Hindawi
Journal of Function Spaces
Volume 2021, Article ID 4103772, 7 pages
https://doi.org/10.1155/2021/4103772

Review Article

Fourth Toeplitz Determinants for Starlike Functions Defined by Using the Sine Function

Hai-Yan Zhang and Huo Tang

School of Mathematics and Computer Sciences, Chifeng University, Chifeng 024000, China

Correspondence should be addressed to Huo Tang; thth2009@163.com

Received 27 April 2021; Accepted 11 August 2021; Published 30 August 2021

Academic Editor: Umair Ali

In this article, we aim to study the upper bounds of the fourth Toeplitz determinant $T_4(2)$ for the function class $\mathcal{S}_s^*$, which are connected with the sine function.

1. Introduction

Suppose that $\mathcal{A}$ represents the class of analytic functions f which in the open unit disk $\mathbb{D} = \{z : |z|<1\}$ of the form

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots (z \in \mathbb{D}), \tag{1}$$

and suppose that $\mathcal{S}$ is the subclass of $\mathcal{A}$ consisting of univalent functions.

Let $\mathcal{P}$ denotes the class of analytic functions p normalized by

$$p(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots, \tag{2}$$

and meeting the condition $\mathfrak{R}(p(z)) > 0 (z \in \mathbb{D})$. Let f and g be analytic functions in $\mathbb{D}$. Then, we say that the function g is subordinate to the function f, and we write

$$g(z) \prec f(z) (z \in \mathbb{D}), \tag{3}$$

if there exists a Schwarz function $\omega(z)$ with $\omega(0) = 0$ and $|\omega(z)| < 1$, such that (see [1])

$$g(z) = f(\omega(z)) (z \in \mathbb{D}). \tag{4}$$

In 2018, Cho et al. [2] introduced the following function class S_s^*:

$$S_s^* := \left\{ f \in \mathcal{A} : \frac{z f'(z)}{f(z)} \prec (1 + \sin z)(z \in \mathbb{D}) \right\}, \tag{5}$$

which means that the quantity $zf'(z)/f(z)$ lies in an eight-shaped region in the right-half plane.

Thomas and Halim [3] defined the symmetric Toeplitz determinant $T_q(n)$ as follows:

$$T_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_n & \cdots & a_{n+q-1} \\ \vdots & & \vdots & \vdots \\ a_{n+q-1} & a_{n+q+2} & \cdots & a_n \end{vmatrix} (n \geq 1, q \geq 1). \tag{6}$$

As a special case, we have

$$T_4(2) = \begin{vmatrix} a_2 & a_3 & a_4 & a_5 \\ a_3 & a_2 & a_3 & a_4 \\ a_4 & a_3 & a_2 & a_3 \\ a_5 & a_4 & a_3 & a_2 \end{vmatrix} (n = 2, q = 4). \tag{7}$$

That is,

$$T_4(2) = \left(a_2^2 - a_3^2\right)^2 + 2\left(a_3^2 - a_2a_4\right)\left(a_2a_4 - a_3a_5\right) - \left(a_2a_3 - a_3a_4\right)^2 + \left(a_4^2 - a_3a_5\right)^2 - \left(a_3a_4 - a_2a_5\right)^2. \tag{8}$$

Many and many researchers have studied several Hankel and Toeplitz determinants for various classes of functions. For example, Janteng et al. [4, 5] investigated second Hankel determinant for a function with a positive real part derivative and starlike and convex functions, respectively; Bansal [6] and Lee et al. [7] discussed the second Hankel determinant for certain analytic functions; Bansal et al. [8], Zaprawa [9], Zhang et al. [10] and Babalola [11] derived third-order Hankel determinant for certain different univalent functions; Raza et al. [12] and Shi et al. [13, 14] studied upper bounds of the third Hankel determinant for some classes of analytic functions related to lemniscate of Bernoulli, cardioid domain and exponential function; Mahmood et al. [15] found third Hankel determinant for a subclass of q-starlike functions. Following the above work, Zhang et al. [16] recently considered fourth-order Hankel determinants of starlike functions related to the sine function. On the other hand, Thomas et al. [3] and Ali et al. [17] studied Toeplitz matrices whose elements are the coefficients of starlike, close-to-convex, and univalent functions. Besides, Tang et al. [18] studied third-order Hankel and Toeplitz determinant for a subclass of multivalent q-starlike functions of order α; Zhang et al. [19] considered third-order Hankel and Toeplitz determinants of starlike functions, which are defined by using the sine function; Ramachandran et al. [20] derived an estimation for the Hankel and Topelitz determinant with domains bounded by conical sections involving Ruscheweygh derivative; Srivastava et al. [21] found the Hankel determinant and the Toeplitz matrices for this newly-defined class of analytic q-starlike functions. Based on the work of Shi et al. [14], Zhang and Tang [16], Thomas and Halim [3], and Ali et al. [17], in the present paper, we aim to investigate the fourth-order Toeplitz determinant $T_4(2)$ for this function class $\mathcal{S}_s^*$ associated with sine function and obtain the upper bounds for the determinants $T_4(2)$.

