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Research Article

A New Regularity Criterion for the Three-Dimensional Incompressible Magnetohydrodynamic Equations in the Besov Spaces

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Regularity criteria of the weak solutions to the three-dimensional (3D) incompressible magnetohydrodynamic (MHD) equations are discussed. Our results imply that the scalar pressure field π plays an important role in the regularity problem of MHD equations. We derive that the weak solution (u, b) is regular on $(0, T]$, which is provided for the scalar pressure field π in the Besov spaces.

1. Introduction

In this article, we consider the global regularity problem concerning the 3D incompressible MHD equations

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla \pi = b \cdot \nabla b + \Delta u, \\ \partial_t b + u \cdot \nabla b = b \cdot \nabla u + \Delta b, \\ \nabla \cdot u = 0, \nabla \cdot b = 0, \end{cases} \quad (1)$$

that satisfy the initial condition

$$u|_{t=0} = u_0, b|_{t=0} = b_0. \quad (2)$$

Here, $u = (u_1, u_2, u_3)$, $b = (b_1, b_2, b_3)$, and $\pi = \pi(x, t)$ represent the velocity field, the magnetic field, and the pressure, respectively; $u_0(x), b_0(x)$ are the corresponding initial data which satisfied $\nabla \cdot u_0 = 0$, $\nabla \cdot b_0 = 0$ in the sense of distribution.

MHD equations govern the dynamics of the velocity and magnetic fields of electrically conducting fluids such as plasmas, liquid metals, and salt water. Besides their important physical applications, the MHD equations also have important mathematical significance. It is well known [1] that problem (1) is locally well-posed for any given initial datum $u_0, b_0 \in H^s(R^3)$. However, whether a local strong solution can exist globally, or equivalently, whether global weak solutions are smooth is still a challenging open problem. Nevertheless, there exist plenty of results in the literature showing that the answer to this problem is positive if some additional conditions are imposed on the weak solutions [2–12]. Some of them are motivated by the works on the Navier-Stokes equations $(b = 0)$

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla \pi = \Delta u, \\ \nabla \cdot u = 0, \\ u|_{t=0} = u_0. \end{cases} \quad (3)$$

Among these results, Zhou [7] and He and Xin [13] obtained some Ladyzhenskaya-Prodi-Serrin-type regularity criteria for the 3D MHD equations in terms of velocity and the gradient of velocity, independently. They proved that the velocity u satisfies

$$\int_0^T \|u(\cdot,t)\|_{L^r}^s dt < +\infty, \frac{2}{s} + \frac{3}{r} \le 1, r > 3, \tag{4}$$

then the weak solution (u, b) is regular on $(0, T]$.

However, we are interested in regularity criteria involving only in terms of the pressure π (for details, refer to [14–17]). For regularity criteria results for the 3D MHD equations, Jia and Zhou [14] obtained the following:

$$\partial_3\pi \in L^p\left(0, T; L^q\left(R^3\right)\right), \frac{2}{p} + \frac{3}{q} = 2, 3 \le q < \infty. \tag{5}$$

In [15], Liu established the new regularity criteria in terms of the pressure as follows:

$$\partial_3\pi \in L^p\left(0, T; L^{q_1}\left(R^2_{x_1,x_2}; L^{q_2}\left(R_{x_3}\right)\right)\right), \frac{2}{p} + \frac{2}{q_1} + \frac{1}{q_2} = s \in [2,3], \frac{3}{s} \le q_2 \le q_1 \le \frac{1}{k-2}. \tag{6}$$

Later, Gala and Ragusa [16] extended the regularity criteria to the BMO space and Besov space. If the pressure π or the pressure gradient $\nabla\pi$ satisfies

$$\pi \in L^2\left(0, T; \dot{B}^{-1}_{\infty,\infty}\left(R^3\right)\right), \tag{7}$$

or

$$\nabla\pi \in L^{2/3}\left(0, T; BMO\left(R^3\right)\right), \tag{8}$$

then the weak solution (u, b) is regular on $(0, T]$.

Very recently, Tong and Wang [17] showed the following regularity criterion for the 3D MHD:

$$\int_0^T \frac{\|\pi\|_{\dot{B}^{-1}_{\infty,\infty}}}{\sqrt{1 + \ln\left(e + \|\pi\|\dot{B}^{-1}_{\infty,\infty}\right)}} dt < \infty, \tag{9}$$

or

$$\int_0^T \frac{\|\nabla u\|^2_{\dot{B}^{-1}_{\infty,\infty}} + \|\nabla u\|^2_{\dot{B}^{-1}_{\infty,\infty}}}{\sqrt{1 + \ln\left(e + \|\nabla u\|\dot{B}^{-1}_{\infty,\infty} + \|\nabla u\|\dot{B}^{-1}_{\infty,\infty}\right)}} dt < \infty. \tag{10}$$

It is the aim of the paper to give the complete description on the regularity criteria of the weak solutions of the 3D MHD equations on the pressure field. Let us introduce the assignment of this paper; we first recall some preliminaries on functional settings and state the main results in Section 2 and prove the main results in Section 3.

