Hindawi
Journal of Function Spaces
Volume 2021, Article ID 4395574, 10 pages
https://doi.org/10.1155/2021/4395574

Research Article

Coefficient Bounds of Kamali-Type Starlike Functions Related with a Limacon-Shaped Domain

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Received 5 September 2021; Accepted 6 October 2021; Published 19 October 2021

Academic Editor: Wasim Ul-Haq

In this article, we familiarize a subclass of Kamali-type starlike functions connected with limacon domain of bean shape. We examine certain initial coefficient bounds and Fekete-Szegö inequalities for the functions in this class. Analogous results have been acquired for the functions f^{-1} and $\xi/f(\xi)$ and also found the upper bound for the second Hankel determinant $a_2a_4 - a_3^2$.

1. Introduction

Denote by $\mathscr{A}$ the class of analytic functions

$$f(\xi) = \xi + a_2\xi^2 + a_3\xi^3 + \cdots, \tag{1}$$

in the open unit disk $U = \{\xi : |\xi|<1\}$. The Hankel determinants $\mathscr{H}_j(n)$, $(n = 1, 2, 3, \cdots; j = 1, 2, 3, \cdots)$ of f are denoted by

$$\mathscr{H}_j(n) = \begin{bmatrix} a_n & a_{n+1} & . & . & a_{n+j-1} \\ a_{n+1} & a_{n+2} & . & . & a_{n+j} \\ . & . & . & . & . \\ a_{n+j-1} & a_{n+j} & . & . & a_{n+2j-2} \end{bmatrix}, \tag{2}$$

where $a_1 = 1$. Hankel determinants are beneficial, for example, in viewing that whether the certain coefficient functionals related to functions are bounded in U or not and do they carry the sharp bounds, see [1]. The applications of Hankel inequalities in the study of meromorphic functions can be seen in [2, 3]. In 1966, Pommerenke [4] inspected $|\mathscr{H}_j(n)|$ of univalent functions and $p-$valent functions as well as starlike functions. In [5], it is evidenced that the Hankel determinants of univalent functions satisfy

$$|\mathscr{H}_j(n)| < kn^{-((1/2)+\beta)j+3/2}\,; (n = 1, 2, 3, \cdots; j = 1, 2, 3, \cdots), \tag{3}$$

where $\beta > 0.00025$ and k depends only on j. Later, Hayman [6] demonstrated that $|\mathscr{H}_2(n)| < A_n^{1/2}$, $(n = 1, 2, 3, \cdots;$ A an absolute constant) for univalent functions. Further, the Hankel determinant bounds of univalent functions with a positive Hayman index α were determined by Elhosh [7], of p-valent functions were seen in [8–10], and of close-to-convex and k-fold symmetric functions were discussed in [11]. Lately, several authors have explored the bounds $|\mathscr{H}_j(n)|$, of functions belonging to various subclasses of univalent and multivalent functions (for details, see [6, 12–27]). The Hankel determinant $\mathscr{H}_2(1) = a_3 - a_2^2$ is the renowned Fekete-Szegö Functional (see [28, 29]) and $H_2(2)$; second, Hankel determinant is presumed by $\mathscr{H}_2(2) = a_2a_4 - a_3^2$.

An analytic function f_1 is subordinate to an analytic function f_2, written as $f_1 \prec f_2$, if there is an analytic function $w : U \longrightarrow U$ with $w(0) = 0$, satisfying $f_1(\xi) = f_2(w(\xi))$.

Let $\mathcal{P}$ be the class of functions with positive real part consisting of all analytic functions $p : U \longrightarrow \mathbb{C}$ satisfying $p(0) = 1$ and $\mathrm{Re}\,p(z) > 0$.

Ma and Minda [30] amalgamated various subclasses of starlike and convex functions which are subordinate to a function $\psi \in \mathcal{P}$ with $\psi(0) = 1, \psi'(0) > 0, \psi$ maps U onto a region starlike with respect to 1 and symmetric with respect to real axis and familiarized the classes as below:

$$\begin{aligned}\mathcal{S}^*(\psi) = \left\{ f \in \mathcal{A} : \frac{\xi f'(\xi)}{f(\xi)} \prec \psi \right\} \text{ and } \mathcal{C}(\psi) \\ = \left\{ f \in \mathcal{A} : 1 + \frac{\xi f''(\xi)}{f'(\xi)} \prec \psi \right\}.\end{aligned} \tag{4}$$

By choosing ψ satisfying Ma-Minda conditions and that maps U on to some precise regions like parabolas, cardioid, lemniscate of Bernoulli, and booth lemniscate in the right-half of the complex plane, several fascinating subclasses of starlike and convex functions are familiarized and studied. Raina and Sokół [31] considered the class $\mathcal{S}^*(\psi)$ for $\psi(z) = \xi + \sqrt{1 + \xi^2}$ and established some remarkable inequalities (also see [32] and references cited therein). Gandhi in [33] considered a class $\mathcal{S}^*(\psi)$ with $\psi = \beta e^{\xi} + (1 - \beta)(1 + \xi)$, $0 \le \beta \le 1$, a convex combination of two starlike functions. Further, coefficient inequalities of functions linked with petal type domains were widely discussed by Malik et al. ([34], see also references cited therein). The region bounded by the cardioid specified by the equation

$$\left(9x^2 + 9y^2 - 18x + 5\right)^2 - 16\left(9x^2 + 9y^2 - 6x + 1\right) = 0, \tag{5}$$

was studied in [35]. Lately, Masih and Kanas [36] introduced novel subclasses $\mathrm{ST}_{L(s)}$ and $\mathrm{CV}_{L(s)}$ of starlike and convex functions, respectively. Geometrically, they consist of functions $f \in A$ such that $\xi f'(\xi)/f(\xi)$ and $(\xi f'(\xi))'/f'(\xi)$, respectively, are lying in the region bounded by the limacon

$$\left[(u-1)^2 + v^2 - s^4\right]^2 = 4s^2\left[\left(u - 1 + s^2\right)^2 + v^2\right], \quad \text{where } 0 < s \le \frac{1}{\sqrt{2}}. \tag{6}$$