2. Main Results

Due to prove our desired results, we require the following lemmas.

Lemma 1 (see [22]). *If $p(z) \in \mathcal{P}$, then exists some x, z with $|x| \le 1, |z| \le 1$, such that*

$$2c_2 = c_1^2 + x\left(4 - c_1^2\right), \tag{9}$$

$$4c_3 = c_1^3 + 2c_1x\left(4 - c_1^2\right) - \left(4 - c_1^2\right)c_1x^2 + 2\left(4 - c_1^2\right)\left(1 - |x|^2\right)z. \tag{10}$$

Lemma 2 (see [23]). *Let $p(z) \in \mathcal{P}$, then*

$$|c_1^4 + c_2^2 + 2c_1c_3 - 3c_1^2c_2 - c_4| \le 2\,; \tag{11}$$

$$|c_1^5 + 3c_1c_2^2 + 3c_1^2c_3 - 4c_1^3c_2 - 2c_1c_4 - 2c_2c_3 + c_5| \le 2\,; \tag{12}$$

$$|c_1^6 + 6c_1^2c_2^2 + 4c_1^3c_3 + 2c_1c_5 + 2c_2c_4 + c_3^2 - c_2^3 - 5c_1^4c_2 - 3c_1^2c_4 - 6c_1c_2c_3 - c_6| \le 2\,; \tag{13}$$

$$|c_n| \le 2, n = 1, 2, \cdots. \tag{14}$$

Lemma 3 (see [24]). *Let $p(z) \in \mathcal{P}$, then, we have*

$$|c_2 - \frac{c_1^2}{2}| \le 2 - \frac{|c_1|^2}{2}\,; \tag{15}$$

$$|c_{n+k} - \mu c_n c_k| < 2, 0 \le \mu \le 1\,; \tag{16}$$

$$|c_{n+2k} - \mu c_n c_k^2| \le 2(1 + 2\mu). \tag{17}$$

The following are the main conclusions of this paper and related proof.

Theorem 1. *Suppose that $f(z) \in \mathcal{S}_s^*$ and of the form (1), then*

$$|a_2| \le 1, |a_3| \le \frac{1}{2}, |a_4| \le 0.344, |a_5| \le \frac{3}{8}, |a_6| \le \frac{67}{120}, |a_7| \le \frac{5587}{10800}. \tag{18}$$

Proof. Because $f(z) \in \mathcal{S}_s^*$, by the definition of subordination, so there exists a Schwarz function $\omega(z)$ with $\omega(0) = 0$ and $|\omega(z)| < 1$, such that

$$\frac{zf'(z)}{f(z)} = 1 + \sin(\omega(z)). \tag{19}$$

□

Now

$$\begin{aligned}\frac{zf'(z)}{f(z)} &= \frac{z + \sum_{n=2}^{\infty} na_nz^n}{z + \sum_{n=2}^{\infty} a_nz^n} = \left(1 + \sum_{n=2}^{\infty} na_nz^{n-1}\right) \\ &\cdot \left[1 - a_2z + \left(a_2^2 - a_3\right)z^2 - \left(a_2^3 - 2a_2a_3 + a_4\right)z^3 + \left(a_2^4 - 3a_2^2a_3 + 2a_2a_4 - a_5\right)z^4 + \cdots\right] \\ &= 1 + a_2z + \left(2a_3 - a_2^2\right)z^2 + \left(a_2^3 - 3a_2a_3 + 3a_4\right)z^3 \\ &+ \left(4a_5 - a_2^4 + 4a_2^2a_3 - 4a_2a_4 - 2a_3^2\right)z^4 \\ &+ \left(5a_6 - 5a_2a_5 + a_2^5 - 5a_3a_4 - 5a_2^3a_3 + 5a_2^2a_4 + 5a_2a_3^2\right)z^5 \\ &+ \left(6a_7 - 6a_2a_6 + 6a_2^2a_5 - 6a_3a_5 + 12a_2a_3a_4 - a_2^6 - 6a_2^3a_4 - 3a_4^2 + 2a_3^3 - 9a_2^2a_3^2 + 6a_2^4a_3\right)z^6 + \cdots.\end{aligned} \tag{20}$$

Define a function

$$p(z) = \frac{1 + \omega(z)}{1 - \omega(z)} = 1 + c_1z + c_2z^2 + \cdots. \tag{21}$$

Apparently so, $p(z) \in \mathscr{P}$ and

$$\omega(z) = \frac{p(z) - 1}{1 + p(z)} = \frac{c_1 z + c_2 z^2 + c_3 z^3 + \cdots}{2 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots}. \tag{22}$$