2. Preliminaries and Main Result

Throughout this text, C stands for a generic positive constant which may differ in value from one line to another. We use $\|\cdot\|_p$ to denote the norm of the Lebesgue space $L^p (1 \le p \le \infty)$, and the norm as follows:

$$\|f\|_p = \begin{cases} \left(\int_{R^3} |f(x)|^p dx\right)^{1/p}, & 1 \le p < \infty, \\ \operatorname{esssup}_{x\in R^3} |f(x)|, & p = \infty. \end{cases} \tag{11}$$

In order to define the Besov space, let us first recall the Littlewood-Paley theory (see Ref. [18]). For a given function $f \in \mathcal{S}$, its Fourier transformation $\hat{f}$ of $\mathcal{F}$ is defined by

$$\mathcal{F}f(\xi) = \hat{f}(\xi) = \int_{R^3} e^{-ix\cdot\xi} f(x) dx, \tag{12}$$

where $\mathcal{S}$ is the Schwartz class of rapidly decreasing functions defined on R^3. Choose a nonnegative radial function $\psi \in \mathcal{S}(R^3)$ supported in $\mathcal{B} \triangleq \{\xi \in R^3; |\xi| \le 2\}$ such that for $\xi \in R^3$ and $|\xi| \le 1$. Setting the radial function

$$\phi(\xi) = \psi\left(\frac{\xi}{2}\right) - \psi(\xi), \quad \xi \in R^3, \tag{13}$$

for the integer set Z, we have

$$\sum_{j\in Z} \phi\left(2^{-j}\xi\right) = 1, \quad \xi \in R^3 \setminus \{0\}. \tag{14}$$

Let $h = \mathcal{F}^{-1}\phi$ and define the dyadic blocks as follows:

$$\Delta_j f(x) = 2^{3j} \int_{R^3} h\left(2^j s\right) f(x - s) ds, \quad x \in R^3. \tag{15}$$

We thus have the following Littlewood-Paley decomposition:

$$f = \sum_{j\in Z} \Delta_j f(x). \tag{16}$$

For $\alpha \in R$, $1 \le p, q \le \infty$, we can now define the homogeneous Besov space $\dot{B}^\alpha_{p,q}(R^3)$ as

$$\dot{B}^\alpha_{p,q}\left(R^3\right) = \left\{ f \in \frac{\mathcal{S}'\left(R^3\right)}{\mathcal{P}\left(R^3\right)}; \quad \|f\|_{\dot{B}^\alpha_{p,q}} < \infty, \right. \tag{17}$$

where

$$\|f\|_{\dot{B}^\alpha_{p,q}} = \begin{cases} \left(\sum_{j=-\infty}^{\infty} 2^{j\alpha q} \left\|\Delta_j f(x)\right\|_p^q\right)^{1/q}, & 1 \le q < \infty, \\ \sup_{j\in Z} \left\|\Delta_j f(x)\right\|_p, & q = \infty. \end{cases} \tag{18}$$

$\mathcal{P}(R^3)$ is the set of all scalar polynomials defined on R^3. $\mathcal{S}'(R^3)$ is the space of all tempered distributions on R^3.

Definition 1. Assuming $(u_0, b_0) \in L^2(\mathbb{R}^3)$, and $\nabla\cdot u_0 = \nabla\cdot b_0 = 0$, $T > 0$, the measurable function (u, b) defined on $(0, T] \times R^3$ is called the weak solution of Equation (1), if

(1) $(u, b) \in L^\infty(0, T; L^2(R^3)) \cap L^2(0, T; H^1(R^3))$.

(2) $\nabla\cdot u = \nabla\cdot b = 0$ and $\forall\varphi \in C_0^\infty((0, T) \times R^3)$ have

$$\int_0^T\int_{R^3} (u, b)\cdot\nabla\varphi dxdt = 0, \tag{19}$$

Equation (1) holds in the sense of distributions.

For $\forall\varphi \in C_0^\infty((0, T) \times R^3)$, $\nabla\cdot\varphi = 0$, we have

$$\begin{aligned}&\int_0^T\int_{R^3} (\partial_t\varphi + (u\cdot\nabla)\varphi)\cdot udxdt + \int_{R^3} u_0\cdot\varphi(x, 0)dx = \int_0^T\int_{R^3} \nabla u : \nabla\varphi dxdt,\\ &\int_0^T\int_{R^3} (\partial_t\varphi + (b\cdot\nabla)\varphi)\cdot bdxdt + \int_{R^3} b_0\cdot\varphi(x, 0)dx = \int_0^T\int_{R^3} \nabla b : \nabla\varphi dxdt,\end{aligned} \tag{20}$$

where $A : B = \sum_{i,j=1}^3 a_{ij}b_{ij}$, $A = (a_{ij})$, $B = (b_{ij})$.

(3) The strong energy inequality, that is,

$$\|u(t), b(t)\|_{L^2}^2 + 2\int_0^t \|\nabla(u, b)(\tau)\|_{L^2}^2 d\tau \le \|u(0), b(0)\|_{L^2}^2, \quad \forall 0 \le t < T. \tag{21}$$

In this paper, we establish the following theorem.

