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Research Article

Global Existence and General Decay of Solutions for a Quasilinear System with Degenerate Damping Terms

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In this work, we consider a quasilinear system of viscoelastic equations with degenerate damping, dispersion, and source terms under Dirichlet boundary condition. Under some restrictions on the initial datum and standard conditions on relaxation functions, we study global existence and general decay of solutions. The results obtained here are generalization of the previous recent work.

1. Introduction

Let Ω be a bounded domain with a sufficiently smooth boundary in $R^n (n \geq 1)$. We investigate a quasilinear system of two viscoelastic equations in the presence of degenerate damping, dispersion, and source terms, namely,

$$\begin{cases} |u_t|^\eta u_{tt} - \Delta u + \int_0^t h_1(t-s)\Delta u(s)ds - \Delta u_{tt} + \left(|u|^k + |v|^l\right)|u_t|^{j-1}u_t = f_1(u,v), (x,t) \in \Omega \times (0,T), \\ |v_t|^\eta v_{tt} - \Delta v +{}_0^t h_2(t-s)\Delta v(s)ds - \Delta v_{tt} + \left(|v|^\theta + |u|^\varrho\right)|v_t|^{s-1}v_t = f_2(u,v), (x,t) \in \Omega \times (0,T), \\ u(x,t) = v(x,t) = 0, (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), x \in \Omega, \\ v(x,0) = v_0(x), v_t(x,0) = v_1(x), x \in \Omega, \end{cases} \tag{1}$$

where, $s \geq 1, \eta > 0, k, l, \theta, \varrho \geq 0$, and $h_i(.): R^+ \longrightarrow R^+ (i = 1, 2)$ are positive relaxation functions which will be specified later. $(|(.)|^a + |(.)|^b)|(.)_t|^{\tau-1}(.)_t$ and $-\Delta(.)_{tt}$ are the degenerate damping term and the dispersion term, respectively.

By taking

$$\begin{aligned} f_1(u,v) &= a|u+v|^{2(\kappa+1)}(u+v) + b|u|^\kappa u|v|^{\kappa+2}, \\ f_2(u,v) &= a|u+v|^{2(\kappa+1)}(u+v) + b|v|^\kappa v|u|^{\kappa+2}, \end{aligned} \tag{2}$$

in which $a > 0, b > 0$, and

$$1 < \kappa < +\infty \text{if } n = 1, 2 \text{ and } 1 < \kappa \leq \frac{3-n}{n-2} \text{ if } n \geq 3. \quad (3)$$

It is simple to show that

$$uf_1(u, v) + vf_2(u, v) = 2(\kappa + 2)F(u, v), \forall (u, v) \in R^2, \quad (4)$$

where

$$F(u, v) = \frac{1}{2(\kappa + 2)} \left[a|u + v|^{2(\kappa+2)} + 2b|uv|^{\kappa+2} \right]. \quad (5)$$

To motivate our problem (1), it can trace back to the initial boundary value problem for the single viscoelastic equation of the form

$$|u_t|^\eta u_{tt} - \Delta u + \int_0^t h(t-s)\Delta u(s)ds - \Delta u_{tt} + g(u, u_t) = f(u). \quad (6)$$

This type problem appears a variety of mathematical models in applied science. For instance, in the theory of viscoelasticity, physics, and material science, problem (5) has been studied by various authors, and several results concerning blow-up and energy decay have been studied case ($\eta \geq 0$). For example, Liu [1] studied a general decay of solutions case ($g(u, u_t) = 0$). Messaoudi and Tatar [2] applied the potential well method to indicate the global existence and uniform decay of solutions ($g(u, u_t) = 0$ instead of Δu_t). Furthermore, the authors obtained a blow-up result for positive initial energy. Wu [3] studied a general decay of solution case ($g(u, u_t) = |u_t|^m u_t$). Later, Wu [4] studied the same problem case ($g(u, u_t) = u_t$) and discussed the decay rate of solution energy. Recently, Yang et al. [5] proved the existence of global solution and asymptotic stability result without restrictive conditions on the relaxation function at infinity case ($f(u) = \sigma(x, t)W_t(t, x)$).

In case $g(u, u_t) = 0$ and without dispersion term, problem (5) has been investigated by Song [6], and the blow-up result for positive initial energy has been proved.

For a coupled system, He [7] investigated the following problem

$$\begin{cases} |u_t|^\eta u_{tt} - \Delta u + \int_0^t h_1(t-s)\Delta u(s)ds - \Delta u_{tt} + |u_t|^{j-2}u_t = f_1(u, v), \\ |v_t|^\eta v_{tt} - \Delta v + \int_0^t h_2(t-s)\Delta v(s)ds - \Delta v_{tt} + |v_t|^{s-2}v_t = f_2(u, v), \end{cases} \quad (7)$$

where $\eta > 0, j, s \geq 2$. The author proved general and optimal decay of solutions. Then, in [8], the author investigated the same problem without damping term and established a general decay of solutions. Furthermore, the author obtained a blow-up of solutions for negative initial energy. In addition, problem (1) with in case $\eta = 0$ and without dispersion term, Wu [9] proved a general decay of solutions. Later, Pişkin and Ekinci [10] studied a general decay and blow-up of solutions with nonpositive initial energy for problem (1) case (Kirchhoff-type instead of Δu and without dispersion term). In recent years, some other authors investigate the hyperbolic type system with degenerate damping term (see [11–14]).

The rest of the paper is arranged as follows: in Section 2, as preliminaries, we give necessary assumptions and lemmas that will be used later and local existence theorem without proof. In Section 3, we prove the global existence of solution. In the last section, we studied the general decay of solutions.

2. Preliminaries

We begin this section with some assumptions, notations, lemmas, and theorems. Denote the standart $L^2(\Omega)$ norm by $\|.\| = \|.\|_{L^2(\Omega)}$ and $L^p(\Omega)$ norm by $\|.\|_p = \|.\|_{L^p(\Omega)}$.