On the other hand,

$$\begin{aligned} 1 + \sin(\omega(z)) = {} & 1 + \frac{1}{2} c_1 z + \left(\frac{c_2}{2} - \frac{c_1^2}{4}\right) z^2 + \left(\frac{5c_1^3}{48} + \frac{c_3 - c_1 c_2}{2}\right) z^3 \\ & + \left(\frac{c_4 - c_1 c_3}{2} + \frac{5c_1^2 c_2}{16} - \frac{c_2^2}{4} - \frac{c_1^4}{32}\right) z^4 \\ & + \left(\frac{c_5 - c_1 c_4 - c_2 c_3}{2} + \frac{5c_1^2 c_3 + c_1 c_2^2}{16} - \frac{c_1^3 c_2}{8} + \frac{c_1^5}{3840}\right) z^5 \\ & + \left(\frac{c_6 - c_1 c_5 - c_2 c_4}{2} + \frac{5c_1 c_2 c_3}{8} + \frac{5c_2^3}{48} - \frac{c_3^2}{4}\right. \\ & \left. + \frac{5c_1^6}{512} + \frac{c_1^4 c_2}{768} - \frac{3c_1^2 c_2^2}{16} + \frac{5c_1^2 c_4}{16} - \frac{c_1^3 c_3}{8}\right) z^6 + \cdots. \end{aligned} \tag{23}$$

Comparing the coefficients of $z, z^2, z^3, z^4, z^5, z^6$ between the equations (20) and (23), we obtain

$$\begin{aligned} a_2 &= \frac{c_1}{2}, a_3 = \frac{c_2}{4}, a_4 = \frac{c_3}{6} - \frac{c_1 c_2}{24} - \frac{c_1^3}{144}, a_5 \\ &= \frac{c_4}{8} - \frac{c_1 c_3}{24} + \frac{5c_1^4}{1152} - \frac{c_1^2 c_2}{192} - \frac{c_2^2}{32}, \end{aligned} \tag{24}$$

$$a_6 = \frac{-3c_1 c_4}{80} - \frac{7c_2 c_3}{120} - \frac{11c_1^5}{4800} - \frac{43c_1 c_2^2}{960} + \frac{71c_1^3 c_2}{5760} + \frac{c_5}{10}, \tag{25}$$

$$\begin{aligned} a_7 = {} & \frac{c_1^2 c_4}{480} + \frac{c_1 c_2 c_3}{480} + \frac{833c_1^6}{691200} - \frac{41c_1^2 c_2^2}{3840} - \frac{109c_1^4 c_2}{11520} \\ & - \frac{c_1 c_5}{30} - \frac{5c_2 c_4}{96} + \frac{5c_2^3}{1152} + \frac{c_6}{12} + \frac{c_1^3 c_3}{144}. \end{aligned} \tag{26}$$

By virtue of Lemma 2, we can obtain

$$|a_2| \le 1, |a_3| \le \frac{1}{2}, \tag{27}$$

$$|a_4| = \left|\frac{c_3}{6} - \frac{c_1 c_2}{24} - \frac{c_1^3}{144}\right| = \left|\frac{1}{6}\left[c_3 - \frac{c_1 c_2}{3}\right] + \frac{c_1}{72}\left[c_2 - \frac{c_1^2}{2}\right]\right|. \tag{28}$$

Let $c_1 = c, c \in [0, 2]$ and using Lemma 3, we get

$$|a_4| = \left|\frac{1}{6}\left[c_3 - \frac{c_1 c_2}{3}\right] + \frac{c_1}{72}\left[c_2 - \frac{c_1^2}{2}\right]\right| \le \frac{1}{3} + \frac{c\left(2 - c^2/2\right)}{72}, \tag{29}$$

setting

$$F(c) = \frac{1}{3} + \frac{c\left(2 - c^2/2\right)}{72}, \tag{30}$$

It can be easily verified that $F(c)$ takes its maximum value at $c = 2\sqrt{3}/3$, that is

$$|a_4| \le F\left(\frac{2\sqrt{3}}{3}\right) = \frac{1}{3} + \frac{\sqrt{3}}{162} \approx 0.344, \tag{31}$$

$$\begin{aligned} |a_5| &= \left|\frac{c_4}{8} - \frac{c_1 c_3}{24} + \frac{5c_1^4}{1152} - \frac{c_1^2 c_2}{192} - \frac{c_2^2}{32}\right| \\ &= \left|\frac{1}{8}\left[c_4 - \frac{c_1 c_3}{3}\right] - \frac{c_1^2}{576}\left[c_2 - \frac{c_1^2}{2}\right] - \frac{c_2}{32}\left(c_2 - \frac{c_1^2}{2}\right) - \frac{7c_1^2 c_2}{576}\right|. \end{aligned} \tag{32}$$