Theorem 2. *Let $u_0, b_0 \in L^2(\mathbb{R}^3) \cap L^4(\mathbb{R}^3)$, with $\nabla\cdot u_0 = 0$, $\nabla\cdot b_0 = 0$, $T > 0$ in the sense of distribution. Let (u, b) be a weak solution of the MHD Equations (1) on $(0, T]$ which satisfies the strong energy inequality (21). If the corresponding pressure π satisfies*

$$\pi \in L^{2/(\alpha+1)}\left(0, T; \dot{B}^{\alpha}_{\infty,\infty}(R^3)\right), \quad 0 < \alpha < 1, \tag{22}$$

then the weak solution (u, b) is regular on $(0, T]$.

When the time critical index $2/(\alpha + 1) = 1$, we can derive the following Theorem 3. And if $\pi \in L^{4/3}(0, T; \dot{B}^0_{\infty,\infty})$, we prove that the weak solution (u, b) is regular on $(0, T]$ in Theorem 4.

Theorem 3. *Under the same assumption in Theorem 2, if the corresponding pressure p satisfies*

$$\pi \in L^1\left(0, T; \dot{B}^{\alpha}_{\infty,\infty}(R^3)\right), \quad \frac{1}{2} \le \alpha < 1, \tag{23}$$

then the weak solution (u, b) is regular on $(0, T]$.

Theorem 4. *Under all the assumptions in Theorem 2, the weak solution (u, b) is regular on $(0, T]$. If the corresponding pressure p satisfies the following condition:*

$$\pi \in L^{4/3}\left(0, T; \dot{B}^{0}_{\infty,\infty}(R^3)\right). \tag{24}$$

Remark 5. In article [17], the regularity condition of $\int_0^T (\|\pi\|_{\dot{B}^{-1}_{\infty,\infty}} / \sqrt{1 + \ln (e + \|\pi\|\dot{B}^{-1}_{\infty,\infty})})dt < \infty$ is better than the regularity condition of Theorems 2–4. However, in article [17], the initial condition is required to satisfy $u_0, b_0 \in H^3(R^3)$, while Theorems 2–4 only need to satisfy $u_0, b_0 \in L^2(R^3) \cap L^4(R^3)$.

3. Pressure Regularity Criteria

In order to avoid the difficulties caused by the magnetic field when we do energy estimates, by adding and subtracting $(1)_1$ with $(1)_2$, we convert the 3D MHD Equations (1) into a mathematically symmetric form as follows:

$$\begin{cases}\partial_t w^+ + w^-\cdot\nabla w^+ + \nabla\pi = \Delta w^+,\\ \partial_t w^- + w^+\cdot\nabla w^- + \nabla\pi = \Delta w^-,\\ \nabla\cdot w^+ = 0, \nabla\cdot w^- = 0,\\ w^+(x, 0) = w_0^+(x), w^-(x, 0) = w_0^-(x),\end{cases} \tag{25}$$

with $w^\pm \triangleq u \pm b$.

Let us introduce some auxiliary results. To establish some new regularity criteria in terms of pressure, an effective method is to find a "bridge" between the desired results and the known criteria. The following lemma plays such a role in the proof of Theorems 2–4.

Lemma 6 (Bernstein inequality [18]). *For $1 \le p \le q \le \infty$ and an integer k, the following estimate is true:*

$$\left\|\nabla^k \Delta_j f\right\|_q \le C2^{jk+3j(1/p-1/q)}\left\|\Delta_j f\right\|_p. \tag{26}$$

Lemma 7. *Let $(u_0, b_0) \in L^2(R^3) \cap L^4(R^3)$ and $\nabla\cdot u_0 = \nabla\cdot b_0 = 0$. Let (u, b) be the corresponding solution to the 3D MHD Equations (1). If w^+ and w^- satisfy*

$$\sup_{0\le t\le T}\left(\|w^+(\cdot, t)\|_4 + \|w^+(\cdot, t)\|_4\right) < +\infty. \tag{27}$$

Then, the weak solution (u, b) is regular on $[0, T]$.

Proof. By the definition of w^+ and w^-, one can deduce that

$$\|u(\cdot, t)\|_4 + \|b(\cdot, t)\|_4 \le \|w^+(\cdot, t)\|_4 + \|w^-(\cdot, t)\|_4. \tag{28}$$

By (4), we can derive that the weak solution (u, b) is regular on $[0, T]$.

Now, we are ready to prove the theorem. Taking the inner product of the first equation of (21) with $|w_3^+|^2 w_3^+$ and the second equation of (21) with $|w_3^-|^2 w_3^-$ in $L^2(R^3)$, adding them together, and noticing that $\nabla\cdot w^\pm = 0$, we have

$$\frac{1}{4}\frac{d}{dt}\left(\|w^+\|_4^4+\|w^-\|_4^4\right)+\frac{1}{2}\int_{R^3}\left(\left|\nabla|w^+|^2\right|^2+\left|\nabla|w^-|^2\right|^2\right)dx + \int_{R^3}\left(|\nabla|w^+||^2|w^+|^2+|\nabla|w^-||^2|w^-|^2\right)dx = -\int_{R^3}\nabla\pi\cdot w^+|w^+|^2dx-\int_{R^3}\nabla\pi\cdot w^-|w^-|^2dx \triangleq I+II. \tag{29}$$