To state and prove our result, we need some assumptions:

(A1) Regarding $h_i : [0, \infty) \longrightarrow (0, \infty), (i = 1, 2)$ is C^1 functions and satisfies

$$h_i(\alpha) > 0, h_i'(\alpha) \leq 0, 1 - {}_0^\infty h_i(\alpha)d\alpha = l_i > 0, \alpha \geq 0 \quad (8)$$

and nonincreasing differentiable positive C^1 functions ς_1 and ς_2 such that

$$h_i'(t) \leq -\varsigma_i(t)h_i(t), t \geq 0, i = 1, 2. \quad (9)$$

(A2) For the nonlinearity, we assume that

$$\begin{cases} 1 \leq j, s \text{ if } n = 1, 2, \\ 1 \leq j, s \leq \dfrac{n+2}{n-2} \text{ if } n \geq 3. \end{cases} \quad (10)$$

(A3) Assume that η satisfies

$$\begin{cases} 0 < \eta \text{ if } n = 1, 2, \\ 0 < \eta \leq \dfrac{2}{n-2} \text{ if } n \geq 3. \end{cases} \quad (11)$$

In addition, we present some notations:

$$(\alpha \diamond \nabla w)(t) = \int_0^t \alpha(t-s)\|\nabla w(t) - \nabla w(s)\|^2 ds, \quad l = \min\{l_1, l_2\}. \quad (12)$$

Lemma 1 (Sobolev-Poincare inequality) [15]. *Let q be a number with $2 \leq q < \infty (n = 1, 2)$ or $2 \leq q \leq 2n/(n-2) (n \geq 3)$, and then there is a constant $C_* = C_*(\Omega, q)$ such that*

$$\|u\|_q \leq C_* \|\nabla u\| \textit{ for } u \in H_0^1(\Omega). \quad (13)$$

Now, we state the local existence theorem that can be established by combining arguments of [7, 10].

Theorem 2. *Assume that (A1)-(A3) and (2) hold. Let $u_0, v_0 \in H_0^1(\Omega)$ and $u_1, v_1 \in L^2(\Omega)$ be given. Then, for some $T > 0$,* *problem (1) has a unique local weak solution in the following class:*

$$u, v \in C([0, T]; H^2(\Omega) \cap H_0^1(\Omega)),$$
$$u_t \in C([0, T]; H_0^1(\Omega)) \cap L^{j+1}(\Omega), \quad (14)$$
$$v_t \in C([0, T]; H_0^1(\Omega)) \cap L^{s+1}(\Omega).$$

We define the energy function as follows:

$$\begin{aligned} E(t) = {} & \frac{1}{\eta+2}\left(\|u_t\|_{\eta+2}^{\eta+2} + \|v_t\|_{\eta+2}^{\eta+2}\right) + \frac{1}{2}[(h_1 \diamond \nabla u)(t) \\ & + (h_2 \diamond \nabla v)(t) + \|\nabla u_t\|^2 + \|\nabla v_t\|^2] \\ & + \frac{1}{2}\left[\left(1 - \int_0^t h_1(s)ds\right)\|\nabla u(t)\|^2\right. \\ & \left. + \left(1 - \int_0^t h_2(s)ds\right)\|\nabla v(t)\|^2\right] - \int_\Omega F(u, v)dx. \end{aligned} \quad (15)$$

Also, we define

$$\begin{aligned} I(t) = {} & \|\nabla u_t\|^2 + \|\nabla v_t\|^2 + \left(1 - \int_0^t h_1(s)ds\right)\|\nabla u(t)\|^2 \\ & + \left(1 - \int_0^t h_2(s)ds\right)\|\nabla v(t)\|^2 + (h_1 \diamond \nabla u)(t) \\ & + (h_2 \diamond \nabla v)(t) - 2(\kappa+2)\int_\Omega F(u, v)dx, \\ J(t) = {} & \frac{1}{2}\left[\left(1 - \int_0^t h_1(s)ds\right)\|\nabla u(t)\|^2 + \left(1\int_0^t h_2(s)ds\right)\right. \\ & \left. \cdot \|\nabla v(t)\|^2\right] + \frac{1}{2}[(h_1 \diamond \nabla u)(t) + (h_2 \diamond \nabla v)(t) \\ & + \|\nabla u_t\|^2 + \|\nabla v_t\|^2] - \int_\Omega F(u, v)dx. \end{aligned} \quad (16)$$

By computation, we get

$$\begin{aligned} \frac{d}{dt}E(t) \le {} & \frac{1}{2}\left[\left(h_1' \diamond \nabla u\right)(t) + \left(h_2' \diamond \nabla v\right)(t)\right] \\ & - \frac{1}{2}\left(h_1(t)\|\nabla u\|^2 + h_2(t)\|\nabla v\|^2\right) \\ & - \int_\Omega \left(|u|^k + |v|^l\right)|u_t|^{j+1}dx \\ & - \int_\Omega \left(|v|^\theta + |u|^\varrho\right)|v_t|^{s+1}dx \le 0. \end{aligned} \quad (17)$$

3. Global Existence

In this part, in order to state and prove the global existence of solution (1), we firstly give two lemmas.

Lemma 3 [16]. *Assume that (4) holds. Then, there exist $\rho > 0$ such that for the solution (u, v),*

$$\|u+v\|_{2(\kappa+2)}^{2(\kappa+2)} + 2\|uv\|_{\kappa+2}^{\kappa+2} \le \rho\left(l_1\|\nabla u\|^2 + l_2\|\nabla v\|^2\right)^{\kappa+2}. \quad (18)$$

Lemma 4. *Let $u_0, v_0 \in H_0^1(\Omega)$, $u_1, v_1 \in L^2(\Omega)$. Suppose that (A1)-(A3) hold. If*

$$I(0) > 0 \text{ and } \beta = \rho\left(\frac{2(\kappa+2)}{\kappa+1}E(0)\right)^{\kappa+1} < 1, \quad (19)$$

then

$$I(t) > 0, \forall t > 0. \quad (20)$$

Proof. We have $I(0) > 0$ and by continuity of $I(t)$ about t, there exist a maximal time $t_m > 0$ such that

$$I(t) \ge 0, \text{ on } t \in [0, t_m]. \quad (21)$$

Let t_0 be as follows:

$$\{I(t_0) = 0 \text{ and } I(t) > 0, \text{ for all } 0 \le t < t_0\}. \quad (22)$$

By using (8), (9), and (A1), we get

$$\begin{aligned} J(t) = {} & \frac{\kappa+1}{2(\kappa+2)}\left\{\left(1 - \int_0^t h_1(s)ds\right)\|\nabla u(t)\|^2\right. \\ & + \left(1 - \int_0^t h_2(s)ds\right)\|\nabla v(t)\|^2 + (h_1 \diamond \nabla u)(t) \\ & \left. + (h_2 \diamond \nabla v)(t) + \|\nabla u_t\|^2 + \|\nabla v_t\|^2\right\} + \frac{1}{2(\kappa+2)}I(t) \\ \ge {} & \frac{\kappa+1}{2(\kappa+2)}\{l_1\|\nabla u(t)\|^2 + l_2\|\nabla v(t)\|^2 + (h_1 \diamond \nabla u)(t) \\ & + (h_2 \diamond \nabla v)(t) + \|\nabla u_t\|^2 + \|\nabla v_t\|^2\}. \end{aligned} \quad (23)$$

