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Research Article

Applications of Mittag-Leffer Type Poisson Distribution to a Subclass of Analytic Functions Involving Conic-Type Regions

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In this article, we introduce a new subclass of analytic functions utilizing the idea of Mittag-Leffler type Poisson distribution associated with the Janowski functions. Further, we discuss some important geometric properties like necessary and sufficient condition, convex combination, growth and distortion bounds, Fekete-Szegö inequality, and partial sums for this newly defined class.

1. Introduction, Definitions, and Motivation

Let $\mathscr{A}$ represent the collections of holomorphic (analytic) functions f defined in the open unit disc:

$$\mathbb{D} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}, \tag{1}$$

such that the Taylor series expansion of f is given by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \; (z \in \mathbb{D}). \tag{2}$$

By convention, $\mathscr{S}$ stands for a subclass of class $\mathscr{A}$ comprising of univalent functions of the form (2) in the open unit disc $\mathbb{D}$. Let $\mathscr{P}$ represent the class of all functions p that are holomorphic in $\mathbb{D}$ with the condition

$$\Re(p(z)) > 0, \tag{3}$$

and has the series representation

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \; (z \in \mathbb{D}). \tag{4}$$

Next, we recall the definition of subordination, for two functions $h_1, h_2 \in \mathscr{A}$, we say h_1 is subordinated to h_2 and is symbolically written as

$$h_1 \prec h_2, \quad (5)$$

if there exists an analytic function $w(z)$ with the properties

$$|w(z)| \le |z|, w(0) = 0, \quad (6)$$

such that

$$h_1(z) = h_2(w(z)). \quad (7)$$

Further if $h_2 \in \mathcal{S}$, then the above condition becomes

$$\begin{gathered} h_1 \prec h_2 \Leftrightarrow h_1(0) = h_2(0), \\ h_1(\mathbb{D}) \le h_2(\mathbb{D}). \end{gathered} \quad (8)$$

Now, recall the definition of convolution, let $f \in \mathcal{A}$ given by (2) and $h(z)$ given by

$$h(z) = z + \sum_{n=2}^{\infty} b_n z^n, \quad (9)$$

then their convolution denoted by $(f * h)(z)$ is given by

$$(f * h)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n \ (z \in \mathbb{D}). \quad (10)$$

The most important and well-known family of analytic functions is the class of starlike functions denoted by $\mathcal{S}^*$ and is defined as

$$\mathcal{S}^* = \left\{ f \in \mathcal{A} : \Re\left(\frac{zf'(z)}{f(z)}\right) > 0 \ (\forall z \in \mathbb{D}) \right\}. \quad (11)$$

Next, for $-1 \le B < A \le 1$, Janowski [1] generalized the class $\mathcal{S}^*$ as follows.

Definition 1. A function h with property that $h(0) = 1$ is placed in the class $\mathcal{P}[A, B]$ if and only if

$$h(z) \prec \frac{1 + Az}{1 + Bz} \ (-1 \le B < A \le 1). \quad (12)$$

Janowski also proved that for a function $p \in \mathcal{P}$, a function $h(z)$ belongs to $\mathcal{P}[A, B]$ if the following relation holds

$$h(z) = \frac{(A + 1)p(z) - (A - 1)}{(B + 1)p(z) - (B - 1)}. \quad (13)$$

Also, function f of form (2) belongs to the class $\mathcal{S}^*[A, B]$ if

$$\frac{zf'(z)}{f(z)} = \frac{(A + 1)p(z) - (A - 1)}{(B + 1)p(z) - (B - 1)} (-1 \le B < A \le 1). \quad (14)$$

Kanas et al. (see [2, 3]; see also [4, 5]) were the first to define the conic domain $\Omega_k (k \geqq 0)$ as follows:

$$\Omega_k = \left\{ u + iv : u > k\sqrt{(u - 1)^2 + v^2} \right\}. \quad (15)$$

Moreover, for fixed k, Ω_k represents the conic region bounded successively by the imaginary axis $(k = 0)$. For $k = 1$, it is a parabola, and for $0 < k < 1$, it is the right-hand branch of the hyperbola, and for $k > 1$, it represents an ellipse.