Here, we have used the following identities due to the fact that the divergence free condition $\nabla w^+=\nabla w^-=0$:

$$\int_{R^3}(w^+\cdot\nabla w^-)\cdot w^-|w^-|^2dx=\int_{R^3}(w^-\cdot\nabla w^+)\cdot w^+|w^+|^2dx=0. \tag{30}$$

By the Littlewood-Paley decomposition, p can be written as follows:

$$\pi=\sum_{j\in Z}\Delta_j\pi=\sum_{j\le N}\Delta_j\pi+\sum_{j\ge N+1}\Delta_j\pi. \tag{31}$$

Hence, by (29), we obtain

$$I=\left|\sum_{j\le N}\int_{R^3}\Delta_j\nabla\pi\cdot w^+|w^+|^2dx\right|+\left|\sum_{j\ge N+1}\int_{R^3}\Delta_j\nabla\pi\cdot w^+|w^+|^2dx\right|\triangleq I_1+I_2. \tag{32}$$

□

3.1. The Proof of Theorem 2. We first consider that the velocity π satisfies the growth condition (22):

$$\pi\in L^{2/(\alpha+1)}\left(0,T;\dot{B}^{\alpha}_{\infty,\infty}\right),\quad 0<\alpha<1. \tag{33}$$

For $0<\alpha<1$, by the Hölder inequality, we estimate $I_i(_i=1,2)$ of the right-hand side of (32) one by one.

$$\begin{aligned} I_1 &\le \sum_{j\le N}\left\|\Delta_j\nabla\pi\right\|_\infty\|w^+\|_2\left\||w^+|^2\right\|_2 \\ &\le \sum_{j\le N}2^{j(-\alpha+1)}2^{j(\alpha-1)}\left\|\Delta_j\nabla\pi\right\|_\infty\|w^+\|_2\|w^+\|_4^2 \\ &\le C2^{N(-\alpha+1)}\|\nabla\pi\|_{\dot{B}^{\alpha-1}_{\infty,\infty}}\|w^+\|_2\|w^+\|_4^2. \end{aligned} \tag{34}$$

Similarly, for I_2, integration by parts gives

$$\begin{aligned} I_2 &\le \sum_{j\ge 1+N}\left\|\Delta_j\pi\right\|_\infty\|w^+\|_2\||w^+|\nabla w^+\|_2 \\ &\le \sum_{j\ge 1+N}2^{-j\alpha}2^{j\alpha}\left\|\Delta_j\pi\right\|_\infty\|w^+\|_2\||w^+|\nabla w^+\|_2 \\ &\le C2^{-\alpha N}\|\pi\|_{\dot{B}^{\alpha}_{\infty,\infty}}\|w^+\|_2\||w^+|\nabla w^+\|_2. \end{aligned} \tag{35}$$

Hence, choosing the integer N such that (34) is equal to (35),

$$2^{N(-\alpha+1)}\|\nabla\pi\|_{\dot{B}^{\alpha-1}_{\infty,\infty}}\|w^+\|_2\|w^+\|_4^2=2^{-\alpha N}\|\pi\|_{\dot{B}^{\alpha}_{\infty,\infty}}\|w^+\|_2\||w^+|\nabla w^+\|_2, \tag{36}$$

then

$$\left[\log_2\frac{\||w^+|\nabla w^+\|_2}{\|w^+\|_4^2}\right]\le N\le\left[\log_2\frac{\||w^+|\nabla w^+\|_2}{\|w^+\|_4^2}\right]+1. \tag{37}$$

Plugging the estimates (34) and (35) into (32), and using Young inequality, we have

$$\begin{aligned} I &\le C2^{-\alpha\log_2\left(\||w^+|\nabla w^+\|_2/\|w^+\|_4^2\right)}\|\pi\|_{\dot{B}^{\alpha}_{\infty,\infty}}\|w^+\|_2\||w^+|\nabla w^+\|_2 \\ &\le C\left(\frac{\||w^+|\nabla w^+\|_2}{\|w^+\|_4^2}\right)^{-\alpha}\|\pi\|_{\dot{B}^{\alpha}_{\infty,\infty}}\||w^+|\nabla w^+\|_2 \\ &\le C\||w^+|\nabla w^+\|_2^{1-\alpha}\|w^+\|_4^{2\alpha}\|\pi\|_{\dot{B}^{\alpha}_{\infty,\infty}} \\ &\le C\left(\|w^+\|_4^{2\alpha}\|\pi\|_{\dot{B}^{\alpha}_{\infty,\infty}}\right)^{2/(\alpha+1)}+\frac{1}{4}\left(\||w^+|\nabla w^+\|_2^{1-\alpha}\right)^{2/(1-\alpha)} \\ &\le C\|w^+\|_4^{4\alpha/(\alpha+1)}\|\pi\|_{\dot{B}^{\alpha}_{\infty,\infty}}^{2/(\alpha+1)}+\frac{1}{4}\||w^+|\nabla w^+\|_2^2. \end{aligned} \tag{38}$$