Let $c_1 = c, c \in [0, 2]$ from Lemma 3, we obtain

$$|a_5| \le \frac{1}{4} + \frac{5c^2\left(2 - c^2/2\right)}{576} + \frac{1}{16}\left(2 - \frac{c^2}{2}\right) + \frac{7c^2}{288}, \tag{33}$$

taking

$$F(c) = \frac{1}{4} + \frac{5c^2\left(2 - c^2/2\right)}{576} + \frac{1}{16}\left(2 - \frac{c^2}{2}\right) + \frac{7c^2}{288}, \tag{34}$$

It can be easily verified that maximum of $F(c)$ occurs at $c = 0$, that is,

$$|a_5| \le F(0) = \frac{3}{8}, \tag{35}$$

$$\begin{aligned} |a_6| &= \left|\frac{-3c_1 c_4}{80} - \frac{7c_2 c_3}{120} - \frac{11c_1^5}{4800} - \frac{43c_1 c_2^2}{960} + \frac{71c_1^3 c_2}{5760} + \frac{c_5}{10}\right| \\ &= \left|\frac{1}{24}\left[c_5 - \frac{9c_1 c_4}{10}\right] + \frac{7}{120}\left[c_5 - c_2 c_3\right] + \frac{11c_1^3}{2400}\left[c_2 - \frac{c_1^2}{2}\right]\right. \\ &\quad \left. - \frac{43c_1 c_2}{960}\left(c_2 - \frac{c_1^2}{2}\right) - \frac{211c_1^3 c_2}{14400}\right|. \end{aligned} \tag{36}$$

Assume $c_1 = c, c \in [0, 2]$, by Lemma 3, we get

$$|a_6| \le \frac{7}{60} + \frac{1}{12} + \frac{11c^3\left(2 - c^2/2\right)}{2400} + \frac{43}{240}\left(2 - \frac{c^2}{2}\right) + \frac{211c^3}{7200}, \tag{37}$$

putting

$$F(c) = \frac{7}{60} + \frac{1}{12} + \frac{11c^3\left(2 - c^2/2\right)}{2400} + \frac{43}{240}\left(2 - \frac{c^2}{2}\right) + \frac{211c^3}{7200}, \tag{38}$$

it is demonstrable that maximum of $F(c)$ occurs at $c = 0$, that is,

$$|a_6| \le F(0) = \frac{67}{120}. \tag{39}$$

$$\begin{aligned}|a_7| &= \left|\frac{c_1^2c_4}{480} + \frac{c_1c_2c_3}{480} + \frac{833c_1^6}{691200} - \frac{41c_1^2c_2^2}{3840} - \frac{109c_1^4c_2}{11520} - \frac{c_1c_5}{30} - \frac{5c_2c_4}{96} + \frac{5c_2^3}{1152} + \frac{c_6}{12} + \frac{c_1^3c_3}{144}\right| \\ &= \left|\frac{-37c_1^6}{691200} - \frac{25c_1^2c_2^2}{5760} - \frac{c_1c_5}{30} + \frac{c_1^2\left[c_4 - c_2^2\right]}{480} + \frac{c_1c_2[c_3 - c_1c_2]}{480} + \frac{c_1^3[c_3 - c_1c_2]}{144} - \frac{29c_1^4\left[c_2 - c_1^2/2\right]}{11520} + \frac{5c_2^2\left[c_2 - c_1^2/2\right]}{1152} + \frac{\left[c_6 - 5/8c_2c_4\right]}{12}\right|.\end{aligned} \tag{40}$$

Let $c_1 = c, c \in [0, 2]$ and applying Lemma 3, we get

$$|a_7| \le \frac{1}{6} + \frac{c^2}{240} + \frac{9c}{120} + \frac{29c^4\left(2 - c^2/2\right)}{11520} + \frac{37c^6}{691200} + \frac{c^3}{72} + \frac{25c^2}{1440} + \frac{5\left(2 - c^2/2\right)}{288}, \tag{41}$$

showing

$$F(c) = \frac{1}{6} + \frac{c^2}{240} + \frac{9c}{120} + \frac{29c^4\left(2 - c^2/2\right)}{11520} + \frac{37c^6}{691200} + \frac{c^3}{72} + \frac{25c^2}{1440} + \frac{5\left(2 - c^2/2\right)}{288}, \tag{42}$$

further, we get

$$F'(c) \ge 0. \tag{43}$$

So, the function $F(c)$ takes its maximum value at $c = 2$, that is,

$$|a_7| \le F(2) = \frac{5587}{10800}. \tag{44}$$

Theorem 2. *Suppose that $f(z) \in \mathcal{S}_s^*$ and of the form (1), then, we get*

$$\left|a_3^2 - a_2^2\right| \le \frac{5}{4}. \tag{45}$$

Proof. According to equation (26), we have

$$\left|a_3^2 - a_2^2\right| = \left|\frac{c_2^2}{16} - \frac{c_1^2}{4}\right|. \tag{46}$$