From (7) and (10), we have

$$\begin{aligned} & l_1\|\nabla u(t)\|^2 + l_2\|\nabla v(t)\|^2 \\ & \le \frac{2(\kappa+2)}{\kappa+1}J(t) \le \frac{2(\kappa+2)}{\kappa+1}E(t) \\ & \le \frac{2(\kappa+2)}{\kappa+1}E(0), \forall t \in [0, t_0]. \end{aligned} \quad (24)$$

By (11) and (12), we infer that

$$\begin{aligned} & 2(\kappa+2)\int_\Omega F(u(t_0), v(t_0))dx \\ & \le \rho\left(l_1\|\nabla u(t_0)\|^2 + l_2\|\nabla v(t_0)\|^2\right)^{\kappa+2} \\ & \le \rho\left(\frac{2(\kappa+2)}{\kappa+1}E(0)\right)^{\kappa+1}\left(l_1\|\nabla u(t_0)\|^2 + l_2\|\nabla v(t_0)\|^2\right) \\ & \le \rho\left(l_1\|\nabla u(t_0)\|^2 + l_2\|\nabla v(t_0)\|^2\right) \\ & \le \left(1 - \int_0^t h_1(s)ds\right)\|\nabla u(t)\|^2 + \left(1 - \int_0^t h_2(s)ds\right) \\ & \cdot \|\nabla v(t)\|^2. \end{aligned} \quad (25)$$

Thus, from (8), we obtain

$$I(t_0) > 0, \tag{26}$$

which contradicts to (13). Thus, $I(t) > 0$ on $[0, T]$.

Theorem 5. *Suppose that the conditions of Lemma 4 hold, then the solution (1) is bounded and global in time.*

Proof. We have

$$\begin{aligned} E(0) \geq E(t) &= J(t) + \frac{1}{\eta+2}\left(\|u_t\|_{\eta+2}^{\eta+2} + \|v_t\|_{\eta+2}^{\eta+2}\right) \\ &\geq \frac{\kappa+1}{2(\kappa+2)}\left(l_1\|\nabla u(t)\|^2 + l_2\|\nabla v(t)\|^2\right. \\ &\quad + \|\nabla u_t\|^2 + \|\nabla v_t\|^2 + (h_1\diamond\nabla u)(t) \\ &\quad \left.+ (h_2\diamond\nabla v)(t)\right) + \frac{1}{\eta+2}\left(\|u_t\|_{\eta+2}^{\eta+2} + \|v_t\|_{\eta+2}^{\eta+2}\right). \end{aligned} \tag{27}$$

Thus,

$$\|\nabla u(t)\|^2 + \|\nabla v(t)\|^2 + \|\nabla u_t\|^2 + \|\nabla v_t\|^2 \leq CE(0), \tag{28}$$

where positive constant C depends only on κ, l_1, l_2. This implies that the solution of problem (1) is global in time.

4. General Decay of Solutions

This section is devoted to show the decay of solution (1). Set

$$\Gamma(t) := ME(t) + \varepsilon\Phi(t) + F(t), \tag{29}$$

where M and ε are positive constants and

$$\begin{aligned} \Phi(t) &= \delta_1(t)\left[\frac{1}{\eta+1}\int_\Omega |u_t|^\eta u_t u dx + \int_\Omega \nabla u_t \nabla u dx\right] \\ &\quad + \delta_2(t)\left[\frac{1}{\eta+1}\int_\Omega |v_t|^\eta v_t v dx + \int_\Omega \nabla v_t \nabla v dx\right], \end{aligned}$$

$$\begin{aligned} F(t) &= \delta_1(t)\left[\int_\Omega\left(\Delta u_t - \frac{|u_t|^\eta u_t}{\eta+1}\right)\int_0^t h_1(t-s)\right. \\ &\quad \left.\cdot (u(t)-u(s))dsdx\right] + \delta_2(t)\left[\int_\Omega\left(\Delta v_t - \frac{|v_t|^\eta v_t}{\eta+1}\right)\right. \\ &\quad \left.\cdot \int_0^t h_2(t-s)(v(t)-v(s))dsdx\right]. \end{aligned} \tag{30}$$

Lemma 6. *For ε which is small enough while M is large enough, the relation*

$$\alpha_1\Gamma(t) \leq E(t) \leq \alpha_2\Gamma(t), \forall t \geq 0 \tag{31}$$

holds for two positive constants α_1 and α_2.

Proof. As references [1, 10], it can be show easily that $\Gamma(t)$ and $E(t)$ are equivalent in the sense that α_1 and α_2 are positive constants, depending on ε and M.

Lemma 7 [3]. *Assume that (12) holds. Let (u, v) be the solution of problem (1). Then, for $\sigma \geq 0$, we get*

$$\begin{cases} \displaystyle\int_\Omega\left(\int_0^t h_1(t-s)(u(t)-u(s))ds\right)^{\sigma+2} dx \leq (1-l_1)^{\sigma+1}c_*^{\sigma+2}\left(\frac{2(\kappa+2)E(0)}{l_1(\kappa+1)}\right)^{\sigma/2}(h_1\diamond\nabla u)(t), \\ \displaystyle\int_\Omega\left(\int_0^t h_2(t-s)(v(t)-v(s))ds\right)^{\sigma+2} dx \leq (1-l_1)^{\sigma+1}c_*^{\sigma+2}\left(\frac{2(\kappa+2)E(0)}{l_2(\kappa+1)}\right)^{\sigma/2}(h_2\diamond\nabla v)(t). \end{cases} \tag{32}$$

Lemma 8 [16]. *Let (A1)-(A3) hold. Assume that $u_0, v_0 \in H_0^1(\Omega)$, $u_1, v_1 \in L^2(\Omega)$, be given and satisfying (12). Then, throughout the solution (u, v) of (1), there exist two positive constants β_1 and β_2 such that for any $\delta > 0$ and for all $t \geq 0$,*

$$\begin{aligned} &\int_\Omega f_1(u,v)\int_0^t h_1(t-s)(u(t)-u(s))dsdx \\ &\quad \leq \beta_1\delta\left(l_1\|\nabla u\|^2 + l_2\|\nabla v\|^2\right) + \frac{(1-l_1)c_*^2(h_1\diamond\nabla u)(t)}{4\delta}, \\ &\int_\Omega f_2(u,v)\int_0^t h_2(t-s)(v(t)-v(s))dsdx \\ &\quad \leq \beta_2\delta\left(l_1\|\nabla u\|^2 + l_2\|\nabla v\|^2\right) + \frac{(1-l_2)c_*^2(h_2\diamond\nabla u)(t)}{4\delta}. \end{aligned} \tag{33}$$

Lemma 9. *Let $u_0, v_0 \in H_0^1(\Omega)$, $u_1, v_1 \in L^2(\Omega)$, be given and satisfying (12). Suppose that (A1)-(A3) hold. Then, for each $t_0 > 0$, the functional $\Gamma(t)$ verifies, throughout the solution of (1)*

$$\Gamma'(t) \leq -\xi_1 E(t) + \xi_2[(h_1\diamond\nabla u)(t) + (h_2\diamond\nabla v)(t)], t \geq t_0, \tag{34}$$

where $\xi_i > 0, (i = 1, 2)$.