For these conic regions, the following functions play the role of extremal functions:

$$p_k(z) = \begin{cases} \chi_1(k, z), & (k = 0), \\ \chi_2(k, z), & (k = 1), \\ \chi_3(k, z), & (0 \leqq k < 1), \\ \chi_4(k, z), & (k > 1), \end{cases} \quad (16)$$

where

$$\chi_1(k, z) = \frac{1 + z}{1 - z} = 1 + 2z + 2z^2 + \cdots,$$

$$\chi_2(k, z) = 1 + \frac{2}{\pi^2}\left(\log \frac{1 + \sqrt{z}}{1 - \sqrt{z}}\right)^2,$$

$$\chi_3(k, z) = 1 + \frac{2}{1 - k^2} \sinh^2\left\{\left(\frac{2}{\pi} \arccos k\right) \operatorname{arctanh}\left(\sqrt{z}\right)\right\},$$

$$\chi_4(k, z) = 1 + \frac{1}{k^2 - 1} \sin\left(\frac{\pi}{2K(\kappa)} \int_0^{u(z)/\sqrt{\kappa}} \frac{dt}{\sqrt{1 - t^2}\sqrt{1 - \kappa^2 t^2}}\right) + \frac{1}{k^2 - 1},$$

$$u(z) = \frac{z - \sqrt{\kappa}}{1 - \sqrt{\kappa z}} (\forall z \in \mathbb{D}), \quad (17)$$

and $\kappa \in (0, 1)$ is chosen such that $\lambda = \cosh(\pi K'(\kappa)/(4K(\kappa)))$. Here $K(\kappa)$ is Legendre's complete elliptic integral of first kind and $K'(\kappa) = K(\sqrt{1 - \kappa^2})$, that is, $K'(\kappa)$ is the complementary integral of $K(\kappa)$. Assume that

$$p_k(z) = 1 + P_1 z + P_2 z^2 + \cdots (\forall z \in \mathbb{D}). \quad (18)$$

Then, in [6], it has been shown that, for (16), one can have

$$P_1 = \begin{cases} \dfrac{2N^2}{1 - k^2}, & (0 \leqq k < 1), \\ \dfrac{8}{\pi^2}, & (k = 1), \\ \dfrac{\pi^2}{4k^2(\kappa)^2(1 + \kappa)\sqrt{\kappa}}, & (k > 1), \end{cases} \quad (19)$$

$$P_2 = D(k) P_1, \quad (20)$$

where

$$D(k)=\begin{cases}\dfrac{N^2+2}{3}, & (0\leqq k<1),\\ \dfrac{2}{3}, & (k=1),\\ \dfrac{[4K(\kappa)]^2\left(\kappa^2+6\kappa+1\right)-\pi^2}{24[K(\kappa)]^2(1+\kappa)\sqrt{\kappa}}, & (k>1),\end{cases} \quad (21)$$

with

$$N=\frac{2}{\pi}\arccos k. \quad (22)$$

Definition 2. A function f of the form (2) is said to be in class $k-\mathcal{ST}$, if and only if

$$l\frac{zf'(z)}{f(z)}\prec p_k(z), k\geq 0. \quad (23)$$

Noor and Malik [7] combined the concepts of the Janowski functions and the conic regions and gave the following definition.

Definition 3. A function $h\in\mathcal{P}$ is said to be in the class $k-\mathcal{P}[A,B]$ if and only if

$$h(z)\prec\frac{(A+1)p_k(z)-(A-1)}{(B+1)p_k(z)-(B-1)},-1\leq B<A\leq 1, k\geq 0. \quad (24)$$

Geometrically, $h(z)\in k-\mathcal{P}[A,B]$ takes all values in domain $\Delta_k[A,B]$, which is defined as follows

$$\Delta_k[A,B]=\left\{w:\Re\left(\frac{(B-1)w-(A-1)}{(B+1)w-(A+1)}\right)>k\left|\frac{(B-1)w-(A-1)}{(B+1)w-(A+1)}-1\right|\right\}, \quad (25)$$

the domain $\Delta_k[A,B]$ represents conic-type regions, which was introduced and studied by Noor and Malik [7] and is further generalized by the many authors, see for example [8] and the references cited therein.

Definition 4 [7]. A function $f\in\mathcal{A}$ is said to be in the class $k-\mathcal{S}^*[A,B]$ if and only if

$$\frac{zf'(z)}{f(z)}\prec\frac{(A+1)p_k(z)-(A-1)}{(B+1)p_k(z)-(B-1)}. \quad (26)$$

The generalized exponential series:

$$E_\alpha(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(1+\alpha k)}, \alpha, z\in\mathbb{C} \text{ and } \Re(\alpha)>0, \quad (27)$$

is one special-type function with single parameter α, was introduced by Mittag-Leffer (see [9]), and is therefore known as the Mittag-Leffler function. Another function $E_{\alpha,\beta}(z)$ with two parameters α and β having similar properties to those of Mittag-Leffler function is given by

$$E_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\beta+\alpha k)}(\alpha,\beta,z\in\mathbb{C}), \quad (28)$$

and was introduced by Wiman [10, 11] Agrawal [12], and by the many other (see for example [13–16]). It can be seen that the series $E_{\alpha;\beta}(z)$ converges for all finite values of z if

$$\Re(\alpha)>0, \Re(\beta)>0. \quad (29)$$

During the last years, the interest in Mittag-Leffler type functions has considerably increased due to their vast potential of applications in applied problems such as fluid flow, electric networks, probability, and statistical distribution theory. For a detailed account of properties, generalizations and applications of functions (27) and (28), one may refer to [17–19] and [20].