By applying Lemma 1, we get

$$\left|a_3^2 - a_2^2\right| = \left|\frac{c_1^4}{64} + \frac{x^2\left(4 - c_1^2\right)^2}{64} + \frac{c_1^2x\left(4 - c_1^2\right)}{32} - \frac{c_1^2}{4}\right|. \tag{47}$$

Let $|x| = t, t \in [0, 1], c_1 = c, c \in [0, 2]$. Then, by the triangle inequality, we obtain

$$\left|a_3^2 - a_2^2\right| \le \frac{c^2t\left(4 - c^2\right)}{32} + \frac{t^2\left(4 - c^2\right)^2}{64} + \frac{c^4}{64} + \frac{c^2}{4}. \tag{48}$$

Suppose that

$$F(c, t) = \frac{c^2t\left(4 - c^2\right)}{32} + \frac{t^2\left(4 - c^2\right)^2}{64} + \frac{c^4}{64} + \frac{c^2}{4}, \tag{49}$$

then $\forall t \in [0, 1], \forall c \in [0, 2]$, the upper bound of $F(c, t)$ corresponds to $t = 1, c = 2$. Hence,

$$\left|a_3^2 - a_2^2\right| \le F(1, 2) = \frac{5}{4}. \tag{50}$$

□

Theorem 3. *Suppose that $f(z) \in \mathcal{S}_s^*$ and of the form (1), then, we have*

$$|a_2a_3 - a_3a_4| \le \frac{25}{36}. \tag{51}$$

Proof. From (26), we have

$$\begin{aligned}|a_2a_3 - a_3a_4| &= \left|\frac{c_1c_2}{8} + \frac{c_1^3c_2}{576} - \frac{c_2c_3}{24} + \frac{c_1c_2^2}{96}\right| \\ &= \left|\frac{c_2}{4}\left[\frac{c_1}{2} - \frac{\left(c_3 - c_1c_2/4\right)}{6} + \frac{c_1^3}{144}\right]\right|.\end{aligned} \tag{52}$$

If we insert $c_1 = c, c \in [0, 2]$ and according to Lemma 3, we get

$$|a_2a_3 - a_3a_4| \le \frac{1}{2}\left[\frac{c}{2} + \frac{1}{3} + \frac{c^3}{144}\right]. \tag{53}$$

Assume that

$$F(c) = \frac{1}{2}\left[\frac{c}{2} + \frac{1}{3} + \frac{c^3}{144}\right]. \tag{54}$$

Therefore, we have $\forall c \in (0, 2)$

$$F'(c) = \frac{1}{4} + \frac{c^2}{96} > 0, \tag{55}$$

namely, the maximum value of $F(c)$ can be obtained at $c = 2$, that is,

$$|a_2a_3 - a_3a_4| \le F(2) = \frac{25}{36}. \tag{56}$$

□

Theorem 4. *Suppose that $f(z) \in \mathcal{S}_s^*$ and of the form (1), then, we get*

$$|a_2a_4 - a_3^2| \le \frac{1}{4}. \tag{57}$$

Proof. Suppose that $f(z) \in \mathcal{S}_s^*$, then, through equation (26), we get

$$|a_2a_4 - a_3^2| = \left|\frac{c_1c_3}{12} - \frac{c_1^2c_2}{48} - \frac{c_1^4}{288} - \frac{c_2^2}{16}\right|. \tag{58}$$

□

Now, according to Lemma 1, we obtain

$$\begin{aligned}|a_2a_4 - a_3^2| &= \left|\frac{c_1c_3}{12} - \frac{c_1^2c_2}{48} - \frac{c_1^4}{288} - \frac{c_2^2}{16}\right| \\ &= \left|-\frac{5c_1^4}{576} - \frac{x^2c_1^2(4-c_1^2)}{48} - \frac{x^2(4-c_1^2)^2}{64} + \frac{c_1(4-c_1^2)(1-|x|^2)z}{24}\right|.\end{aligned} \tag{59}$$

If we insert $c_1 = c, c \in [0,2], |x| = t, t \in [0,1]$. Then, by the triangle inequality, we get

$$|a_2a_4 - a_3^2| \le \frac{t^2c^2(4-c^2)}{48} + \frac{(1-t^2)c(4-c^2)}{24} + \frac{t^2(4-c^2)^2}{64} + \frac{5c^4}{576}. \tag{60}$$