Proof. By applying (18) and Eq.(1) and getting $\delta_i \equiv 1 (i = 1, 2)$ in (18), we have

$$\Phi'(t) \leq \frac{1}{\eta+1}\left(\|u_t\|_{\eta+2}^{\eta+2} + \|v_t\|_{\eta+2}^{\eta+2}\right) - \|\nabla u\|^2 - \|\nabla v\|^2$$

$$+\|\nabla u_t\|^2+\|\nabla v_t\|^2\int_\Omega \nabla u(t)\int_0^t h_1(t-s)\nabla u(s)dsdx + \int_\Omega \nabla v(t)\int_0^t h_2(t-s)\nabla v(s)dsdx + 2(\kappa+2) \cdot \int_\Omega F(u,v)dx - \int_\Omega u\left(|u|^k+|v|^l\right)u_t|u_t|^{j-1}dx - \int_\Omega v\left(|v|^\theta+|u|^\rho\right)v_t|v_t|^{s-1}dx. \tag{35}$$

For estimating the seventh term in the right side of (22) as follows (see [17]):

$$\int_\Omega \nabla u(t)\int_0^t h_1(t-s)\nabla u(s)dsdx \le \frac{1}{2}\|\nabla u\|^2+\frac{1}{2}\int_\Omega\left(\int_0^t h_1(t-s)(|\nabla u(s) -\nabla u(t)|+|\nabla u(t)|)ds\right)^2dx. \tag{36}$$

By exploiting Young's inequality and the assumption that $\int_0^t h_1(s)ds \le \int_0^\infty h_1(s)ds \le 1-l_1$, for $\gamma_1>0$,

$$\begin{aligned}\int_\Omega \nabla u(t)\int_0^t h_1(t-s)\nabla u(s)dsdx &\le \frac{1}{2}\|\nabla u\|^2+\frac{1}{2}(1+\gamma_1)\int_\Omega\left(\int_0^t h_1(t-s)|\nabla u(s)|ds\right)^2dx \\ &\quad+\frac{1}{2}\left(1+\frac{1}{\gamma_1}\right)\int_\Omega\left(\int_0^t h_1(t-s)|\nabla u(s)-\nabla u(t)|ds\right)^2dx \\ &\le \frac{1}{2}\|\nabla u\|^2+\frac{1}{2}(1+\gamma_1)\left(\int_0^t h_1(s)ds\right)^2\|\nabla u\|^2 \\ &\quad+\frac{1}{2}\left(1+\frac{1}{\gamma_1}\right)\left(\int_0^t h_1(s)ds\right)(h_1\diamond\nabla u)(t) \\ &\le \frac{1+(1+\gamma_1)(1-l_1)^2}{2}\|\nabla u\|^2 + \frac{(1+(1/\gamma_1))(1-l_1)}{2}(h_1\diamond\nabla u)(t).\end{aligned} \tag{37}$$

Similarly with $\gamma_2>0$,

$$\int_\Omega \nabla v(t)\int_0^t h_2(t-s)\nabla v(s)dsdx \le \frac{1+(1+\gamma_2)(1-l_2)^2}{2}\|\nabla v\|^2 + \frac{(1+(1/\gamma_2))(1-l_2)}{2}(h_2\diamond\nabla v)(t). \tag{38}$$

By estimating the following terms in (22), we have

$$\int_\Omega u\left(|u|^k+|v|^l\right)u_t|u_t|^{j-1}dx \le {}_\Omega|u|^{k+1}|u_t|^j dx + \int_\Omega |u||v|^l|u_t|^j dx. \tag{39}$$

Exploiting Young's inequality, Hölder's inequality, Sobolev-Poincare inequality, (A3), and (15) for $\beta_1>0$, one has

$$\begin{aligned}\int_\Omega |u|^{k+1}|u_t|^j dx &\le \left(\int_\Omega |u|^k|u_t|^{j+1}dx\right)^{j/j+1}\left(\int_\Omega |u|^{k+j+1}dx\right)^{j/j+1} \\ &\le \frac{j\beta_1^{-j+1/j}}{j+1}\int_\Omega |u|^k|u_t|^{j+1}dx+\frac{\beta_1^{j+1}C_*^{k+j+1}}{j+1}\|\nabla u\|^{k+j+1} \\ &\le \frac{j\beta_1^{-j+1/j}}{j+1}\int_\Omega\left(|u|^k+|v|^l\right)|u_t|^{j+1}dx + \frac{\beta_1^{j+1}C_*^{k+j+1}\chi_1^{k+j-1/2}}{j+1}\|\nabla u\|^2, \\ \left|\int_\Omega |u||v|^l|u_t|^j dx\right| &\le \left(\int_\Omega |v|^l|u_t|^{j+1}dx\right)^{j/j+1}\left(\int_\Omega |v|^l|u|^{j+1}dx\right)^{j/j+1} \\ &\le \frac{j\beta_1^{-j+1/j}}{j+1}\int_\Omega\left(|u|^k+|v|^l\right)|u_t|^{j+1}dx + \frac{\beta_1^{j+1}}{2(j+1)}\left(\|v\|_{2l}^{2l}+\|u\|_{2(j+1)}^{2(j+1)}\right) \\ &\le \frac{j\beta_1^{-j+1/j}}{j+1}\int_\Omega\left(|u|^k+|v|^l\right)|u_t|^{j+1}dx + \frac{\beta_1^{j+1}C_*^{2l}\chi_2^{l-1}}{2(j+1)} \cdot \left(\|\nabla v\|^2\right)+\frac{\beta_1^{j+1}C_*^{2j+2}\chi_1^j}{2(j+1)}\left(\|\nabla u\|^2\right),\end{aligned} \tag{40}$$

where

$$\chi_1=\frac{2(\kappa+2)E(0)}{l_1(\kappa+1)} \text{ and } \chi_2=\frac{2(\kappa+2)E(0)}{l_2(\kappa+1)}. \tag{41}$$

By inserting (27) and (28) into (26), we have

$$\begin{aligned}\int_\Omega u\left(|u|^k+|v|^l\right)u_t|u_t|^{j-1}dx &\le \frac{2j\beta_1^{-j+1/j}}{j+1}\int_\Omega\left(|u|^k+|v|^l\right)|u_t|^{j+1}dx \\ &\quad+\frac{\beta_1^{j+1}C_*^{2l}\chi_2^{l-1}}{2(j+1)}\left(\|\nabla v\|^2\right)+\frac{\beta_1^{j+1}}{j+1} \cdot\left(C_*^{k+j+1}\chi_1^{(k+j-1)/2}+\frac{C_*^{2j+2}\chi_1^j}{2}\right)\|\nabla u\|^2.\end{aligned} \tag{42}$$