Lately, Yuzaimi et al. [37] defined a region bounded by the bean-shaped limacon region as below:

$$\begin{aligned}\Omega(U) = \Big\{ w = x + iy : \left(4x^2 + 4y^2 - 8x - 5\right)^2 \\ + 8\left(4x^2 + 4y^2 - 12x - 3\right) = 0 \Big\}, s \in [-1, 1] \setminus \{0\}.\end{aligned} \tag{7}$$

Suppose that

$$\varphi(\xi) \colon U \longrightarrow \mathbb{C}, \tag{8}$$

is the function defined by

$$\varphi(\xi) = 1 + \sqrt{2}\xi + \frac{1}{2}\xi^2, \tag{9}$$

is preferred so that the limacon is in the bean shape [37]. Motivated by this present work and other aforesaid articles, the goal in this paper is to examine some coefficient inequalities and bounds on Hankel determinants of the Kamali-type class of starlike functions satisfying the conditions as given in Definition 1.

Definition 1. Let $\varphi : U \longrightarrow \mathbb{C}$ be analytic and for $0 \le \vartheta \le 1$, we let the class $\mathcal{M}(\vartheta, \varphi)$ *as*

$$\mathcal{M}(\vartheta, \varphi) = \left\{ f \in A : \frac{\vartheta \xi^3 f'''(\xi) + (1 + 2\vartheta) f''(\xi)\xi^2 + \xi f'(\xi)}{\vartheta \xi^2 f''(\xi) + \xi f'(\xi)} \prec \varphi(\xi), \xi \in U \right\}, \tag{10}$$

where $\varphi(\xi) = 1 + \sqrt{2}\xi + (1/2)\xi^2$ as in (9).

We include the following results which are needed for the proofs of our main results.

Lemma 2 see [38]. *Suppose that* $p(\xi) = 1 + c_1\xi + c_2\xi^2 + \cdots$, $\Re(p_1) > 0$, $\xi \in U$, *then*

$$\begin{aligned}|c_n| \le 2 (n = 1, 2, 3, \cdots), \\ \left|c_2 - v c_1^2\right| \le 2 \max\{1, |2v - 1|\},\end{aligned} \tag{11}$$

and the outcome is sharp for the functions formulated by

$$\begin{aligned}p(\xi) = \frac{1 + \xi^2}{1 - \xi^2}, \\ p(\xi) = \frac{1 + \xi}{1 - \xi}.\end{aligned} \tag{12}$$

Lemma 3 see [30]. *Suppose that* $p_1(\xi) = 1 + c_1\xi + c_2\xi^2 + \cdots$, $\Re(p_1) > 0$, $\xi \in U$. *Then,*

(i) For $v < 0$ *or* $v > 1$, *we have*

$$|c_2 - v c_1^2| \le \begin{cases} -4v + 2 & \text{if } v \le 0, \\ 2 & \text{if } 0 \le v \le 1, \\ 4v - 2 & \text{if } v \ge 1. \end{cases} \tag{13}$$

Equality occurs when $p_1(\xi) = (1 + \xi)/(1 - \xi)$ *or one of its rotations.*

(ii) For $v \in (0, 1)$, *the equality exists when* $p_1(\xi) = (1 + \xi^2)/(1 - \xi^2)$ *or one of its rotations*

(iii) For $v = 0$, *the equality happens when*

$$p_1(\xi) = \left(\frac{1}{2} + \frac{1}{2}\vartheta\right)\frac{1+\xi}{1-\xi} + \left(\frac{1}{2} - \frac{1}{2}\vartheta\right)\frac{1-\xi}{1+\xi}\,(0 \le \vartheta \le 1), \tag{14}$$

or one of its rotations.

Lemma 4 see [39]. *If $p \in \mathscr{P}$ and is given by $p(\xi) = 1 + c_1\xi + c_2\xi^2 + \cdots$ then*

$$2c_2 = c_1^2 + x\left(4 - c^2\right), \tag{15}$$

$$4c_3 = c_1^3 + 2\left(4 - c_1^2\right)c_1 x - c_1\left(4 - c_1^2\right)x^2 + 2\left(4 - c_1^2\right)\left(1 - |x|^2\xi\right), \tag{16}$$

for some x, ξ with $|x| \le 1$ and $|\xi| \le 1$.

Theorem 5. *Let the function $f \in \mathscr{M}(\vartheta, \varphi)$ be given by (1) then*

$$\begin{aligned} |a_2| &\le \frac{1}{\sqrt{2}(\vartheta+1)}, \\ |a_3| &\le \frac{\sqrt{2}}{6(2\vartheta+1)} \max\left\{1, \left|\frac{5}{2\sqrt{2}}\right|\right\} = \frac{5}{12(2\vartheta+1)}. \end{aligned} \tag{17}$$

Proof. Since $f \in M(\vartheta, \varphi)$, there exists an analytic function w with $w(0) = 0$ and $|w(\xi)| < 1$ in U such that

$$\frac{\vartheta\xi^3 f'''(\xi) + (1+2\vartheta)f''(\xi)\xi^2 + \xi f'(\xi)}{\vartheta\xi^2 f''(\xi) + \xi f'(\xi)} = \varphi(w(\xi)). \tag{18}$$