Geometric properties including starlikeness, convexity, and close-to-convexity for the Mittag-Leffler function $E_{\alpha,\beta}(z)$ were recently investigated by Bansal and Prajapat in [21]. Differential subordination results associated with generalized Mittag-Leffler function were also obtained in [22].

A variable $\mathcal{N}$ is said to be Poisson distributed if it takes the values $0, 1, 2, 3, \cdots$ with probabilities $e^{-\psi}, \psi e^{-\psi}/1!, \psi^2 e^{-\psi}/2!, \psi^3 e^{-\psi}/3!, \cdots$ respectively, where ψ is called the parameter. Thus,

$$P_n(\mathcal{N}=n)=\frac{\psi^n e^{-\psi}}{n!}(n=0,1,2,3,\cdots). \quad (30)$$

It is easy to see that (30) is a mass probability function because

$$\begin{aligned}&P(\psi,\alpha,\beta;n)(z)\geq 0,\\ &\sum_{n=0}^{\infty}P(\psi,\alpha,\beta;n)(z)=1.\end{aligned} \quad (31)$$

The power series $Y(\psi,z)$ given by

$$Y(\psi,z)=z+\sum_{n=2}^{\infty}\frac{\psi^{n-1}e^{-\psi}}{(n-1)!}z^n \,(\forall z\in\mathbb{D} \text{ and } \psi>0), \quad (32)$$

which coefficients are probabilities of Poisson distribution is introduced by Porwal [23]. We can see that by ratio test the radius of convergence of $Y(\psi,z)$ is infinity. Porwal [23] also defined and introduced the following series:

$$\mathcal{G}(\psi,z)=2z-Y(\psi,z)=z-\sum_{n=2}^{\infty}\frac{\psi^{n-1}e^{-\psi}}{(n-1)!}z^n \,(\forall z\in\mathbb{D} \text{ and } \psi>0). \quad (33)$$

The works of Porwal [23] motivate researchers to introduced a new probability distribution if it assumes the positive values and its mass function is given by (30) (see for example [24–26]).

It was Porwal and Dixit [24] who studied and connected the Poisson distribution and the well-known Mittag-Leffer function systematically. They called it the Mittag-Leffer type Poisson distribution and prevailed moments. The Mittag-Leffer type Poisson distribution is given by (see [24])

$$Y(\psi, \alpha, \beta)(z) = z + \sum_{n=2}^{\infty} \frac{\psi^{n-1}}{\Gamma(\alpha(n-1)+\beta)E_{\alpha;\beta}(\psi)} z^n, \quad (34)$$

where $Y(\psi, \alpha, \beta)(z)$ is a normalized function of class $\mathcal{S}$, since

$$Y(\psi, \alpha, \beta)(0) = 0 = Y'(\psi, \alpha, \beta)(0) - 1. \quad (35)$$

The probability mass function for the Mittag-Leffer type Poisson distribution series is given by

$$P(\psi, \alpha, \beta\,; n)(z) = \frac{\psi^n}{E_{\alpha;\beta}(\psi)\Gamma(\alpha n + \beta)} \; (n = 0, 1, 2, 3, \cdots), \quad (36)$$

where $E_{\alpha;\beta}(\psi)$ is given by (28). It is worthy to note that the Mittag-Leffer type Poisson distribution is a generalization of Poisson distribution. Furtheremore, Bajpai [27] also studied and obtain some necessary and sufficient conditions for this distribution series.

Very recently, using the Mittag-Leffer type Poisson distribution series, Alessa et al. [28] defined the convolution operator as

$$\Omega(\psi, \alpha, \beta)f(z) = Y(\psi, \alpha, \beta) * f(z) = z + \sum_{n=2}^{\infty} \varphi_{\psi}^{n}(\alpha, \beta) a_n z^n, \quad (37)$$

where

$$\varphi_{\psi}^{n}(\alpha, \beta) = \frac{\psi^{n-1}}{\Gamma(\alpha(n-1)+\beta)E_{\alpha;\beta}(\psi)}. \quad (38)$$

Using this convolution operator, they defined and studied a new subclass of analytic function systematically. They obtained certain coefficient estimates, neighborhood results, partial sums, and convexity and compactness properties for their defined functions class.