Similarly, for $\beta_2 > 0$, we have

$$\begin{aligned}
&\int_\Omega v\left(|v|^\theta + |u|^\rho\right) v_t |v_t|^{s-1} dx \\
&\quad \le \frac{2s\beta_2^{-s+1/s}}{s+1}\int_\Omega \left(|v|^\theta + |u|^\rho\right)|v_t|^{s+1} dx \\
&\qquad + \frac{\beta_2^{s+1} C_*^{2\rho}\chi_1^{\rho-1}}{2(s+1)}\left(\|\nabla u\|^2\right) + \frac{\beta_2^{s+1}}{s+1} \\
&\qquad \cdot \left(C_*^{\theta+s+1}\chi_2^{(\theta+s-1)/2} + \frac{C_*^{2s+2}\chi_2^s}{2}\right)\|\nabla v\|^2.
\end{aligned} \tag{43}$$

Thus, inserting (24) and (25) and (29) and (30) into (22), we obtain

$$\begin{aligned}
\Phi'(t) \le{}& \frac{1}{\eta+1}\left(\|u_t\|_{\eta+2}^{\eta+2} + \|v_t\|_{\eta+2}^{\eta+2}\right) - a_1\|\nabla u\|^2 \\
&- a_2\|\nabla v\|^2 + 2(\kappa+2)\int_\Omega F(u,v)dx + \|\nabla u_t\|^2 \\
&+ \|\nabla v_t\|^2 + \frac{(1+(1/\gamma_1))(1-l_1)}{2}(h_1\diamond\nabla u)(t) \\
&+ \frac{(1+(1/\gamma_2))(1-l_2)}{2}(h_2\diamond\nabla v)(t) \\
&+ \frac{2j\beta_1^{-j+1/j}}{j+1}\int_\Omega\left(|u|^k + |v|^l\right)|u_t|^{j+1}dx \\
&+ \frac{2s\beta_2^{-s+1/s}}{s+1}\int_\Omega\left(|v|^\theta + |u|^\rho\right)|v_t|^{s+1}dx,
\end{aligned} \tag{44}$$

where

$$\begin{aligned}
a_1 ={}& \frac{1-(1+\gamma_1)(1-l_1)^2}{2} - \frac{\beta_1^{j+1}}{j+1} \\
&\cdot\left(C_*^{k+j+1}\chi_1^{(k+j-1)/2} + \frac{C_*^{2j+2}\chi_1^j}{2}\right) - \frac{\beta_2^{s+1}C_*^{2\rho}\chi_1^{\rho-1}}{2(s+1)}, \\
a_2 ={}& \frac{1-(1+\gamma_2)(1-l_2)^2}{2} - \frac{\beta_2^{s+1}}{s+1} \\
&\cdot\left(C_*^{\theta+s+1}\chi_2^{(\theta+s-1)/2} + \frac{C_*^{2s+2}\chi_2^s}{2}\right) - \frac{\beta_1^{j+1}C_*^{2l}\chi_2^{l-1}}{2(j+1)}.
\end{aligned} \tag{45}$$

At this moment, choosing $\gamma_1 = l_1/1-l_1, \gamma_2 = l_2/1-l_2$, and picking β_1 and β_2 small enough such that

$$\begin{aligned}
&\frac{\beta_1^{j+1}}{j+1}\left(C_*^{k+j+1}\chi_1^{(k+j-1)/2} + \frac{C_*^{2j+2}\chi_1^j}{2}\right) + \frac{\beta_2^{s+1}C_*^{2\rho}\chi_1^{\rho-1}}{2(s+1)} \le \frac{l_1}{4}, \\
&\frac{\beta_2^{s+1}}{s+1}\left(C_*^{\theta+s+1}\chi_2^{(\theta+s-1)/2} + \frac{C_*^{2s+2}\chi_2^s}{2}\right) + \frac{\beta_1^{j+1}C_*^{2l}\chi_2^{l-1}}{2(j+1)} \le \frac{l_2}{4}.
\end{aligned} \tag{46}$$

Consequently, (31) yields

$$\begin{aligned}
\Phi'(t) \le{}& \frac{1}{\eta+1}\left(\|u_t\|_{\eta+2}^{\eta+2} + \|v_t\|_{\eta+2}^{\eta+2}\right) - \frac{l_1}{4}\|\nabla u\|^2 \\
&- \frac{l_2}{4}\|\nabla v\|^2 + 2(\kappa+2)\int_\Omega F(u,v)dx \\
&+ \frac{1-l_1}{2l_1}(h_1\diamond\nabla u)(t) + \frac{1-l_2}{2l_2}(h_2\diamond\nabla v)(t) + \|\nabla u_t\|^2 \\
&+ \|\nabla v_t\|^2 + \frac{2j\beta_1^{-j+1/j}}{j+1}\int_\Omega\left(|u|^k + |v|^l\right)|u_t|^{j+1}dx \\
&+ \frac{2s\beta_2^{-s+1/s}}{s+1}\int_\Omega\left(|v|^\theta + |u|^\rho\right)|v_t|^{s+1}dx.
\end{aligned} \tag{47}$$

In order to estimate the $F'(t)$, we set

$$F_1(t) = \int_\Omega\left(\Delta u_t - \frac{1}{\eta+1}|u_t|^\eta u_t\right)\int_0^t h_1(t-s)(u(t)-u(s))dsdx, \tag{48}$$

Then, by using equations (1), we have

$$\begin{aligned}
F_1'(t) ={}& -\int_\Omega\left(\int_0^t h_1(t-s)\nabla u(s)ds\right) \\
&\cdot\left(\int_0^t h_1(t-s)(\nabla u(t)-\nabla u(s))ds\right)dx \\
&+ \int_\Omega\left(|u|^k + |v|^l\right)|u_t|^{j-1}u_t\int_0^t h_1(t-s) \\
&\cdot(u(t)-u(s))dsdx + \int_\Omega \nabla u(t)\int_0^t h_1(t-s) \\
&\cdot(\nabla u(t)-\nabla u(s))dsdx - \int_\Omega \nabla u_t(t)\int_0^t h_1'(t-s) \\
&\cdot(\nabla u(t)-\nabla u(s))dsdx - \frac{1}{\eta+1}\int_\Omega |u_t|^\eta u_t \\
&\cdot\int_0^t h_1'(t-s)(u(t)-u(s))dsdx \\
&- \int_\Omega f_1(u,v)\int_0^t h_1(t-s)(u(t)-u(s))dsdx \\
&- \frac{1}{\eta+1}\left(\int_0^t h_1(s)\right)\|u_t\|_{\eta+2}^{\eta+2} - \left(\int_0^t h_1(s)\right)\|\nabla u_t\|^2.
\end{aligned} \tag{49}$$