Putting

$$F(c,t) = \frac{t^2c^2(4-c^2)}{48} + \frac{(1-t^2)c(4-c^2)}{24} + \frac{t^2(4-c^2)^2}{64} + \frac{5c^4}{576}, \tag{61}$$

then, $\forall t \in [0,1], \forall c \in [0,2]$, the upper bound of $F(c,t)$ corresponds to $t=1, c=0$. Hence,

$$|a_2a_4 - a_3^2| \le F(0,1) = \frac{1}{4}. \tag{62}$$

Theorem 5. *Suppose that $f(z) \in \mathcal{S}_s^*$ and of the form (1), then, we get*

$$|a_2a_5 - a_3a_4| \le \frac{11}{36}. \tag{63}$$

Proof. Assume that $f(z) \in \mathcal{S}_s^*$, then, on the basis of equation (26), we obtain

$$\begin{aligned}|a_2a_5 - a_3a_4| &= \left|\frac{5c_1^5}{2304} + \frac{c_1c_4}{16} - \frac{c_1c_2^2}{192} - \frac{c_1^2c_3}{48} - \frac{c_1^3c_2}{1152} - \frac{c_2c_3}{24}\right| \\ &= \left|-\frac{c_1^3[c_2 - c_1^2/2]}{1152} - \frac{c_3[c_2 - c_1^2/2]}{24} + \frac{c_1[c_4 - c_1c_3]}{24}\right. \\ &\quad \left. + \frac{c_1^5}{576} + \frac{c_1[c_4 - 1/4c_2^2]}{48}\right|.\end{aligned} \tag{64}$$

□

If we insert $c_1 = c, c \in [0,2]$, from Lemma 3, we obtain

$$|a_2a_5 - a_3a_4| \le \frac{c^3[2 - c^2/2]}{1152} + \frac{[2 - c^2/2]}{12} + \frac{c}{8} + \frac{c^5}{576}. \tag{65}$$

Taking

$$F(c) = \frac{c^3[2 - c^2/2]}{1152} + \frac{[2 - c^2/2]}{12} + \frac{c}{8} + \frac{c^5}{576}. \tag{66}$$

Then, easy to show that maximum of $F(c)$ occurs at $c = 2, \forall c \in [0,2]$, also which is

$$|a_2a_5 - a_3a_4| \le F(2) = \frac{11}{36}. \tag{67}$$

Theorem 6. *Suppose that $f(z) \in \mathcal{S}_s^*$ and in the form (1), then, we get*

$$|a_3a_5 - a_2a_4| \le \frac{9}{16}. \tag{68}$$

Proof. Assume that $f(z) \in \mathcal{S}_s^*$, then, according to equation (26), we get

$$\begin{aligned}|a_3a_5 - a_2a_4| &= \left|\frac{c_2^3}{128} + \frac{c_1c_3}{12} - \frac{c_1^2c_2}{48} - \frac{c_1^4}{288} - \frac{5c_1^4c_2}{4608}\right. \\ &\quad \left. - \frac{c_2c_4}{32} + \frac{c_1c_2c_3}{96} + \frac{c_1^2c_2^2}{768}\right| \\ &= \left|\frac{[c_1[c_3 - c_1c_2/4]}{12} + \frac{5c_1^2c_2[c_2 - c_1^2/2]}{2304}\right. \\ &\quad \left. - \frac{c_2[c_4 - 1/3c_1c_3]}{32} + \frac{c_2^2[c_2 - c_1^2/2]}{128} + \frac{7c_1^2c_2^2}{2304} - \frac{c_1^4}{288}\right|.\end{aligned} \tag{69}$$

□

If we insert $c_1 = c, c \in [0,2]$ and in view of Lemma 3, we have

$$|a_3a_5 - a_2a_4| \le \frac{c}{6} + \frac{1}{8} + \frac{5c^2[2 - c^2/2]}{1152} + \frac{[2 - c^2/2]}{32} + \frac{7c^2}{576} + \frac{c^4}{288}. \tag{70}$$

Taking

$$F(c)=\frac{c}{6}+\frac{1}{8}+\frac{5c^2\left[2-c^2/2\right]}{1152}+\frac{\left[2-c^2/2\right]}{32}+\frac{7c^2}{576}+\frac{c^4}{288}. \tag{71}$$