By letting $0<\alpha<1$, such that $4\alpha/(\alpha+1)\le 4$, and by using the Young inequality, it follows that

$$\|w^+\|_4^{4\alpha/(\alpha+1)}\le C\left(\|w^+\|_4^4+1\right). \tag{39}$$

Hence, we obtain

$$I\le C\|\pi\|_{\dot{B}^{\alpha}_{\infty,\infty}}^{2/(\alpha+1)}\left(\|w^+\|_4^4+1\right)+\frac{1}{4}\||w^+|\nabla w^+\|_2^2. \tag{40}$$

Using the similar way with the estimate II, we see that

$$II\le C\|\pi\|_{\dot{B}^{\alpha}_{\infty,\infty}}^{2/(\alpha+1)}\left(\|w^-\|_4^4+1\right)+\frac{1}{4}\||w^-|\nabla w^-\|_2^2. \tag{41}$$

Combination of (40), (41), and (29) implies that

$$\begin{aligned} &\frac{1}{4}\frac{d}{dt}\left(\|w^+\|_4^4+\|w^-\|_4^4+1\right)+\frac{1}{2}\int_{R^3}\left(\left|\nabla|w^+|^2\right|^2+\left|\nabla|w^-|^2\right|^2\right)dx \\ &\quad+\int_{R^3}\left(|\nabla|w^+||^2|w^+|^2+|\nabla|w^-||^2|w^-|^2\right)dx \\ &\quad=-\int_{R^3}\nabla\pi\cdot w^+|w^+|^2dx \\ &\quad-\int_{R^3}\nabla\pi\cdot w^-|w^-|^2dx \\ &\quad=C\left(\|w^+\|_{L^4}^4+\|w^-\|_4^4+1\right)\|\pi\|_{\dot{B}^{\alpha}_{\infty,\infty}}^{2/(\alpha+1)}. \end{aligned} \tag{42}$$

Using the Gronwall inequality, we have

$$\sup_{0<t<T}\left(\|w^+\|_4^4+\|w^-\|_4^4+1\right)\le C\left(\|w_0^+\|_4^4+\|w_0^-\|_4^4+1\right)\exp\cdot\left\{C\int_0^t\|\pi\|_{\dot{B}^{\alpha}_{\infty,\infty}}^{2/(\alpha+1)}\right\}. \tag{43}$$

By assuming Theorem 2 and the definition of $W_0^{\pm}$, we have

$$\|w^+(0)\|_4+\|w^-(0)\|_4\le C. \tag{44}$$

By using Lemma 6 and (22), we have

$$\sup_{0<t<T}\left(\|w^+\|_4^4+\|w^-\|_4^4+1\right)\le C. \tag{45}$$

This completes the proof of Theorem 2.

3.2. The Proof of Theorem 3. Next, we are ready to prove Theorem 3. Let us begin with the case of growth condition (23):

$$\pi\in L^1\left(0,T;\dot{B}^{\alpha}_{\infty,\infty}\right),\quad \frac{1}{2}\le\alpha<1. \tag{46}$$

For $1/2\le\alpha<1$, by the Hölder inequality, we estimate I_i $({}_i=1,2)$ one by one.

$$\begin{aligned}I_1&\le\sum_{j\le N}\left\|\Delta_j\nabla\pi\right\|_{\infty}\|w^+\|_2\left\||w^+|^2\right\|_2\\&\le\sum_{j\le N}2^{j(-\alpha+1)}2^{j(\alpha-1)}\left\|\Delta_j\nabla\pi\right\|_{\infty}\|w^+\|_2\left\||w^+|^2\right\|_2\\&\le2^{N(-\alpha+1)}\|\nabla\pi\|_{\dot{B}^{\alpha-1}_{\infty,\infty}}\|w^+\|_2\left\||w^+|^2\right\|_2\\&\le C2^{N(-\alpha+1)}\|\pi\|_{\dot{B}^{\alpha}_{\infty,\infty}}\|w^+\|_4^2.\end{aligned} \tag{47}$$

Similarly, for I_2 of (32), by (26) and integration by parts, gives

$$\begin{aligned}I_2&\le\sum_{j\ge1+N}\left\|\Delta_j\pi\right\|_4\|w^+\|_4\||w^+|\nabla w^+\|_2\\&\le\sum_{j\ge1+N}2^{3j/4}\left\|\Delta_j\pi\right\|_2\|w^+\|_4\||w^+|\nabla w^+\|_2\\&\le\left\{\sum_{j\ge1+N}2^{-j/2}\right\}^{1/2}\left\{\sum_{j\le1+N}2^{2j}\left\|\Delta_j\pi\right\|_2^2\right\}^{1/2}\|w^+\|_4\||w^+|\nabla w^+\|_2\\&\le C2^{-N/4}\|\pi\|_{\dot{B}^{1}_{2,2}}\|w^+\|_4\||w^+|\nabla w^+\|_2\\&\le C2^{-N/4}\|w^+\cdot\nabla w^-\|_2\|w^+\|_4\||w^+|\nabla w^+\|_2.\end{aligned} \tag{48}$$