For the first term of (33), by applying (A1), Hölder's inequality, and Young's inequality, we deduce

$$\begin{aligned}&\left|-\int_\Omega\left(\int_0^t h_1(t-s)\nabla u(s)ds\right)\left(\int_0^t h_1(t-s)(\nabla u(t)-\nabla u(s))ds\right)dx\right|\\&\quad\le\delta\int_\Omega\left(\int_0^t h_1(t-s)\nabla u(s)ds\right)^2dx\\&\qquad+\frac{1}{4\delta}\int_\Omega\left(\int_0^t h_1(t-s)(\nabla u(t)-\nabla u(s))ds\right)^2dx\\&\quad\le\delta\int_\Omega\left[\int_0^t h_1(t-s)(|\nabla u(s)-\nabla u(t)|+|\nabla u(t)|)ds\right]^2dx\\&\qquad+\frac{1}{4\delta}\left(\int_0^t h_1(s)ds\right)(h_1\diamond\nabla u)(t)\le 2\delta(1-l_1)^2\|\nabla u\|^2\\&\qquad+\left(2\delta+\frac{1}{4\delta}\right)(1-l_1)(h_1\diamond\nabla u)(t),\forall\delta>0.\end{aligned}\tag{50}$$

Then, in order to estimate the following term, we seperate such that

$$\begin{aligned}&\left|\int_\Omega\left(|u|^k+|v|^l\right)|u_t|^{j-1}u_t\int_0^t h_1(t-s)(u(t)-u(s))dsdx\right|\\&\quad=I_1+I_2,\end{aligned}\tag{51}$$

where

$$\begin{aligned}I_1&=\int_\Omega|u|^k|u_t|^{j-1}u_t\int_0^t h_1(t-s)(u(t)-u(s))dsdx,\\I_2&=\int_\Omega|v|^l|u_t|^{j-1}u_t\int_0^t h_1(t-s)(u(t)-u(s))dsdx.\end{aligned}\tag{52}$$

By Hölder's inequality, Young's inequality, (15), and (21), we get

$$\begin{aligned}|I_1|&\le\left(\int_\Omega|u|^k|u_t|^{j+1}dx\right)^{j/j+1}\left(\int_\Omega|u|^k\left(\int_0^t h_1(t-s)\right.\right.\\&\qquad\left.\left.\cdot(u(t)-u(s))ds\right)^{j+1}dx\right)^{1/j+1}\\&\le\frac{j\delta^{-j+1/j}}{j+1}\int_\Omega\left(|u|^k+|v|^l\right)|u_t|^{j+1}dx+\frac{\delta^{j+1}}{j+1}\left(\frac{1}{2}\|u\|_{2k}^{2k}\right.\\&\qquad\left.+\frac{1}{2}\int_\Omega\left(\int_0^t h_1(t-s)(u(t)-u(s))ds\right)^{2(j+1)}dx\right)\\&\le\frac{j\delta^{-j+1/j}}{j+1}\int_\Omega\left(|u|^k+|v|^l\right)|u_t|^{j+1}dx+\frac{\delta^{j+1}}{j+1}\left(\frac{c_*^{2k}}{2}\chi_1^{k-1}\|\nabla u\|^2\right.\\&\qquad\left.+\frac{c_*^{2j+2}(2\chi_1)^j(1-l_1)^{2j+1}}{2}(h_1\diamond\nabla u)(t)\right)\\|I_2|&\le\frac{j\delta^{-j+1/j}}{j+1}\int_\Omega\left(|u|^k+|v|^l\right)|u_t|^{j+1}dx+\frac{\delta^{j+1}}{j+1}\left(\frac{c_*^{2l}}{2}\chi_2^{l-1}\|\nabla v\|^2\right.\\&\qquad\left.+\frac{c_*^{2j+2}(2\chi_1)^j(1-l_1)^{2j+1}}{2}(h_1\diamond\nabla u)(t)\right).\end{aligned}\tag{53}$$

From (A1) assumption, Hölder's inequality, and Young's inequality, we get

$$\begin{aligned}&\left|\int_\Omega\nabla u(t)\int_0^t h_1(t-s)(\nabla u(t)-\nabla u(s))dsdx\right|\\&\quad\le\delta\|\nabla u\|^2+\frac{1-l_1}{4\delta}(h_1\diamond\nabla u)(t).\end{aligned}\tag{54}$$

In order to estimate the forth term, we use Young's inequality, Sobolev-Poincare inequality, Hölder's inequality, and (A1) assumption

$$\begin{aligned}&\left|-\int_\Omega\nabla u_t(t)\int_0^t h_1'(t-s)(\nabla u(t)-\nabla u(s))dsdx\right|\\&\quad\le\delta\|\nabla u_t\|^2-\frac{h_1(0)}{4\delta}\left(h_1'\diamond\nabla u\right)(t).\\&\left|-\frac{1}{\eta+1}\int_\Omega|u_t|^\eta u_t\int_0^t h_1'(t-s)(u(t)-u(s))dsdx\right|\\&\quad\le\frac{1}{\eta+1}\left[\delta\|u_t\|_{2(\eta+1)}^{2(\eta+1)}+\frac{1}{4\delta}\int_\Omega\left(\int_0^t h_1'(t-s)(u(t)-u(s))ds\right)^2dx\right]\\&\quad\le\frac{1}{\eta+1}\left[\delta\|u_t\|_{2(\eta+1)}^{2(\eta+1)}-\frac{h_1(0)c_*^2}{4\delta}\int_\Omega\int_0^t h_1'(t-s)|\nabla u(t)-\nabla u(s)|^2dsdx\right]\\&\quad\le\frac{\delta c_*^{2(\eta+1)}}{\eta+1}\left(\frac{2(\kappa+2)E(0)}{\kappa+1}\right)^\eta\|\nabla u_t\|^2-\frac{h_1(0)c_*^2}{4\delta(\eta+1)}\left(h_1'\diamond\nabla u\right)(t).\end{aligned}\tag{55}$$