In recent years, binomial distribution series, Pascal distribution series, Poisson distribution series, and Mittag-Leffer type Poisson distribution series play important role in the geometric function theory of complex analysis. The sufficient ways were innovated for certain subclasses of starlike and convex functions involving these special functions (see for example [26, 29–32]). Motivated by the abovementioned works and from the work of Alessa et al. [28], in this article, by mean of certain convolution operator for Mittag-Leffer type Poisson distribution, we shall define a new subclass of starlike functions involving both the conic-type regions and the Janowski functions. We then obtain some interesting properties for this newly defined function class including for example necessary and sufficient condition, convex combination, growth and distortion bounds, Fekete-Szegö inequality, and partial sums. We now define a subclass of Janowski-type starlike functions involving the conic domains by mean of certain convolution operator for Mittag-Leffer type Poisson distribution as follows.

Definition 5. For $-1 \le B < A \le 1$, a function $f \in \mathcal{A}$ is in class $k-\Omega\mathcal{S}^*(\alpha, \beta, A, B)$ if

$$\Re\left(\frac{(B-1)\vartheta(f\,;\psi,\alpha,\beta)-(A-1)}{(B+1)\vartheta(f\,;\psi,\alpha,\beta)-(A+1)}-1\right) \ge k\left|\frac{(B-1)\vartheta(f\,;\psi,\alpha,\beta)-(A-1)}{(B+1)\vartheta(f\,;\psi,\alpha,\beta)-(A+1)}-1\right|, \quad (39)$$

where

$$\vartheta(f\,;\psi,\alpha,\beta) = \frac{z(\Omega(\psi,\alpha,\beta)f(z))'}{\Omega(\psi,\alpha,\beta)f(z)}. \quad (40)$$

For the proofs of our key findings, we need the following lemma.

Lemma 6 [33]. *Let $p \in \mathcal{P}$ have the series expansion of form (4), then*

$$\left|a_3 - \zeta a_2^2\right| \le 2 \max\{1, |2\zeta - 1|\}, \text{ where } \zeta \in \mathbb{C}. \quad (41)$$

2. Main Results

Theorem 7. *Let $f \in k-\Omega\mathcal{S}^*(\alpha, \beta, A, B)$ and is of the form (2), then*

$$\sum_{n=2}^{\infty}[2(k+1)|1-n| + |n(1+B)-(1+A)|]\varphi_{\psi}^{n}(\alpha,\beta)|a_n| < |B-A|. \quad (42)$$

The result is sharp for the function given in (51).

Proof. Suppose that inequality (42) holds true, then it is enough to show that

$$k\left|\frac{(B-1)\vartheta(f\,;\psi,\alpha,\beta)-(A-1)}{(B+1)\vartheta(f\,;\psi,\alpha,\beta)-(A+1)}-1\right| - \Re\left(\frac{(B-1)\vartheta(f\,;\psi,\alpha,\beta)-(A-1)}{(B+1)\vartheta(f\,;\psi,\alpha,\beta)-(A+1)}-1\right) < 1. \quad (43)$$

For this, consider

$$k\left|\frac{(B-1)\vartheta(f\,;\psi,\alpha,\beta)-(A-1)}{(B+1)\vartheta(f\,;\psi,\alpha,\beta)-(A+1)}-1\right| - \Re\left(\frac{(B-1)\vartheta(f\,;\psi,\alpha,\beta)-(A-1)}{(B+1)\vartheta(f\,;\psi,\alpha,\beta)-(A+1)}-1\right). \quad (44)$$

As we have set

$$\vartheta(f\,;\psi,\alpha,\beta)=\frac{z(\Omega(\psi,\alpha,\beta)f(z))'}{\Omega(\psi,\alpha,\beta)f(z)}, \quad (45)$$