Hence, choosing the integer N such that (47) is equal to (48)

$$2^{N(-\alpha+1)}\|\pi\|_{\dot{B}^{\alpha}_{\infty,\infty}}\|w^+\|_4^2=2^{-N/4}\|w^+\cdot\nabla w^-\|_2\|w^+\|_4\||w^+|\nabla w^+\|_2, \tag{49}$$

then

$$\left[\log_2\left\{\frac{\|w^+\cdot\nabla w^-\|_2\||w^+|\nabla w^+\|_2}{\|\pi\|_{\dot{B}^{\alpha}_{\infty,\infty}}\|w^+\|_4}\right\}^{4/(5-4\alpha)}\right]\le N\le\left[\log_2\left\{\frac{\|w^+\cdot\nabla w^-\|_2\||w^+|\nabla w^+\|_2}{\|\pi\|_{\dot{B}^{\alpha}_{\infty,\infty}}\|w^+\|_4}\right\}^{4/(5-4\alpha)}\right]+1. \tag{50}$$

Plugging the estimates (47) and (48) into (32), we have

$$\begin{aligned}I&\le C2^{\log_2\left\{\|w^+\cdot\nabla w^-\|_2\||w^+|\nabla w^+\|_2/\|\pi\|_{\dot{B}^{\alpha}_{\infty,\infty}}\|w^+\|_4\right\}^{4(1-\alpha)/(5-4\alpha)}}\|\pi\|_{\dot{B}^{\alpha}_{\infty,\infty}}\|w^+\|_4^2\\&\le C\|\pi\|_{\dot{B}^{\alpha}_{\infty,\infty}}^{1/(5-4\alpha)}\|w^+\|_4^{(6-4\alpha)/(5-4\alpha)}\|w^+\cdot\nabla w^-\|_2^{4(1-\alpha)/(5-4\alpha)}\\&\quad\cdot\||w^+|\nabla w^+\|_2^{4(1-\alpha)/(5-4\alpha)}\le C\left\{\|\pi\|_{\dot{B}^{\alpha}_{\infty,\infty}}^{1/(5-4\alpha)}\|w^+\|_4^{(6-4\alpha)/(5-4\alpha)}\right\}^{5-4\alpha}\\&\quad+\frac{1}{4}\left\{\|w^+\cdot\nabla w^-\|_2^{4(1-\alpha)/(5-4\alpha)}\||w^+|\nabla w^+\|_2^{4(1-\alpha)/(5-4\alpha)}\right\}^{(5-4\alpha)/4(1-\alpha)}\\&\le C\|\pi\|_{\dot{B}^{\alpha}_{\infty,\infty}}\|w^+\|_4^{6-4\alpha}+\frac{1}{4}\|w^+\cdot\nabla w^-\|_2\||w^+|\nabla w^+\|_2\\&\le C\|\pi\|_{\dot{B}^{\alpha}_{\infty,\infty}}\|w^+\|_4^{6-4\alpha}+\frac{1}{8}\|w^+\cdot\nabla w^-\|_2^2+\frac{1}{8}\||w^+|\nabla w^+\|_2^2.\end{aligned} \tag{51}$$

Using the similar way with the estimate II, we see that

$$II\le C\|\pi\|_{\dot{B}^{\alpha}_{\infty,\infty}}\|w^-\|_4^{6-4\alpha}+\frac{1}{8}\|w^-\cdot\nabla w^+\|_2^2+\frac{1}{8}\||w^-|\nabla w^-\|_2^2. \tag{52}$$

Combination of (51), (52), and (29) implies that

$$\begin{aligned}&\frac{1}{4}\frac{d}{dt}\left(\|w^+\|_{L^4}^4+\|w^-\|_{L^4}^4\right)+\frac{1}{4}\int_{R^3}\left(\left|\nabla|w^+|^2\right|^2+\left|\nabla|w^-|^2\right|^2\right)dx\\&\quad+\int_{R^3}\left(|\nabla|w^+||^2|w^+|^2+|\nabla|w^-||^2|w^-|^2\right)dx\\&\quad=C\|\pi\|_{\dot{B}^{\alpha}_{\infty,\infty}}\left(\|w^+\|_4^{6-4\alpha}+\|w^-\|_4^{6-4\alpha}\right)\\&\quad+\frac{1}{4}\left(\|w^-\cdot\nabla w^+\|_2^2+\|w^-\cdot\nabla w^+\|_2^2\right).\end{aligned} \tag{53}$$

By letting $1/2\le\alpha<1$, such that $6-4\alpha\le4$, and by using the Young inequality, it follows that

$$\|w^{\pm}\|_4^{6-4\alpha}\le C\left(\|w^{\pm}\|_4^4+1\right). \tag{54}$$

Using the triangle inequality

$$\begin{aligned}
\|u\|_r &= \frac{1}{2}\|w^+ + w^-\|_r = \frac{1}{2}\left(\|w^+\|_r + \|w^-\|_r\right),\\
\|b\|_r &= \frac{1}{2}\|w^+ - w^-\|_r = \frac{1}{2}\left(\|w^+\|_r + \|w^-\|_r\right),\\
\|w^+\|_r &= \|u+b\|_r \le \|u\|_r + \|b\|_r,\\
\|w^-\|_r &= \|u-b\|_{L^r} \le \|u\|_r + \|b\|_r,
\end{aligned} \tag{55}$$