Define the function p_1 by

$$p_1(\xi) = \frac{1 + w(\xi)}{1 - w(\xi)} = 1 + c_1\xi + c_2\xi^2 + c_3\xi^3 + \cdots, \tag{19}$$

or, equivalently

$$\begin{aligned} w(\xi) &= \frac{p_1(\xi) - 1}{p_1(\xi) + 1} \\ &= \frac{1}{2}\left[c_1\xi + \left(c_2 - \frac{c_1^2}{2}\right)\xi^2 + \left(c_3 - c_1c_2 + \frac{c_1^3}{4}\right)\xi^3 + \cdots\right], \end{aligned} \tag{20}$$

then p_1 is analytic in U with $p_1(0) = 1$ and has a positive real part in U. By using (20) together with (9), it is evident that

$$\begin{aligned} \varphi(w(\xi)) = \varphi\left(\frac{p_1(\xi) - 1}{p_1(\xi) + 1}\right) &= 1 + \frac{c_1\xi}{\sqrt{2}} + \left(\frac{1}{\sqrt{2}}\left(c_2 - \frac{c_1^2}{2}\right) + \frac{c_1^2}{8}\right)\xi^2 \\ &+ \left\{\frac{1}{\sqrt{2}}\left(c_3 - c_1c_2 + \frac{c_1^3}{4}\right) + \frac{c_1}{4}\left(c_2 - \frac{c_1^2}{2}\right)\right\}\xi^3 + \cdots. \end{aligned} \tag{21}$$

Since

$$\begin{aligned} &\frac{\vartheta\xi^3 f'''(\xi) + (1+2\vartheta)f''(\xi)\xi^2 + \xi f'(\xi)}{\vartheta\xi^2 f''(\xi) + \xi f'(\xi)} \\ &= 1 + 2(\vartheta+1)a_2\xi + \left(-4(\vartheta+1)^2 a_2^2 + 6(2\vartheta+1)a_3\right)\xi^2 \\ &\quad + \left[8(\vartheta+1)^3 a_2^3 - 18\left(2\vartheta^2 + 3\vartheta + 1\right)a_2a_3 + 12(3\vartheta+1)a_4\right]\xi^3 + \cdots, \end{aligned} \tag{22}$$

and equating coefficients of ξ, ξ^2, ξ^3 from (21) to (22), we get

$$a_2 = \frac{c_1}{2\sqrt{2}(\vartheta+1)}, \tag{23}$$

$$a_3 = \frac{1}{24(2\vartheta+1)}\left[c_1^2\left(\frac{5}{2} - \sqrt{2}\right) + 2\sqrt{2}c_2\right], \tag{24}$$

$$a_4 = \frac{1}{192(3\vartheta+1)}\left[c_1^3\left(\frac{11}{\sqrt{2}} - 8\right) + 4\left(2 - \sqrt{2}\right)c_1c_2 + 8\sqrt{2}c_3\right]. \tag{25}$$

Now by applying Lemma 2, we get

$$|a_2| = \frac{1}{\sqrt{2}(\vartheta+1)}, \tag{26}$$

and also,

$$\begin{aligned} |a_3| &= \frac{1}{24(2\vartheta+1)}\left|2\sqrt{2}c_2 + c_1^2\left(\frac{5}{2} - \sqrt{2}\right)\right| \\ &= \frac{\sqrt{2}}{12(2\vartheta+1)}\left|c_2 - \frac{c_1^2}{2}\left(1 - \frac{5}{2\sqrt{2}}\right)\right| \\ &= \frac{\sqrt{2}}{12(2\vartheta+1)}\left|c_2 - \kappa c_1^2\right|, \end{aligned} \tag{27}$$

where $\kappa = 1/2(1 - (5/2\sqrt{2}))$. Now by applying Lemma 2, we get

$$|a_3| \le \frac{\sqrt{2}}{6(2\vartheta+1)} \max\left\{1, \left|\frac{5}{2\sqrt{2}}\right|\right\} = \frac{5}{12(2\vartheta+1)}. \tag{28}$$

To show these bounds are sharp, we define the function $K_{\phi_n}(\xi)$, $\phi_n = q(\xi^{n-1})(n = 2, 3, 4, \cdots)$ with $K_{\phi_n}(0) = 0 = K'_{\phi_n}(0) - 1$ by

$$\frac{\vartheta\xi^3 K'''_{\phi_n}(\xi) + (1+2\vartheta)K''_{\phi_n}(\xi)\xi^2 + \xi K'_{\phi_n}(\xi)}{\vartheta\xi^2 K''_{\phi_n}(\xi) + \xi K'_{\phi_n}(\xi)} = \varphi\left(\xi^{n-1}\right). \tag{29}$$

Clearly, the function $K_{\phi_n} \in \mathscr{M}(\vartheta, \varphi)$. This completes the proof. □

Theorem 6. *Let the function $f \in \mathscr{M}(\vartheta, \varphi)$ be given by (1) and for any $\varpi \in \mathbb{C}$ then*

$$|a_3 - \varpi a_2^2| \leq \frac{\sqrt{2}}{6(2\vartheta+1)} \max\left\{1, \left|-\frac{5}{2\sqrt{2}} + \varpi\frac{3(2\vartheta+1)}{\sqrt{2}(\vartheta+1)^2}\right|\right\}. \tag{30}$$

Proof. Let the function $f \in \mathscr{M}(\vartheta, \varphi)$ be given by (1), as in Theorem 5, from (23) to (24), we have

$$\begin{aligned} a_3 - \varpi a_2^2 &= \frac{\sqrt{2}}{12(2\vartheta+1)}\left[c_2 - \frac{c_1^2}{2}\left(1 - \frac{5}{2\sqrt{2}}\right)\right] - \varpi\frac{c_1^2}{8(\vartheta+1)^2} \\ &= \frac{\sqrt{2}}{12(2\vartheta+1)}\left[c_2 - \frac{c_1^2}{2}\left(1 - \frac{5}{2\sqrt{2}} + \varpi\frac{3(2\vartheta+1)}{\sqrt{2}(\vartheta+1)^2}\right)\right] \\ &= \frac{\sqrt{2}}{12(2\vartheta+1)}\left[c_2 - \aleph c_1^2\right], \end{aligned} \tag{31}$$

where $\aleph = 1/2(1 - (5/2\sqrt{2}) + \varpi(3(2\vartheta+1)/\sqrt{2}(\vartheta+1)^2))$. Now by Lemma 2, we get

$$\left|a_3 - \varpi a_2^2\right| \leq \frac{\sqrt{2}}{6(2\vartheta+1)} \max\left\{1, \left|-\frac{5}{2\sqrt{2}} + \varpi\frac{3(2\vartheta+1)}{\sqrt{2}(\vartheta+1)^2}\right|\right\}. \tag{32}$$

□

The result is sharp.