Combining these estimates (34)-(40) and (33) becomes

$$\begin{aligned}F_1'(t)&\le\left[\left(2\delta+\frac{1}{2\delta}\right)(1-l_1)+\frac{\delta^{j+1}c_*^{2j+2}(2\chi_1)^j(1-l_1)^{2j+1}}{j+1}\right]\\&\quad\cdot(h_1\diamond\nabla u)(t)+\left(\delta+2\delta(1-l_1)^2+\frac{\delta^{j+1}}{j+1}\frac{c_*^{2k}}{2}\chi_1^{k-1}\right)\\&\quad\cdot\|\nabla u\|^2+\frac{\delta^{j+1}}{j+1}\frac{c_*^{2l}}{2}\chi_2^{l-1}\|\nabla v\|^2-\frac{h_1(0)}{4\delta}\left(\frac{c_*^2}{\eta+1}+1\right)\\&\quad\cdot\left(h_1'\diamond\nabla u\right)(t)+\left(\delta+\frac{\delta c_*^{2(\eta+1)}}{\eta+1}\left(\frac{2(\kappa+2)E(0)}{\kappa+1}\right)^\eta\right.\\&\quad\left.-\int_0^t h_1(s)ds\right)\|\nabla u_t\|^2+\frac{2j\delta^{-j+1/j}}{j+1}\int_\Omega\left(|u|^k+|v|^l\right)\\&\quad\cdot|u_t|^{j+1}dx-\frac{1}{\eta+1}\left(\int_0^t h_1(s)ds\right)\|u_t\|_{\eta+2}^{\eta+2}\\&\quad-\int_\Omega f_1(u,v)\int_0^t h_1(t-s)(u(t)-u(s))dsdx.\end{aligned}\tag{56}$$

Similarly, let

$$F_2(t)={}_\Omega\left(\Delta v_t-\frac{1}{\eta+1}|v_t|^\eta v_t\right)_0^t h_2(t-s)(v(t)-v(s))dsdx,\tag{57}$$

then

$$F_2'(t) \le \left[\left(2\delta + \frac{1}{2\delta}\right)(1-l_2) + \frac{\delta^{s+1} c_*^{2s+2} (2\chi_2)^s (1-l_2)^{2s+1}}{s+1}\right] \cdot (h_2 \diamond \nabla v)(t) + \left(\delta + 2\delta(1-l_2)^2 + \frac{\delta^{s+1}}{s+1}\frac{c_*^{2\theta}}{2}\delta_2^{\theta-1}\right) \cdot \|\nabla v\|^2 + \frac{\delta^{s+1}}{s+1}\frac{c_*^{2\varrho}}{2}\chi_1^{\varrho-1}\|\nabla u\|^2 - \frac{h_2(0)}{4\delta}\left(1+\frac{c_*^2}{\eta+1}\right) \cdot \left(h_2' \diamond \nabla v\right)(t) + \left(\delta + \frac{\delta c_*^{2(\eta+1)}}{\eta+1}\left(\frac{2(\kappa+2)E(0)}{\kappa+1}\right)^{\eta} - \int_0^t h_2(s)ds\right)\|\nabla v_t\|^2 + \frac{2s\delta^{-s+1/s}}{s+1}\int_\Omega \left(|v|^\theta + |u|^\varrho\right) \cdot |v_t|^{s+1}dx - \int_\Omega f_2(u,v)\int_0^t h_2(t-s)(v(t)-v(s))dsdx - \frac{1}{\eta+1}\left(\int_0^t h_2(s)ds\right)\|v_t\|_{\eta+2}^{\eta+2}. \quad (58)$$

Since the function h_i $(i=1,2)$ is positive, then for any $t_0 > 0$,

$$\int_0^t h_i(s)ds \ge \int_0^{t_0} h_i(s)ds = h_i(0) \ge h_3 > 0, \forall t \ge t_0. \quad (59)$$

Hence, we conclude from (17), (10), (32), (41), and (42) that

$$\Gamma'(t) \le -\frac{1}{\eta+1}(h_3-\varepsilon)\left(\|u_t\|_{\eta+2}^{\eta+2} + \|v_t\|_{\eta+2}^{\eta+2}\right) - \left(h_3 - \varepsilon - \delta - \frac{\delta c_*^{2(\eta+1)}}{\eta+1}\left(\frac{2(\kappa+2)E(0)}{\kappa+1}\right)^{\eta}\right)\left(\|\nabla u_t\|^2 + \|\nabla v_t\|^2\right) - \left(\frac{\varepsilon l_1}{4} - c_2\right)\|\nabla u\|^2 - \left(\frac{\varepsilon l_2}{4} - c_3\right)\|\nabla v\|^2 + \left(\frac{\varepsilon(1-l_1)}{2l_1} + c_4\right)(h_1 \diamond \nabla u)(t) + \left(\frac{\varepsilon(1-l_2)}{2l_2} + c_5\right) \cdot (h_2 \diamond \nabla v)(t) + 2\varepsilon(\kappa+2)\int_\Omega F(u,v)dx + \left[\frac{M}{2} - \frac{h_1(0)}{4\delta}\left(\frac{c_*^2}{\eta+1}+1\right)\right]\left(h_1' \diamond \nabla u\right)(t) + \left[\frac{M}{2} - \frac{h_2(0)}{4\delta}\left(\frac{c_*^2}{\eta+1}+1\right)\right]\left(h_2' \diamond \nabla v\right)(t) + \left(\frac{2j\beta_1^{-j+1/j}}{j+1}(\varepsilon+1) - M\right)\int_\Omega \left(|u|^k + |v|^l\right)|u_t|^{j+1}dx + \left(\frac{2s\beta_2^{-s+1/s}}{s+1}(\varepsilon+1) - M\right)\int_\Omega \left(|v|^\theta + |u|^\rho\right)|v_t|^{s+1}dx - \int_\Omega f_1(u,v)\int_0^t h_1(t-s)(u(t)-u(s))dsdx - \int_\Omega f_2(u,v)\int_0^t h_2(t-s)(v(t)-v(s))dsdx, \quad (60)$$

where

$$\begin{aligned} c_2 &= 2\delta(1-l_1)^2 + \frac{\delta^{j+1} c_*^{2k}\chi_1^{k-1}}{2(j+1)} + \frac{\delta^{s+1} c_*^{2\rho}\chi_1^{\rho-1}}{2(s+1)}, \\ c_3 &= 2\delta(1-l_2)^2 + \frac{\delta^{s+1} c_*^{2\theta}\chi_2^{\theta-1}}{2(s+1)} + \frac{\delta^{j+1} c_*^{2l}\chi_2^{l-1}}{2(j+1)}, \\ c_4 &= \left(2\delta + \frac{1}{2\delta}\right)(1-l_1) + \frac{\delta^{j+1} c_*^{2j+2}(2\chi_1)^j (1-l_1)^{2j+1}}{j+1}, \\ c_5 &= \left(2\delta + \frac{1}{2\delta}\right)(1-l_2) + \frac{\delta^{s+1} c_*^{2s+2}(2\chi_2)^s (1-l_2)^{2s+1}}{s+1}. \end{aligned} \quad (61)$$