therefore, after some straightforward simplifications, we have

$$\begin{aligned}
&k\left|\frac{(B-1)\vartheta(f\,;\psi,\alpha,\beta)-(A-1)}{(B+1)\vartheta(f\,;\psi,\alpha,\beta)-(A+1)}-1\right|-\Re\left(\frac{(B-1)\vartheta(f\,;\psi,\alpha,\beta)-(A-1)}{(B+1)\vartheta(f\,;\psi,\alpha,\beta)-(A+1)}-1\right),\\
&\leq(k+1)\left|\frac{(B-1)z(\Omega(\psi,\alpha,\beta)f(z))'-(A-1)\Omega(\psi,\alpha,\beta)f(z)}{(B+1)z(\Omega(\psi,\alpha,\beta)f(z))'-(A+1)\Omega(\psi,\alpha,\beta)f(z)}-1\right|,\\
&=2(k+1)\left|\frac{\Omega(\psi,\alpha,\beta)f(z)-z(\Omega(\psi,\alpha,\beta)f(z))'}{(B+1)z(\Omega(\psi,\alpha,\beta)f(z))'-(A+1)\Omega(\psi,\alpha,\beta)f(z)}\right|,\\
&=2(k+1)\left|\frac{\sum_{n=2}^{\infty}(1-n)\varphi_{\psi}^{n}(\alpha,\beta)a_nz^n}{(B-A)z+\sum_{n=2}^{\infty}[n(1+B)-(1+A)]\varphi_{\psi}^{n}(\alpha,\beta)a_nz^n}\right|,\\
&\leq\frac{2(k+1)\sum_{n=2}^{\infty}|1-n|\varphi_{\psi}^{n}(\alpha,\beta)|a_n|}{|B-A|-\sum_{n=2}^{\infty}|n(1+B)-(1+A)|\varphi_{\psi}^{n}(\alpha,\beta)|a_n|}.
\end{aligned} \quad (46)$$

By using (42), the above inequality is bounded above by 1, and hence, the proof is completed. □

Example 8. For the function

$$f(z)=z+\sum_{n=2}^{\infty}\frac{|B-A|}{[2(k+1)|1-n|+|n(1+B)-(1+A)|]\varphi_{\psi}^{n}(\alpha,\beta)}x_nz^n\ (z\in\mathbb{D}), \quad (47)$$

such that

$$\sum_{n=2}^{\infty}|x_n|=1, \quad (48)$$

we have

$$\begin{aligned}
&\sum_{n=2}^{\infty}[2(k+1)|1-n|+|n(1+B)-(1+A)|]\varphi_{\psi}^{n}(\alpha,\beta)|a_n|\\
&=\sum_{n=2}^{\infty}[2(k+1)|1-n|+|n(1+B)-(1+A)|]\varphi_{\psi}^{n}(\alpha,\beta)\\
&\quad\cdot\frac{|B-A|}{[2(k+1)|1-n|+|n(1+B)-(1+A)|]\varphi_{\psi}^{n}(\alpha,\beta)}|x_n|\\
&=|B-A|\sum_{n=2}^{\infty}|x_n|=|B-A|.
\end{aligned} \quad (49)$$

Hence, $f\in k-\Omega\mathcal{S}^*(\alpha,\beta,A,B)$ and the result is sharp.

Corollary 9. *Let the function f of the form (2) be in the class $k-\Omega\mathcal{S}^*(\alpha,\beta,A,B)$. Then,*

$$|a_n|\leq\frac{|B-A|}{[2(k+1)|1-n|+|n(1+B)-(1+A)|]\varphi_{\psi}^{n}(\alpha,\beta)}. \quad (50)$$

The result is sharp for the function $f_t(z)$ given by

$$f_t(z)=z+\frac{|B-A|}{[2(k+1)|1-n|+|n(1+B)-(1+A)|]\varphi_{\psi}^{n}(\alpha,\beta)}z^n. \quad (51)$$

Theorem 10. *The class $k-\Omega\mathcal{S}^*(\alpha,\beta,A,B)$ is closed under convex combination.*

Proof. Let $f_k(z)\in k-\Omega\mathcal{S}^*(\alpha,\beta,A,B)$ such that

$$lf_k(z)=z+\sum_{n=2}^{\infty}a_{n,k}z^n,\ k\in\{1,2\}. \quad (52)$$

It is enough to show that

$$tf_1(z)+(1-t)f_2(z)\in k-\Omega\mathcal{S}^*(\alpha,\beta,A,B)\ (t\in[0,1]). \quad (53)$$

As

$$ltf_1(z)+(1-t)f_2(z)\quad=z+\sum_{n=2}^{\infty}[ta_{n,1}+(1-t)a_{n,2}]z^n. \quad (54)$$