Then, $\forall c\in[0,2]$, the demonstrable function $F(c)$ obtains the maximum value at $c=2$, that is,

$$|a_3a_5-a_2a_4|\le F(2)=\frac{9}{16}. \tag{72}$$

Theorem 7. *Suppose that $f(z)\in\mathcal{S}_s^*$ and of the form (1), then, we get*

$$|a_5a_3-a_4^2|\le\frac{97}{324}. \tag{73}$$

Proof. Suppose that $f(z)\in\mathcal{S}_s^*$, then, by the equation (26), we obtain

$$\begin{aligned}|a_5a_3-a_4^2|&=\left|\frac{7c_1^4c_2}{13824}+\frac{c_2c_4}{32}+\frac{c_1c_2c_3}{288}-\frac{c_2^3}{128}+\frac{c_1^3c_3}{432}-\frac{7c_1^2c_2^2}{2304}-\frac{c_3^2}{36}-\frac{c_1^6}{20736}\right|\\&=\left|\frac{c_2[c_4-c_1c_3/9]}{32}-\frac{c_3[c_3-c_1c_2/4]}{36}-\frac{c_2^2\left[c_2-c_1^2/2\right]}{128}\right.\\&\quad\left.-\frac{c_1^2c_2\left[c_2-c_1^2/2\right]}{144}+\frac{c_1^3[c_3-31/32c_1c_2]}{432}-\frac{5c_1^4c_2}{6912}-\frac{c_1^6}{20736}\right|.\end{aligned} \tag{74}$$

□

If we insert $c_1=c, c\in[0,2]$ and by Lemma 3, we obtain

$$|a_5a_3-a_4^2|\le\frac{1}{8}+\frac{1}{9}+\frac{\left[2-c^2/2\right]}{32}+\frac{c^2\left[2-c^2/2\right]}{72}+\frac{c^3}{216}+\frac{5c^4}{3456}+\frac{c^6}{20736}. \tag{75}$$

Putting

$$F(c)=\frac{1}{8}+\frac{1}{9}+\frac{\left[2-c^2/2\right]}{32}+\frac{c^2\left[2-c^2/2\right]}{72}+\frac{c^3}{216}+\frac{5c^4}{3456}+\frac{c^6}{20736}. \tag{76}$$

$\forall c\in(0,2), F'(c)>0$, Then, maximum of $F(c)$ occurs at $c=2$, that is

$$|a_5a_3-a_4^2|\le F(2)=\frac{97}{324}. \tag{77}$$

Theorem 8. *Suppose that $f(z)\in\mathcal{S}_s^*$ and of the form (1), then, we get*

$$|T_4(2)|\le\frac{263384.5}{104976}\approx 2.51. \tag{78}$$

Proof. Since

$$\begin{aligned}T_4(2)&=\left(a_2^2-a_3^2\right)^2+2\left(a_3^2-a_2a_4\right)(a_2a_4-a_3a_5)\\&\quad-(a_2a_3-a_3a_4)^2+\left(a_4^2-a_3a_5\right)^2-(a_3a_4-a_2a_5)^2,\end{aligned} \tag{79}$$

then, by applying the triangle inequality, we get

$$\begin{aligned}|T_4(2)|&\le\left|a_2^2-a_3^2\right|^2+2\left|a_3^2-a_2a_4\right||a_2a_4-a_3a_5|\\&\quad+|a_2a_3-a_3a_4|^2+\left|a_4^2-a_3a_5\right|^2+|a_3a_4-a_2a_5|^2.\end{aligned} \tag{80}$$

Now, substituting (18), (45)–(73) into (80), we easily obtain the desired assertion (78). □

3. Conclusion

In this paper, based on the paper [15], we continuously discuss the problem of the fourth-order Toeplitz determinant of starlike functions, which are connected with the sine function and get the upper bounds of the determinant. In the next step, we can consider the fourth-order Toeplitz determinant of other function classes defined by various linear or nonlinear operators and also make the related discussion on the fifth-order Toeplitz determinant for certain function classes.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

The present investigation was partly supported by the Natural Science Foundation of the People's Republic of China under Grant 11561001, the Program for Young Talents of Science and Technology in Universities of Inner Mongolia Autonomous Region under Grant NJYT-18-A14, the Natural Science Foundation of Inner Mongolia of the People's Republic of China under Grants 2018MS01026 and 2017MS0113, the Higher School Foundation of Inner Mongolia of the People's Republic of China under Grant NJZY20200, the Natural Science Foundation of Chifeng of Inner Mongolia, the Program for Key Laboratory Construction of Chifeng University (no. CFXYZD202004), and the Research and Innovation.

References

[1] S. S. Miller and P. T. Mocanu, *Differential Subordinations: Theory and Applications*, Series on Monographs and Textbooks in Pure and Applied Mathematics, no. 225, 2000Marcel Dekker Incorporated, New York and Basel, 2000.

[2] N. E. Cho, V. Kumar, S. S. Kumar, and V. Ravichandran, "Radius problems for starlike functions associated with the sine function," *Bulletin of the Iranian Mathematical Society*, vol. 45, no. 1, pp. 213–232, 2019.

[3] D. K. Thomas and S. Abdul Halim, "Retracted article: Toeplitz matrices whose elements are the coefficients of starlike and close-to-convex functions," *Bulletin of the Malaysian Mathematical Sciences Society*, vol. 40, no. 4, article 385, pp. 1781–1790, 2017.