In particular, by taking $\varpi = 1$, we get

$$|a_3 - a_2^2| \leq \frac{\sqrt{2}}{6(2\vartheta+1)} \max\left\{1, \left|-\frac{5}{2\sqrt{2}} + \frac{3(2\vartheta+1)}{\sqrt{2}(\vartheta+1)^2}\right|\right\}. \tag{33}$$

Theorem 7. *Let the function $f \in \mathscr{A}$ be given by (1) belongs to the class $\mathscr{M}(\vartheta, \varphi)(0 \leq \vartheta \leq 1)$. Then, for any real number μ, we have*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \dfrac{5(1+\vartheta)^2 - 3(1+2\vartheta)\varpi}{12(2\vartheta+1)(1+\vartheta)^2} & \mu \leq \delta_1, \\ \dfrac{\sqrt{2}}{6(2\vartheta+1)} & \delta_1 \leq \mu \leq \delta_2, \\ \dfrac{3(1+2\vartheta)\varpi - 5(1+\vartheta)^2}{12(2\vartheta+1)(1+\vartheta)^2} & \mu \geq \delta_2, \end{cases} \tag{34}$$

where for convenience

$$\begin{aligned} \delta_1 &= \frac{\left(5 - 2\sqrt{2}\right)(1+\vartheta)^2}{3(1+2\vartheta)}, \\ \delta_2 &= \frac{\left(5 + 2\sqrt{2}\right)(1+\vartheta)^2}{3(1+2\vartheta)}, \end{aligned} \tag{35}$$

$$|a_3 - \mu a_2^2| + \frac{(1+\vartheta)^2 + (1+2\vartheta)\mu}{1+2\vartheta}|a_2|^2 \leq \frac{1}{2(1+2\vartheta)}.$$

If $\delta_3 \leq \mu \leq \delta_2$, then

$$|a_3 - \mu a_2^2| + \frac{3(1+\vartheta)^2 - (1+2\vartheta)\mu}{1+2\vartheta}|a_2|^2 \leq \frac{1}{2(1+2\vartheta)}. \tag{36}$$

These results are sharp.

Proof. Between (23) and (24) and (31), we have

$$\begin{aligned} a_3 - \varpi a_2^2 &= \frac{\sqrt{2}}{12(2\vartheta+1)}\left[c_2 - \frac{c_1^2}{2}\left(1 - \frac{5}{2\sqrt{2}}\right)\right] - \varpi\frac{c_1^2}{8(\vartheta+1)^2} \\ &= \frac{\sqrt{2}}{12(2\vartheta+1)}\left[c_2 - \frac{c_1^2}{2}\left(1 - \frac{5(\vartheta+1)^2 - 3\varpi(2\vartheta+1)}{2\sqrt{2}(\vartheta+1)^2}\right)\right] \\ &= \frac{\sqrt{2}}{12(2\vartheta+1)}\left[c_2 - \hbar c_1^2\right], \end{aligned} \tag{37}$$

where $\hbar = 1/2(1 - ((5(\vartheta+1)^2 - 3\varpi(2\vartheta+1))/2\sqrt{2}(\vartheta+1)^2))$. Our result now follows by virtue of Lemma 3. To show that these bounds are sharp, we define the function $K_{\phi_n}(n = 2, 3, \cdots)$ by

$$\begin{aligned} &\frac{\vartheta\xi^3 K_{\phi_n}''(\xi) + (1+2\vartheta)K_{\phi_n}''(\xi)\xi^2 + \xi K_{\phi_n}'(\xi)}{\vartheta\xi^2 K_{\phi_n}''(\xi) + \xi K_{\phi_n}'(\xi)} = \phi_n\left(\xi^{n-1}\right), \\ &K_{\phi_n}(0) = 0 = K_{\phi_n}'(0) - 1, \end{aligned} \tag{38}$$

and the functions F_η and $G_\eta (0 \leq \eta \leq 1)$ by

$$\begin{aligned} &\frac{\vartheta\xi^3 F_\eta''(\xi) + (1+2\vartheta)F_\eta''(\xi)\xi^2 + \xi F_\eta'(\xi)}{\vartheta\xi^2 F_\eta''(\xi) + \xi F_\eta'(\xi)} = \phi\left(\frac{\xi(\xi+\eta)}{1+\eta\xi}\right)F_\eta(0) \\ &= 0 = F_\eta'(0) - 1, \\ &\frac{\vartheta\xi^3 G_\eta''(\xi) + (1+2\vartheta)G_\eta''(\xi)\xi^2 + \xi G_\eta'(\xi)}{\vartheta\xi^2 G_\eta''(\xi) + \xi G_\eta'(\xi)} = \phi\left(\frac{-\xi(\xi+\eta)}{1+\eta\xi}\right)G_\eta(0) \\ &= 0 = G_\eta'(0) - 1. \end{aligned} \tag{39}$$

Clearly, the functions $K_{\phi_n} = \varphi(\xi^{n-1})$, $F_\eta, G_\eta \in \mathscr{M}(\vartheta, \varphi)$. Also, we write $K_\phi = K_{\phi_2} = 1 + \sqrt{2}\xi + (1/2)\xi^2$. If $\mu < \delta_1$ or μ