we obtain

$$\begin{aligned}
&\frac{1}{4}\frac{d}{dt}\left(\|u\|_4^4 + \|b\|_4^4 + 1\right) + \frac{1}{4}\int_{R^3}\left(\left|\nabla|u|^2\right|^2 + \left|\nabla|b|^2\right|^2\right)dx\\
&\quad + \int_{R^3}\left(|\nabla|u||^2|u|^2 + |\nabla|u||^2|b|^2 + |\nabla|b||^2|b|^2 + |\nabla|b||^2|u|^2\right)dx\\
&\le C\|\pi\|_{\dot{B}^{\alpha}_{\infty,\infty}}\left(\|u\|_4^4 + \|b\|_4^4 + 1\right) + \frac{1}{2}\\
&\quad \cdot\left(\|u\cdot\nabla u\|_2^2 + \|u\cdot\nabla b\|_2^2 + \|b\cdot\nabla b\|_2^2 + \|b\cdot\nabla u\|_2^2\right).
\end{aligned} \tag{56}$$

Hence,

$$\begin{aligned}
&\frac{1}{4}\frac{d}{dt}\left(\|u\|_4^4 + \|b\|_4^4 + 1\right) + \frac{1}{4}\int_{R^3}\left(\left|\nabla|u|^2\right|^2 + \left|\nabla|b|^2\right|^2\right)dx\\
&\quad + \frac{1}{2}\left(\|u\cdot\nabla u\|_2^2 + \|u\cdot\nabla b\|_2^2 + \|b\cdot\nabla b\|_2^2 + \|b\cdot\nabla u\|_2^2\right)\\
&\le C\|\pi\|_{\dot{B}^{\alpha}_{\infty,\infty}}\left(\|u\|_4^4 + \|b\|_4^4 + 1\right).
\end{aligned} \tag{57}$$

Then, the bounds for the L^4-norms of u and b follow from the standard Gronwall inequality. Thanks to (4), this completes the proof of Theorem 3.

3.3. The Proof of Theorem 4. For the final assumptions (24), that is,

$$\pi \in L^{4/3}\left(0, T; \dot{B}^0_{\infty,\infty}(R^3)\right). \tag{58}$$

After applying the Hölder inequality, we estimate $I_i (_i = 1, 2)$ one by one.

$$\begin{aligned}
I_1 &\le \sum_{j\le N}\left\|\Delta_j\nabla\pi\right\|_{\infty}\|w^+\|_2\left\||w^+|^2\right\|_2\\
&\le \sum_{j\le N}2^j 2^{-j}\left\|\Delta_j\nabla\pi\right\|_{\infty}\|w^+\|_2\|w^+\|_4^2\\
&\le 2^N\|\nabla\pi\|_{\dot{B}^{-1}_{\infty,\infty}}\|w^+\|_2\|w^+\|_4^2 \le C2^N\|\pi\|_{\dot{B}^0_{\infty,\infty}}\|w^+\|_4^2.
\end{aligned} \tag{59}$$

Similarly, for I_2 of (32), by (27), the Hölder inequality, and integration by parts, we have

$$\begin{aligned}
I_2 &\le \sum_{j\ge 1+N}\left\|\Delta_j\pi\right\|_3\|w^+\|_6\||w^+|\nabla w^+\|_2\\
&\le \sum_{j\ge 1+N}2^{j/2}\left\|\Delta_j\pi\right\|_2\|w^+\|_4^{1/2}\|w^+\|_{12}^{1/2}\||w^+|\nabla w^+\|_2\\
&\le \left\{\sum_{j>1+N}2^{-j}\right\}^{1/2}\left\{\sum_{j>1+N}2^j\left\|\Delta_j\pi\right\|_2^2\right\}^{1/2}\|w^+\|_4^{1/2}\left\||w^+|^2\right\|_6^{1/4}\||w^+|\nabla w^+\|_2\\
&\le C2^{-N/2}\|\pi\|_{\dot{B}^1_{2,2}}\|w^+\|_4^{1/2}\||w^+|\nabla w^+\|_2^{5/4}\\
&\le C2^{-N/2}\|w^+\cdot\nabla w^-\|_2\|w^+\|_4^{1/2}\||w^+|\nabla w^+\|_2^{5/4}.
\end{aligned} \tag{60}$$

Hence, choosing the integer N such that (59) is equal to (60),

$$2^N\|\pi\|_{\dot{B}^0_{\infty,\infty}}\|w^+\|_4^2 = 2^{-N/2}\|w^+\cdot\nabla w^-\|_2\|w^+\|_4^{1/2}\||w^+|\nabla w^+\|_2^{5/4}, \tag{61}$$

then

$$\begin{aligned}
&\left[\log_2\frac{\|w^+\cdot\nabla w^-\|_2^{2/3}\||w^+|\nabla w^+\|_2^{5/6}}{\|\pi\|_{\dot{B}^0_{\infty,\infty}}^{2/3}\|w^+\|_4}\right]\\
&\quad \le N \le \left[\log_2\frac{\|w^+\cdot\nabla w^-\|_2^{2/3}\||w^+|\nabla w^+\|_2^{5/6}}{\|\pi\|_{\dot{B}^0_{\infty,\infty}}^{2/3}\|w^+\|_4}\right] + 1.
\end{aligned} \tag{62}$$