[4] A. Janteng, S. Halim, and M. Darus, "Coefficient inequality for a function whose derivative has a positive real part," *Journal of Inequalities in Pure and Applied Mathematics*, vol. 7, no. 2, article 50, 2006.

[5] A. Janteng, S. A. Halim, and M. Darus, "Hankel determinant for starlike and convex functions," *International Journal of Mathematical Analysis*, vol. 13, no. 1, pp. 619–625, 2007.

[6] D. Bansal, "Upper bound of second Hankel determinant for a new class of analytic functions," *Applied Mathematics Letters*, vol. 26, no. 1, pp. 103–107, 2013.

[7] S. K. Lee, V. Ravichandran, and S. Supramaniam, "Bounds for the second Hankel determinant of certain univalent functions," *Journal of inequalities and Applications*, vol. 2013, no. 1, Article ID 281, 2013.

[8] D. Bansal, S. Maharana, and J. K. Prajapat, "Third order Hankel determinant for certain univalent functions," *Journal of the Korean Mathematical Society*, vol. 52, no. 6, pp. 1139–1148, 2015.

[9] P. Zaprawa, "Third Hankel determinants for subclasses of univalent functions," *Mediterranean Journal of Mathematics*, vol. 14, no. 1, 2017.

[10] H. Y. Zhang, H. Tang, and L. N. Ma, "Upper bound of third Hankel determinant for a class of analytic functions," *Pure and Applied Mathematics*, vol. 33, no. 2, pp. 211–220, 2017.

[11] K. O. Babalola, "On $H_3(1)$ Hankel determinant for some classes of univalent functions," 2009, https://arxiv.org/abs/0910.3779.

[12] M. Raza and S. N. Malik, "Upper bound of the third Hankel determinant for a class of analytic functions related with lemniscate of Bernoulli," *Journal of Inequalities and Applications*, vol. 2013, no. 1, Article ID 412, 8 pages, 2013.

[13] L. Shi, I. Ali, M. Arif, N. E. Cho, S. Hussain, and H. Khan, "A study of third Hankel determinant problem for certain subfamilies of analytic functions involving cardioid domain," *Mathematics*, vol. 7, no. 5, p. 418, 2019.

[14] L. Shi, H. M. Srivastava, M. Arif, S. Hussain, and H. Khan, "An investigation of the third Hankel determinant problem for certain subfamilies of univalent functions involving the exponential function," *Symmetry*, vol. 11, no. 5, p. 598, 2019.

[15] S. Mahmood, H. M. Srivastava, N. Khan, Q. Z. Ahmad, B. Khan, and I. Ali, "Upper bound of the third Hankel determinant for a subclass of q-starlike functions," *Symmetry*, vol. 11, no. 3, article 347, 2019.

[16] H.-Y. Zhang and H. Tang, "A study of fourth-order Hankel determinants for starlike functions connected with the sine function," *Journal of Function Spaces*, vol. 2021, Article ID 9991460, 8 pages, 2021.

[17] M. F. Ali, D. K. Thomas, and A. Vasudevarao, "Toeplitz determinants whose elements are the coefficients of analytic and univalent functions," *Bulletin of the Australian Mathematical Society*, vol. 97, no. 2, pp. 253–264, 2018.

[18] H. Tang, S. Khan, S. Hussain, and N. Khan, "Hankel and Toeplitz determinant for a subclass of multivalent q-starlike functions of order α," *AIMS Mathematics*, vol. 6, no. 6, pp. 5421–5439, 2021.

[19] H.-Y. Zhang, R. Srivastava, and H. Tang, "Third-order Hankel and Toeplitz determinants for starlike functions connected with the sine function," *Mathematics*, vol. 7, no. 5, p. 404, 2019.

[20] C. Ramachandran and S. Annamalai, "On Hankel and Toeplitz determinants for some special class of analytic functions involving conical domains defined by subordination," *International Journal of Engineering Research & Technology (IJERT)*, vol. 5, pp. 553–561, 2016.

[21] H. M. Srivastava, Q. Z. Ahmad, N. Khan, B. Khan, and B. Khan, "Hankel and Toeplitz determinants for a subclass of q-starlike functions associated with a general conic domain," *Mathematics*, vol. 7, no. 2, pp. 181–215, 2019.

[22] R. J. Libera and E. J. Zlotkiewicz, "Coefficient bounds for the inverse of a function with derivative in P," *Proceedings of the American Mathematical Society*, vol. 87, no. 2, pp. 251–257, 1983.

[23] V. Ravichandran and S. Verma, "Borne pour le cinquieme coefficient des fonctions etoilees," *Comptes Rendus Mathematique*, vol. 353, no. 6, pp. 505–510, 2015.

[24] C. Pommerenke, *Univalent Functions*, Math, Lehrbucher, vandenhoeck and Ruprecht, Gottingen, 1975.