By using Lemma 8 and (15) for the last two terms of (43), we obtain

$$\Gamma'(t) \le -\frac{1}{\eta+1}(\varepsilon - h_3)\left(\|u_t\|_{\eta+2}^{\eta+2} + \|v_t\|_{\eta+2}^{\eta+2}\right) - \left(\varepsilon + \delta + \frac{\delta c_*^{2(\eta+1)}}{\eta+1}\left(\frac{2(\kappa+2)E(0)}{\kappa+1}\right)^{\eta} - h_3\right) \cdot \left(\|\nabla u_t\|^2 + \|\nabla v_t\|^2\right) - \left(\frac{\varepsilon l_1}{4} - (c_2 + (\beta_1+\beta_2)\delta l_1)\right) \cdot \|\nabla u\|^2 - \left(\frac{\varepsilon l_2}{4} - \varepsilon(c_3 + (\beta_1+\beta_2)\delta l_2)\right)\|\nabla v\|^2 + \left[\left(\frac{\varepsilon}{2l_1} + \frac{c_*^2}{4\delta}\right)(1-l_1) + c_4\right](h_1 \diamond \nabla u)(t) + \left[\left(\frac{\varepsilon}{2l_2} + \frac{c_*^2}{4\delta}\right)(1-l_2) + c_5\right](h_2 \diamond \nabla v)(t) + \left[\frac{M}{2} - \frac{h_1(0)}{4\delta}\left(\frac{c_*^2}{\eta+1}+1\right)\right]\left(h_1' \diamond \nabla u\right)(t) + \left[\frac{M}{2} - \frac{h_2(0)}{4\delta}\left(\frac{c_*^2}{\eta+1}+1\right)\right]\left(h_2' \diamond \nabla u\right)(t) + 2\varepsilon(\kappa+2)\int_\Omega F(u,v)dx + \left(\frac{2j\beta_1^{-j+1/j}}{j+1}(\varepsilon+1) - M\right) \cdot \int_\Omega \left(|u|^k + |v|^l\right)|u_t|^{j+1}dx + \left(\frac{2s\beta_2^{-s+1/s}}{s+1}(\varepsilon+1) - M\right) \cdot \int_\Omega \left(|v|^\theta + |u|^\rho\right)|v_t|^{s+1}dx. \quad (62)$$

At this point, we choose ε and δ which are small enough, and we have

$$\begin{aligned} &\frac{1}{\eta+1}(h_3 - \varepsilon) > 0, \\ &h_3 - \varepsilon - \delta - \frac{\delta c_*^{2(\eta+1)}}{\eta+1}\left(\frac{2(\kappa+2)E(0)}{\kappa+1}\right)^{\eta} > 0, \\ &\frac{\varepsilon l_1}{4} - (c_2 + (\beta_1+\beta_2)\delta l_1) > 0, \\ &\frac{\varepsilon l_2}{4} - (c_2 + (\beta_1+\beta_2)\delta l_2) > 0. \end{aligned} \quad (63)$$

Further, we pick ε so small and

$$\frac{2j\beta_1^{-j+1/j}}{j+1}(\varepsilon+1)-M<0, \frac{2s\beta_2^{-s+1/s}}{s+1}(\varepsilon+1)-M<0. \quad (64)$$

Once δ is fixed, we choose M that is sufficently large so that

$$\frac{M}{2}-\frac{h_1(0)}{4\delta}\left(\frac{c_*^2}{\eta+1}+1\right)\geq 0, \frac{M}{2}-\frac{h_2(0)}{4\delta}\left(\frac{c_*^2}{\eta+1}+1\right)\geq 0. \quad (65)$$

Consequently, for all $t \geq t_0$, we reach at

$$\Gamma'(t) \leq -\xi_1 E(t) + \xi_2((h_1 \diamond \nabla u)(t) + (h_2 \diamond \nabla v)(t)), \quad (66)$$

where there are positive constants ξ_i, $i = 1, 2$.

Now, we are ready to state our stability result.

Theorem 10. *Suppose that (4) and (A1)-(A3) hold, and that* $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$ *and* $(v_0, v_1) \in H_0^1(\Omega) \times L^2(\Omega)$ *satisfy* $E(0) < E_1$ *and*

$$\left(l_1\|\nabla u_0\|^2 + l_2\|\nabla v_0\|^2\right)^{1/2} < \alpha_*. \quad (67)$$

Then for each, the energy of (1) satisfies

$$E(t) \leq K e^{-k\int_{t_0}^t \delta(s)ds}, t \geq t_0, \quad (68)$$

where $\delta(t) := \min\{\delta_1(t), \delta_2(t)\}$ *and* K *and* k *are positive constants.*

Proof. Multiplying (46) by $\delta(t)$, we get

$$\delta(t)\Gamma'(t) \leq -\xi_1\delta(t)E(t) + \xi_2\delta(t)[(h_1 \diamond \nabla u)(t) + (h_2 \diamond \nabla v)(t)]. \quad (69)$$

Applying (A2) and $\delta(t) := \min\{\delta_1(t), \delta_2(t)\}$ and since $-[(h_1' \diamond \nabla u)(t) + (h_2' \diamond \nabla v)(t)] \leq -2E'(t)$ by (10), we obtain

$$\begin{aligned}\delta(t)\Gamma'(t) &\leq -\xi_1\delta(t)E(t) - \xi_2\delta(t)\left[\left(h_1' \diamond \nabla u\right)(t) + \left(h_2' \diamond \nabla v\right)(t)\right] \\ &\leq -\xi_1\delta(t)E(t) - 2\xi_2 E'(t), \forall t \geq t_0.\end{aligned} \quad (70)$$

That is

$$G'(t) \leq -c_*\delta(t)E(t) \leq -k\delta(t)G(t), \forall t \geq t_0. \quad (71)$$

And here, $G(t) = \delta(t)\Gamma(t) + CE(t)$ is equivalent to $E(t)$ due to (20), and k is a positive constant. A simple integration of (50) leads to

$$G(t) \leq G(t_0)e^{-k\int_{t_0}^t \delta(s)ds}, \forall t \geq t_0. \quad (72)$$

This completes the proof.

5. Conclusion

As far as we know, there have not been any global existences and general decay results in the literature known for quasilinear viscoelastic equations with degenerate damping terms. Our work extends the works for some quasilinear viscoelastic equations treated in the literature to the quasilinear viscoelastic equation with degenerate damping terms.

Data Availability

No data were used to support the study.

Disclosure

Title for this paper "Global existence and general decay of solutions for a quasilinear system with degenerate damping terms" has been submitted in "Conference Proceeding of 9th International Eurasian Conference on Mathematical Sciences and Applications."

Conflicts of Interest

The authors declare that they have no competing interests.

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