$> \delta_2$, then the equality holds if and only if f is K_ϕ or one of its rotations. When $\delta_1 < \mu < \delta_2$, then the equality holds if and only if f is $K_{\phi_3} = \varphi(\xi^2) = 1 + \sqrt{2}\xi^2 + (1/2)\xi^4$ or one of its rotations. If $\mu = \delta_1$, then the equality holds if and only if f is F_η or one of its rotations. If $\mu = \delta_2$, then the equality holds if and only if f is G_η or one of its rotation. □

2. Coefficient Estimates for the Function f^{-1}

Theorem 8. *If $f \in \mathcal{M}(\vartheta, \varphi)$ and $f^{-1}(w) = w + \sum_{n=2}^{\infty} d_n w^n$ is the inverse function of f with $|w| < r_0$ where r_0 is the greater than the radius of the Koebe domain of the class $\mathcal{M}(\vartheta, \varphi)$, then for any complex number μ, we have*

$$|d_2| \le \frac{1}{\sqrt{2}(\vartheta + 1)},$$
$$|d_3| \le \frac{\sqrt{2}}{6(2\vartheta + 1)} \max\left\{1, \left|-\frac{5(\vartheta+1)^2 + 12(2\vartheta+1)}{2\sqrt{2}(\vartheta+1)^2}\right|\right\}. \tag{40}$$

Also, for any complex number μ, we have

$$|d_3 - \mu d_2^2| \le \frac{\sqrt{2}}{6(2\vartheta+1)} \max\left\{1, \left|\frac{5(\vartheta+1)^2 + (12+6\mu)(2\vartheta+1)}{2\sqrt{2}(\vartheta+1)^2}\right|\right\}. \tag{41}$$

The result is sharp. In particular,

$$|d_3 - d_2^2| \le \frac{\sqrt{2}}{6(2\vartheta+1)} \max\left\{1, \left|\frac{5(\vartheta+1)^2 + 18(2\vartheta+1)}{2\sqrt{2}(\vartheta+1)^2}\right|\right\}. \tag{42}$$

Proof. Since

$$f^{-1}(w) = w + \sum_{n=2}^{\infty} d_n w^n, \tag{43}$$

is the inverse function of f, we have

$$f^{-1}(f(\xi)) = f\left(f^{-1}(\xi)\right) = \xi. \tag{44}$$

From equations (23) to (24), we get

$$\xi + (a_2 + d_2)\xi^2 + (a_3 + 2a_2 d_2 + d_3)\xi^3 + \cdots = \xi. \tag{45}$$

Equating the coefficients of ξ and ξ^2 on both sides of (45) and simplifying, we get

$$d_2 = -a_2 = -\frac{c_1}{2\sqrt{2}(\vartheta+1)},$$

$$\begin{aligned} d_3 = 2a_2^2 - a_3 &= \frac{c_1^2}{4(\vartheta+1)^2} - \frac{\sqrt{2}}{12(2\vartheta+1)}\left[c_2 - \frac{c_1^2}{2}\left(1 - \frac{5}{2\sqrt{2}}\right)\right] \\ &= -\frac{\sqrt{2}}{12(2\vartheta+1)}\left[c_2 - \frac{c_1^2}{2}\left(1 - \frac{5}{2\sqrt{2}} - \frac{6(2\vartheta+1)}{\sqrt{2}(\vartheta+1)^2}\right)\right] \\ &= -\frac{\sqrt{2}}{12(2\vartheta+1)}\left[c_2 - \frac{c_1^2}{2}\left(1 - \frac{5(\vartheta+1)^2 + 12(2\vartheta+1)}{2\sqrt{2}(\vartheta+1)^2}\right)\right]. \end{aligned} \tag{46}$$

By applying Lemma 2, we get

$$|d_2| \le \frac{1}{\sqrt{2}(\vartheta+1)},$$
$$|d_3| \le \frac{\sqrt{2}}{6(2\vartheta+1)} \max\left\{1, \left|-\frac{5(\vartheta+1)^2 + 12(2\vartheta+1)}{2\sqrt{2}(\vartheta+1)^2}\right|\right\}. \tag{47}$$

For any complex number μ, consider

$$\begin{aligned} d_3 - \mu d_2^2 &= -\frac{\sqrt{2}}{12(2\vartheta+1)}\left[c_2 - \frac{c_1^2}{2}\left(1 - \frac{5(\vartheta+1)^2 + 12(2\vartheta+1)}{2\sqrt{2}(\vartheta+1)^2}\right)\right] \\ &\quad - \mu\frac{c_1^2}{8(1+\vartheta)^2} = -\frac{\sqrt{2}}{12(2\vartheta+1)}\left[c_2 - \rho^* c_1^2\right], \end{aligned} \tag{48}$$

where

$$\rho^* = \frac{1}{2}\left(1 - \frac{5(\vartheta+1)^2 + 3(4+\mu)(2\vartheta+1)}{2\sqrt{2}(\vartheta+1)^2}\right). \tag{49}$$

Taking modulus on both sides of (49) and applying Lemma 2, we get the estimate as stated in (41). This completes the proof of Theorem 8. □

3. The Logarithmic Coefficients

The logarithmic coefficients e_n of f defined in U are given by

$$\log\frac{f(\xi)}{\xi} = 2\sum_{n=1}^{\infty} e_n \xi^n. \tag{50}$$

Using series expansion of $\log(1+\xi)$ on the left hand side of (50) and equating various coefficients give

$$e_1 = \frac{a_2}{2}, \tag{51}$$

$$e_2 = \frac{1}{2}\left(a_3 - \frac{a_2^2}{2}\right). \tag{52}$$

Theorem 9. *Let $f \in \mathcal{M}(\vartheta, \varphi)$ with logarithmic coefficients given by (51) and (52). Then,*

$$|e_1| \le \frac{1}{2\sqrt{2}(\vartheta+1)},$$
$$|e_2| \le \frac{\sqrt{2}}{12(2\vartheta+1)} \max\left\{1, \left|\frac{5(\vartheta+1)^2+3(2\vartheta+1)}{2\sqrt{2}(\vartheta+1)^2}\right|\right\}, \tag{53}$$

and for any $\nu \in \mathbb{C}$, then

$$|e_2 - \nu e_1^2| \le \frac{\sqrt{2}}{12(2\vartheta+1)} \max\left\{1, \left|-\frac{5(\vartheta+1)^2+(3+\nu)(2\vartheta+1)}{2\sqrt{2}(\vartheta+1)^2}\right|\right\}. \tag{54}$$