Now, by Theorem 7, we have

$$\begin{aligned}
&\sum_{n=2}^{\infty}[2(k+1)|1-n|+|n(1+B)-(1+A)|]\\
&\quad\cdot\varphi_{\psi}^{n}(\alpha,\beta)\left|ta_{n,1}+(1-t)a_{n,2}\right|\\
&\leq\sum_{n=2}^{\infty}[2(k+1)|1-n|+|n(1+B)-(1+A)|]\\
&\quad\cdot\varphi_{\psi}^{n}(\alpha,\beta)\left[t\left|a_{n,1}\right|+(1-t)\left|a_{n,2}\right|\right]\\
&\leq t\sum_{n=2}^{\infty}[2(k+1)|1-n|+|n(1+B)-(1+A)|]\\
&\quad\cdot\varphi_{\psi}^{n}(\alpha,\beta)\left|a_{n,1}\right|+(1-t)\\
&\quad\cdot\sum_{n=2}^{\infty}[2(k+1)|1-n|+|n(1+B)-(1+A)|]\varphi_{\psi}^{n}(\alpha,\beta)\left|a_{n,2}\right|\\
&<t|B-A|+(1-t)|B-A|=|B-A|.
\end{aligned} \quad (55)$$

Hence,

$$tf_1(z)+(1-t)f_2(z)\in k-\Omega\mathcal{S}^*(\alpha,\beta,A,B), \quad (56)$$

which completes the proof. □

Theorem 11. *Let $f \in k - \Omega\mathcal{S}^*(\alpha, \beta, A, B)$, then for $|z| = r$*

$$r - \frac{|B-A|}{2(k+1) + |2B - A + 1|\varphi_\psi^2(\alpha, \beta)} r^2 \le |f(z)| \le r + \frac{|B-A|}{2(k+1) + |2B - A + 1|\varphi_\psi^2(\alpha, \beta)} r^2. \quad (57)$$

The result is sharp for the function given in (51) for $n = 2$.

Proof. Let $f \in k - \Omega\mathcal{S}^*(\alpha, \beta, A, B)$. Using Theorem 7, we can deduce the following inequity:

$$|f(z)| \le |z| + \sum_{n=2}^{\infty} |a_n||z^n| \le |z| + |z|^2 \sum_{n=2}^{\infty} |a_n| \le r + \frac{|B-A|}{2(k+1) + |2B - A + 1|\varphi_\psi^2(\alpha, \beta)} r^2. \quad (58)$$

Similarly,

$$|f(z)| \ge |z| - \sum_{n=2}^{\infty} |a_n||z^n| \ge |z| - |z|^2 \sum_{n=2}^{\infty} |a_n| \ge r - \frac{|B-A|}{2(k+1) + |2B - A + 1|\varphi_\psi^2(\alpha, \beta)} r^2. \quad (59)$$

□

Theorem 12. *Let $f \in k - \Omega\mathcal{S}^*(\alpha, \beta, A, B)$, then for $|z| = r$*

$$\left|f'(z)\right| \le 1 + \frac{2|B-A|}{2(k+1) + |2B - A + 1|\varphi_\psi^2(\alpha, \beta)} r, \quad \left|f'(z)\right| \ge 1 - \frac{2|B-A|}{2(k+1) + |2B - A + 1|\varphi_\psi^2(\alpha, \beta)} r. \quad (60)$$

The result is sharp for the function given in (51) for $n = 2$.

Proof. The proof is quite similar to Theorem 11, so left for reader. □

Now, we evaluate a kind of Hankel determinant problem, which is also known as the Fekete-Szegö functional.

Theorem 13. *If f is of the form (2) and belongs to $k - \Omega\mathcal{S}^*(\alpha, \beta, A, B)$, then*

$$\left|a_3 - \xi a_2^2\right| \le \frac{P_1(A-B)}{4\varphi_\psi^3(\alpha, \beta)} \max\left\{1, \left|\frac{(B - 2P_2 + 1)\varphi_\psi(\alpha, \beta) - 2\xi(A-B)P_1^2}{2P_1\varphi_\psi(\alpha, \beta)}\right|\right\}, \quad (61)$$

where P_1 and P_2 are defined by (19) and (20), respectively.

Proof. To prove inequality (61), we let

$$\vartheta(f; \psi, \alpha, \beta) = \frac{z(\Omega(\psi, \alpha, \beta)f(z))'}{\Omega(\psi, \alpha, \beta)f(z)}, \quad (62)$$

then from (26), we have

$$\vartheta(f; \psi, \alpha, \beta) \prec \frac{(A+1)p_k(z) - (A-1)}{(B+1)p_k(z) - (B-1)} = \Phi(z) \text{ (say)}. \quad (63)$$

Thus, if

$$p_k(z) = 1 + P_1 z + P_2 z^2 + \cdots, \quad (64)$$

then by simple computation, we get

$$\Phi(z) = 1 + \frac{1}{2}P_1(A-B)z + \frac{1}{4}(A-B)\left(2P_2 - (1+B)P_1^2\right)z^2 + \cdots. \quad (65)$$

