This study is assumed to be a generalization of the scalar cases [27, 33, 34] to the matrix setting. In addition, this approach allows to derive several new integral and differential representations that can be used in theoretical, applicable aspects like the boundary value problems and the numerical algorithms. Additional research and application on this topic is now under preparation and will be presented in forthcoming works.

Data Availability

No data were used to support the study.

Conflicts of Interest

This work does not have any conflicts of interest.

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