These inequalities are sharp. In particular, for $\nu = 1$, we get

$$|e_2 - e_1^2| \le \frac{\sqrt{2}}{12(2\vartheta+1)} \max\left\{1, \left|-\frac{5(\vartheta+1)^2+4(2\vartheta+1)}{2\sqrt{2}(\vartheta+1)^2}\right|\right\}. \tag{55}$$

Proof. Using (23) and (24) in (51) and (52) and after simplification, one may have

$$e_1 = \frac{c_1}{4\sqrt{2}(\vartheta+1)}, \tag{56}$$

$$e_2 = \frac{\sqrt{2}}{24(2\vartheta+1)}\left[c_2 - \frac{c_1^2}{2}\left(1 - \frac{5(\vartheta+1)^2+3(2\vartheta+1)}{2\sqrt{2}(\vartheta+1)^2}\right)\right]. \tag{57}$$

To determine the bounds for e_2, we express (57) in the form

$$e_2 = \frac{\sqrt{2}}{24(2\vartheta+1)}\left[c_2 - \mu^* c_1^2\right], \tag{58}$$

where

$$\mu^* = \left(1 - \frac{5(\vartheta+1)^2+3(2\vartheta+1)}{2\sqrt{2}(\vartheta+1)^2}\right), \tag{59}$$

then by applying Lemma 2, we get

$$|e_2| \le \frac{\sqrt{2}}{12(2\vartheta+1)} \max\left\{1, \left|\frac{5(\vartheta+1)^2+3(2\vartheta+1)}{2\sqrt{2}(\vartheta+1)^2}\right|\right\}. \tag{60}$$

For any $\nu^* \in \mathbb{C}$, from (56) to (57), we have

$$\begin{aligned} e_2 - \nu e_1^2 &= \frac{\sqrt{2}}{24(2\vartheta+1)}\left[c_2 - \frac{c_1^2}{2}\left(1 - \frac{5(\vartheta+1)^2+3(2\vartheta+1)}{2\sqrt{2}(\vartheta+1)^2}\right)\right] \\ &\quad - \nu \frac{c_1^2}{32(\vartheta+1)^2} = \frac{\sqrt{2}}{24(2\vartheta+1)} \\ &\quad \cdot \left[c_2 - \frac{c_1^2}{2}\left(1 - \frac{5(\vartheta+1)^2+(3+\nu)(2\vartheta+1)}{2\sqrt{2}(\vartheta+1)^2}\right)\right] \\ &= \frac{\sqrt{2}}{24(2\vartheta+1)}\left[c_2 - \mu_1^* c_1^2\right], \end{aligned} \tag{61}$$

where

$$\mu_1^* = \frac{1}{2}\left(1 - \frac{5(\vartheta+1)^2+(3+\nu)(2\vartheta+1)}{2\sqrt{2}(\vartheta+1)^2}\right). \tag{62}$$

An application of Lemma 2 gives the desired estimate. □

4. Coefficients Associated with $\xi/f(\xi)$

In this section, we determine the coefficient bounds and Fekete-Szegö problem associated with the function $H(\xi)$ given by

$$H(\xi) = \frac{\xi}{f(\xi)} = 1 + \sum_{n=1}^{\infty} u_n \xi^n \xi \in U), \tag{63}$$

where $f \in \mathcal{M}(\vartheta, \varphi)$.

Theorem 10. *Let $f \in \mathcal{M}(\vartheta, \varphi)$ and $H(\xi)$ are given by (63). Then*

$$|u_1| \le \frac{1}{\sqrt{2}(\vartheta+1)},$$
$$|u_2| \le \frac{\sqrt{2}}{6(2\vartheta+1)} \max\left\{1, \left|-\frac{5}{2\sqrt{2}} + \frac{3(2\vartheta+1)}{\sqrt{2}(\vartheta+1)^2}\right|\right\}. \tag{64}$$

The results are sharp.

Proof. By routine calculation, one may have

$$H(\xi) = \frac{\xi}{f(\xi)} = 1 - a_2\xi + \left(a_2^2 - a_3\right)\xi^2 + \left(a_2^3 + 2a_2a_3 - a_4\right)\xi^3 + \cdots. \tag{65}$$

Comparing the coefficients of ξ and ξ^2 on both sides of (63) and (65), we get

$$u_1 = -a_2, \tag{66}$$

$$u_2 = a_2^2 - a_3. \tag{67}$$

Using (23) and (24) in (66) and (67), we obtain

$$u_1 = -\frac{c_1}{2\sqrt{2}(\vartheta+1)}, \tag{68}$$

By Lemma 2, we get

$$|u_1| \le \frac{1}{\sqrt{2}(\vartheta+1)}. \tag{69}$$

Now,

$$\begin{aligned} u_2 &= \frac{c_1^2}{8(\vartheta+1)^2} - \frac{\sqrt{2}}{12(2\vartheta+1)}\left[c_2 - \frac{c_1^2}{2}\left(1 - \frac{5}{2\sqrt{2}}\right)\right] \\ &= -\frac{\sqrt{2}}{12(2\vartheta+1)}\left[c_2 - \frac{c_1^2}{2}\left(1 - \frac{5}{2\sqrt{2}} + \frac{3(2\vartheta+1)}{\sqrt{2}(\vartheta+1)^2}\right)\right] \\ &= -\frac{\sqrt{2}}{12(2\vartheta+1)}\left[c_2 - \aleph^* c_1^2\right], \end{aligned} \tag{70}$$

where $\aleph^* = 1/2(1 - (5/2\sqrt{2}) + (3(2\vartheta+1)c_1^2/\sqrt{2}(\vartheta+1)^2))$. Again by using Lemma 2, we get

$$|u_2| \le \frac{\sqrt{2}}{6(2\vartheta+1)} \max\left\{1, |- \frac{5}{2\sqrt{2}} + \frac{3(2\vartheta+1)}{\sqrt{2}(\vartheta+1)^2}|\right\}. \tag{71}$$