Now, from (63), there exists an analytic function $h(z)$ such that

$$h(z) = \frac{1 + \Phi^{-1}(\vartheta(f; \psi, \alpha, \beta))}{1 - \Phi^{-1}(\vartheta(f; \psi, \alpha, \beta))} = 1 + c_1 z + c_2 z^2 + \cdots, \quad (66)$$

is analytic and

$$\Re(h(z)) > 0, \quad (67)$$

in open unit disc $\mathbb{D}$. Also, we have

$$\vartheta(f; \psi, \alpha, \beta) = \Phi\left(\frac{h(z) - 1}{h(z) + 1}\right), \quad (68)$$

where

$$\frac{z(\vartheta(f; \psi, \alpha, \beta))'}{\vartheta(f; \psi, \alpha, \beta)} = 1 + \varphi_\psi^2(\alpha, \beta)a_2 z + \left(2\varphi_\psi^3(\alpha, \beta)a_3 - \varphi_\psi^4(\alpha, \beta)a_2^2\right)z^2 + \cdots. \quad (69)$$

$$\Phi\left(\frac{h(z) - 1}{h(z) + 1}\right) = 1 + \frac{1}{4}(A-B)P_1 c_1 z + \frac{1}{4}(A-B) \cdot \left[P_1 c_2 + \left(\frac{P_2}{2} - \frac{1+B}{4} - \frac{P_1}{2}\right)c_1^2\right]z^2 + \cdots. \quad (70)$$

After comparing the (69) and (70), we get

$$a_2 = \frac{1}{4\varphi_\psi^2(\alpha, \beta)}(A-B)P_1 c_1, \quad (71)$$

$$a_3 = \frac{1}{8\varphi_\psi^3(\alpha, \beta)}(A-B)\left(P_1 c_2 + \left(\frac{P_2}{2} - \frac{1+B}{4} - \frac{P_1}{2}\right)c_1^2\right). \quad (72)$$

Now, by making use of (71) and (72), in conjunction with

Lemma, we have

$$\left|a_3 - \xi a_2^2\right| \leq \frac{P_1(A-B)}{4\varphi_\psi^3(\alpha,\beta)} \max\left\{1, \left|\frac{(B-2P_2+1)\varphi_\psi(\alpha,\beta) - 2\xi(A-B)P_1^2}{2P_1\varphi_\psi(\alpha,\beta)}\right|\right\}, \tag{73}$$

which is the required result. □

3. Partial Sums

In this section, we will examine the ratio of a function of form (2) to its sequence of partial sums

$$f_j(z) = z + \sum_{n=2}^{j} a_n z^n, \tag{74}$$

when the coefficients of f are sufficiently small to satisfy condition (42). We will determine sharp lower bounds for

$$\begin{gathered}\Re\left(\frac{f(z)}{f_j(z)}\right),\\ \Re\left(\frac{f_j(z)}{f(z)}\right),\\ \Re\left(\frac{f'(z)}{f_j'(z)}\right),\\ \Re\left(\frac{f_j'(z)}{f'(z)}\right).\end{gathered} \tag{75}$$

Theorem 14. *If f of form (2) satisfies condition (42), then*

$$\Re\left(\frac{f(z)}{f_j(z)}\right) \geq 1 - \frac{1}{\rho_{j+1}} \quad (\forall z \in \mathbb{D}), \tag{76}$$

$$\Re\left(\frac{f_j(z)}{f(z)}\right) \geq \frac{\rho_{j+1}}{1+\rho_{j+1}} \quad (\forall z \in \mathbb{D}), \tag{77}$$

where

$$\rho_j = \frac{[2(k+1)|1-n| + |n(1+B) - (1+A)|]\varphi_\psi^n(\alpha,\beta)}{|A-B|}. \tag{78}$$

The result is sharp for the function given in (51).