Plugging the estimates (59) and (60) into (32), we have

$$\begin{aligned}
I &\le C2^{\log_2\left(\|w^+\cdot\nabla w^-\|_2^{2/3}\||w^+|\nabla w^+\|_2^{5/6}/\|\pi\|_{\dot{B}^0_{\infty,\infty}}^{2/3}\|w^+\|_4\right)}\|\pi\|_{\dot{B}^0_{\infty,\infty}}\|w^+\|_4^2\\
&\le C\|\pi\|_{\dot{B}^0_{\infty,\infty}}^{1/3}\|w^+\|_4\|w^+\cdot\nabla w^-\|_2^{2/3}\||w^+|\nabla w^+\|_2^{5/6}\\
&\le C\left\{\|\pi\|_{\dot{B}^0_{\infty,\infty}}^{1/3}\|w^+\|_4\right\}^4 + \frac{1}{4}\left\{\|w^+\cdot\nabla w^-\|_2^{2/3}\||w^+|\nabla w^+\|_2^{5/6}\right\}^{4/3}\\
&\le C\|\pi\|_{\dot{B}^0_{\infty,\infty}}^{4/3}\|w^+\|_4^4 + \frac{1}{4}\|w^+\cdot\nabla w^-\|_2^{8/9}\||w^+|\nabla w^+\|_2^{10/9}\\
&\le C\|\pi\|_{\dot{B}^0_{\infty,\infty}}^{4/3}\|w^+\|_4^4 + \frac{1}{8}\|w^+\cdot\nabla w^-\|_2^2 + \frac{1}{8}\||w^+|\nabla w^+\|_2^2.
\end{aligned} \tag{63}$$

Using the similar way with the estimate II, we see that

$$II \le C\|\pi\|_{\dot{B}^0_{\infty,\infty}}^{4/3}\|w^-\|_4^4 + \frac{1}{8}\|w^-\cdot\nabla w^+\|_2^2 + \frac{1}{8}\||w^-|\nabla w^-\|_2^2. \tag{64}$$

Combination of (63), (64), and (29) implies that

$$\begin{aligned}
&\frac{1}{4}\frac{d}{dt}\left(\|w^+\|_4^4 + \|w^-\|_4^4\right) + \frac{1}{4}\int_{R^3}\left(\left|\nabla|w^+|^2\right|^2 + \left|\nabla|w^-|^2\right|^2\right)dx\\
&\quad + \int_{R^3}\left(|\nabla|w^+||^2|w^+|^2 + |\nabla|w^-||^2|w^-|^2\right)dx\\
&\quad = C\|\pi\|_{\dot{B}^0_{\infty,\infty}}^{4/3}\left(\|w^+\|_4^4 + \|w^-\|_4^4\right)\\
&\quad + \frac{1}{4}\left(\|w^-\cdot\nabla w^+\|_2^2 + \|w^-\cdot\nabla w^+\|_2^2\right).
\end{aligned} \tag{65}$$

Using the same triangle inequality

$$\begin{aligned} \|u\|_r &= \frac{1}{2}\|w^+ + w^-\|_r \le \frac{1}{2}\left(\|w^+\|_4 + \|w^-\|_r\right), \\ \|b\|_r &= \frac{1}{2}\|w^+ - w^-\|_r \le \frac{1}{2}\left(\|w^+\|_4 + \|w^-\|_r\right), \\ \|w^+\|_r &= \|u+b\|_r \le \|u\|_r + \|b\|_r, \\ \|w^-\|_r &= \|u-b\|_r \le \|u\|_r + \|b\|_r, \end{aligned} \tag{66}$$

we obtain

$$\begin{aligned} &\frac{1}{4}\frac{d}{dt}\left(\|u\|_4^4 + \|b\|_4^4\right) + \frac{1}{4}\int_{R^3}\left(\left|\nabla|u|^2\right|^2 + \left|\nabla|b|^2\right|^2\right)dx \\ &\quad + \int_{R^3}\left(|\nabla|u||^2|u|^2 + |\nabla|b||^2|b|^2 + |\nabla|u||^2|b|^2 + |\nabla|b||^2|u|^2\right)dx \\ &\le C\|\pi\|_{\dot{B}^0_{\infty,\infty}}^{4/3}\left(\|u\|_4^4 + \|b\|_4^4\right) + \frac{1}{2} \\ &\quad \cdot\left(\|u\cdot\nabla u\|_2^2 + \|u\cdot\nabla b\|_2^2 + \|b\cdot\nabla b\|_2^2 + \|b\cdot\nabla u\|_2^2\right). \end{aligned} \tag{67}$$

Hence,

$$\frac{1}{4}\frac{d}{dt}\left(\|u\|_4^4 + \|b\|_4^4\right) \le C\|\pi\|_{\dot{B}^0_{\infty,\infty}}^{4/3}\left(\|u\|_4^4 + \|b\|_4^4\right). \tag{68}$$

By Gronwall's inequality and (4), Theorem 4 is proven.

Data Availability

No data.

Conflicts of Interest

The authors declare that they have no conflict of interest.

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