For any $\nu \in \mathbb{C}$, between (68) and (70), we get

$$\begin{aligned} u_2 - \nu u_1^2 &= -\frac{\sqrt{2}}{12(2\vartheta+1)}\left[c_2 - \frac{c_1^2}{2}\left(1 - \frac{5}{2\sqrt{2}} + \frac{3(2\vartheta+1)}{\sqrt{2}(\vartheta+1)^2}\right)\right] \\ &\quad - \nu\frac{c_1^2}{8(\vartheta+1)^2} = -\frac{\sqrt{2}}{12(2\vartheta+1)} \\ &\quad \cdot \left[c_2 - \frac{c_1^2}{2}\left(1 - \frac{5}{2\sqrt{2}} + \frac{3(1-\nu)(2\vartheta+1)}{\sqrt{2}(\vartheta+1)^2}\right)\right. \\ &\quad \left. + \nu\frac{3(2\vartheta+1)c_1^2}{4\sqrt{2}(\vartheta+1)^2}\right] - = -\frac{\sqrt{2}}{12(2\vartheta+1)}\left[c_2 - \aleph_1^* c_1^2\right]. \end{aligned} \tag{72}$$

That is,

$$|u_2 - \nu u_1^2| = \frac{1}{2(2+\lambda)}\left|c_2 - \aleph^* c_1^2\right|, \tag{73}$$

where

$$\aleph_1^* = \frac{1}{2}\left(1 - \frac{5}{2\sqrt{2}} + \frac{3(1-\nu)(2\vartheta+1)}{\sqrt{2}(\vartheta+1)^2}\right). \tag{74}$$

The result follows by application of Lemma 2 and therefore completes the proof. □

5. Second Hankel Inequality for $f \in \mathcal{M}(\vartheta, \varphi)$

Theorem 11. *Let the function $f \in \mathcal{M}(\vartheta, \varphi)$ be given by (1), then*

$$|a_2 a_4 - a_3^2| \le \frac{1}{18(2\vartheta+1)^2}. \tag{75}$$

Proof. Since $f \in \mathcal{M}(\vartheta, \varphi)$, there exists an analytic function w with $w(0) = 0$ and $|w(\xi)| < 1$ in U such that,

$$\frac{\vartheta\xi^3 f'''(\xi) + (1+2\vartheta)f''(\xi)\xi^2 + \xi f'(\xi)}{\vartheta\xi^2 f''(\xi) + \xi f'(\xi)} = \varphi(w(\xi)). \tag{76}$$

Therefore, between (23), (24), and (25), we get

$$\begin{aligned} a_2 a_4 - a_3^2 = \frac{\sqrt{2}}{768}\Bigg[& c_1^4\Bigg\{\left(6\vartheta^2 + 4\vartheta + 1\right)\left(\frac{-2}{3} + \frac{2\sqrt{2}}{3}\right) \\ & - \frac{1}{3\sqrt{2}}\left(3\vartheta^2 + 4\vartheta + 1\right) + \left(12\vartheta^2 + 4\vartheta + 1\right) \\ & \cdot \left(\frac{-2}{3} + \frac{1}{3\sqrt{2}}\right) - \frac{2\sqrt{2}(4\vartheta+1)}{3}\Bigg\} \\ & + \frac{2c_2c_1^2}{3}\Big\{2\left(12\vartheta^2 + 4\vartheta + 1\right) + 4\left(6\vartheta^2 + 4\vartheta + 1\right) \\ & \cdot \left(-\sqrt{2} + \frac{1}{2}\right)\Big\} + 8\sqrt{2}c_1c_3(2\vartheta+1)^2 \\ & - \frac{16\sqrt{2}c_2^2}{3}\left(3\vartheta^2 + 4\vartheta + 1\right)\Bigg]. \end{aligned} \tag{77}$$

By writing

$$d_1 = \frac{8\sqrt{2}}{(\vartheta+1)(3\vartheta+1)}, \tag{78}$$

$$d_2 = \frac{4\left\{\left(18\vartheta^2 + 8\vartheta + 2\right) - 2\sqrt{2}\left(6\vartheta^2 + 4\vartheta + 1\right)\right\}}{3(\vartheta+1)(3\vartheta+1)(2\vartheta+1)^2}, \tag{79}$$

$$d_3 = -\frac{16\sqrt{2}}{3(2\vartheta+1)^2}, \tag{80}$$

$$d_4 = \frac{1}{(\vartheta+1)(3\vartheta+1)(2\vartheta+1)^2}\left\{\left(6\vartheta^2+4\vartheta+1\right)\left(\frac{-2}{3}+\frac{2\sqrt{2}}{3}\right) - \left(3\vartheta^2+4\vartheta+1\right)\frac{1}{3\sqrt{2}} + \left(12\vartheta^2+4\vartheta+1\right)\left(\frac{-2}{3}+\frac{1}{3\sqrt{2}}\right) - \frac{2\sqrt{2}(4\vartheta+1)}{3}\right\} = \frac{1}{(\vartheta+1)(3\vartheta+1)(2\vartheta+1)^2} \cdot \left(\frac{28-16\sqrt{2}}{3\sqrt{2}}\vartheta^2 - \frac{11}{3}\vartheta - \frac{4}{3}\right), \tag{81}$$

and $T = \sqrt{2}/768$, we have

$$\left|a_2a_4 - a_3^2\right| = T\left|d_1c_1c_3 + d_2c_1^2c_2 + d_3c_2^2 + d_4c_1^4\right|. \tag{82}$$