Proof. It is easy to verify that

$$\rho_{n+1} > \rho_n > 1 \text{ for } n > 2. \tag{79}$$

Thus, in order to prove the inequality (76), we set

$$\begin{aligned}\rho_{j+1}\left[\frac{f(z)}{f_j(z)} - \left(1 - \frac{1}{\rho_{j+1}}\right)\right] &= \frac{1 + \sum_{n=2}^{j} a_n z^{n-1} + \rho_{j+1}\sum_{n=j+1}^{\infty} a_n z^{n-1}}{1 + \sum_{n=2}^{j} a_n z^{n-1}}\\ &= \frac{1+h_1(z)}{1+h_2(z)}.\end{aligned} \tag{80}$$

We now set

$$\frac{1+h_1(z)}{1+h_2(z)} = \frac{1+w(z)}{1-w(z)}. \tag{81}$$

Then, we find after some suitable simplification that

$$w(z) = \frac{h_1(z) - h_2(z)}{2 + h_1(z) + h_2(z)}. \tag{82}$$

Thus, clearly, we find that

$$w(z) = \frac{\rho_{j+1}\sum_{n=j+1}^{\infty} a_n z^{n-1}}{2 + 2\sum_{n=2}^{j} a_n z^{n-1} + \rho_{j+1}\sum_{n=j+1}^{\infty} a_n z^{n-1}}. \tag{83}$$

By applying the trigonometric inequalities with $|z| < 1$, we arrived at the following inequality:

$$|w(z)| \leq \frac{\rho_{j+1}\sum_{n=j+1}^{\infty}|a_n|}{2 - 2\sum_{n=2}^{j}|a_n| - \rho_{j+1}\sum_{n=j+1}^{\infty}|a_n|}. \tag{84}$$

We can now see that

$$|w(z)| \leq 1, \tag{85}$$

if and only if

$$2\rho_{j+1}\sum_{n=j+1}^{\infty}|a_n| \leq 2 - 2\sum_{n=2}^{j}|a_n|, \tag{86}$$

which implies that

$$\sum_{n=2}^{j}|a_n| + \rho_{j+1}\sum_{n=j+1}^{\infty}|a_n| \leq 1. \tag{87}$$

Finally, to prove the inequality in (76), it suffices to show that the left-hand side of (87) is bounded above by the following sum:

$$\sum_{n=2}^{\infty}\rho_n|a_n|, \tag{88}$$

which is equivalent to

$$\sum_{n=2}^{j}(\rho_n - 1)|a_n| + \sum_{n=j+1}^{\infty}\left(\rho_n - \rho_{j+1}\right)|a_n| \geq 0. \tag{89}$$

In virtue of (89), the proof of inequality in (76) is now completed.

Next, in order to prove the inequality (77), we set

$$\left(1+\rho_{j+1}\right)\left(\frac{f_j(z)}{f(z)} - \frac{\rho_{j+1}}{1+\rho_{j+1}}\right) = \frac{1+\sum_{n=2}^{j}a_n z^{n-1} - \rho_{j+1}\sum_{n=j+1}^{\infty}a_n z^{n-1}}{1+\sum_{n=2}^{\infty}a_n z^{n-1}}, = \frac{1+w(z)}{1-w(z)}, \tag{90}$$

where

$$|w(z)| \leq \frac{\left(1+\rho_{j+1}\right)\sum_{n=j+1}^{\infty}|a_n|}{2-2\sum_{n=2}^{j}|a_n| - \left(\rho_{j+1}-1\right)\sum_{n=j+1}^{\infty}|a_n|} \leq 1. \tag{91}$$

This last inequality in (91) is equivalent to

$$\sum_{n=2}^{j}|a_n| + \rho_{j+1}\sum_{n=j+1}^{\infty}|a_n| \leq 1. \tag{92}$$

Finally, we can see that the left-hand side of the inequality in (92) is bounded above by the following sum:

$$\sum_{n=2}^{\infty}\rho_n|a_n|, \tag{93}$$

so we have completed the proof of the assertion (77). ☐

We next turn to ratios involving derivatives.

Theorem 15. *If f of the form (2) satisfies condition (42), then*

$$\Re\left(\frac{f'(z)}{f_j'(z)}\right) \geq 1 - \frac{j+1}{\rho_{j+1}} \ (\forall z \in \mathbb{D}),$$
$$\Re\left(\frac{f_j'(z)}{f'(z)}\right) \geq \frac{\rho_{j+1}}{\rho_{j+1}+j+1} \ (\forall z \in \mathbb{D}), \tag{94}$$

where ρ_j is given by (78). The result is sharp for the function given in (51).

Proof. The proof of Theorem 15 is similar to that of Theorem 14; we here choose to omit the analogous details. ☐

4. Concluding Remarks and Observations

In our present work, by making use of the idea of Mittag-Leffler type Poisson distribution, we have defined and studied certain new subclasses of starlike functions involving the Janowski functions. Further, we have discussed some important geometric properties like necessary and sufficient condition, convex combination, growth and distortion bounds, Fekete-Szegö inequality, and partial sums for this newly defined functions class.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

All authors jointly worked on the results, and they read and approved the final manuscript.

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