From (15) to (16), it follows that

$$\left|a_2a_4 - a_3^2\right| = \frac{T}{4}\left|c^4(d_1+2d_2+d_3+4d_4) + 2xc^2\left(4-c^2\right) \cdot (d_1+d_2+d_3) + \left(4-c^2\right)x^2\left(-d_1c^2 + d_3\left(4-c^2\right)\right) + 2d_1c\left(4-c^2\right)\left(1-|x|^2\right)\right|. \tag{83}$$

Replacing $|x|$ by μ and then substituting the values of d_1, d_2, d_3, and d_4 from (81) yield

$$\begin{aligned}\left|a_2a_4 - a_3^2\right| &\le \frac{T}{4(\vartheta+1)(3\vartheta+1)(2\vartheta+1)^2} \cdot \left[c^4\left|(4\vartheta+1) + \frac{4}{3\sqrt{2}}\left(12\vartheta^2+4\vartheta+1\right) - \frac{4}{3\sqrt{2}}(\vartheta+1)(3\vartheta+1)\right| + 16\sqrt{2}(2\vartheta+1)^2 \cdot c\left(4-c^2\right)\left(1-\mu^2\right) + 2\mu c^2\left(4-c^2\right) \cdot \left(\frac{4}{3}\left(12\vartheta^2+4\vartheta+1\right) + \frac{4}{3}\left(6\vartheta^2+4\vartheta+1\right)\right) - \mu^2\left(4-c^2\right)\left(\frac{8\sqrt{2}}{3}\left(6\vartheta^2+4\vartheta+1\right)c^2 + \frac{64\sqrt{2}}{3}(3\vartheta+1)(\vartheta+1)\right)\right] \\ &= \frac{T}{(\vartheta+1)(3\vartheta+1)(2\vartheta+1)^2}\left[\frac{c^4}{3}\left|-2\sqrt{2}(4\vartheta+1) + \frac{1}{\sqrt{2}} \cdot \left(12\vartheta^2+4\vartheta+1\right) - \frac{1}{\sqrt{2}}(\vartheta+1)(3\vartheta+1)\right| + 4\sqrt{2}(2\vartheta+1)^2c\left(4-c^2\right) - 4\sqrt{2}(2\vartheta+1)^2 \cdot c\left(4-c^2\right)\mu^2 + \frac{2}{3}\mu c^2\left(4-c^2\right)\left(2\left(12\vartheta^2+4\vartheta+1\right) + \left(12\vartheta^2+8\vartheta+2\right)\right) - \frac{2\sqrt{2}}{3}\mu^2\left(4-c^2\right)\left(c^2\left(6\vartheta^2+4\vartheta+1\right) + 8\left(3\vartheta^2+4\vartheta+2\right)\right)\right] \\ &= \frac{T}{(\vartheta+1)(3\vartheta+1)(2\vartheta+1)^2} \cdot \left[\frac{c^4}{3}\left|-2\sqrt{2}(4\vartheta+1) + \frac{9}{\sqrt{2}}\vartheta^2\right| + 4\sqrt{2}(2\vartheta+1)^2c\left(4-c^2\right) + \frac{1}{3}\mu c^2\left(4-c^2\right)\left(36\vartheta^2+16\vartheta+4\right) - \frac{2\sqrt{2}}{3}\mu^2\left(4-c^2\right) \cdot \left(c^2\left(6\vartheta^2+4\vartheta+1\right) + 6(2\vartheta+1)^2c + 8\left(3\vartheta^2+4\vartheta+2\right)\right)\right] \\ &\equiv F(c,\mu,\vartheta).\end{aligned} \tag{84}$$

Differentiating $F(c,\mu,\vartheta)$ in (84) partially with respect to μ yields

$$\frac{\partial F}{\partial \mu} = \frac{T}{(\vartheta+1)(3\vartheta+1)(2\vartheta+1)^2}\left[\frac{1}{3}c^2\left(4-c^2\right)\left(36\vartheta^2+16\vartheta+4\right) - \frac{4\sqrt{2}}{3}\mu\left(4-c^2\right)\left(c^2\left(6\vartheta^2+4\vartheta+1\right) + 6(2\vartheta+1)^2c + 8\left(3\vartheta^2+4\vartheta+2\right)\right)\right]. \tag{85}$$

It is clear from (85) that $\partial F/\partial\mu > 0$; thus, $F(c,\mu,\vartheta)$ is an increasing function of μ for $0 < \mu < 1$ and for any fixed c with $0 < c < 2$. So, the maximum of $F(c,\mu,\vartheta)$ occurs at $\mu = 1$ and

$$\max F(c,\mu,\vartheta) = F(c,1,\vartheta) \equiv G(c,\vartheta). \tag{86}$$

Note that

$$\begin{aligned}G(c,\vartheta) &= \frac{T}{(\vartheta+1)(3\vartheta+1)(2\vartheta+1)^2}\left[\frac{c^4}{3}\left(\left|-2\sqrt{2}(4\vartheta+1) + \left(12\vartheta^2+4\vartheta+1\right)\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}(\vartheta+1)(3\vartheta+1)\right| - 2\left(12\vartheta^2+4\vartheta+1\right) - \frac{1}{2}\left(24\vartheta^2+16\vartheta+4\right) + 2\sqrt{2}\left(1+4\vartheta+6\vartheta^2\right)\right) + \frac{8}{3}c^2\left(\left(12\vartheta^2+4\vartheta+1\right) + \left(12\vartheta^2+8\vartheta+2\right) - \sqrt{2}\left(12\vartheta^2+8\vartheta+2\right)\right) + \frac{64\sqrt{2}}{3}(3\vartheta+1)(\vartheta+1)\right] \\ &= \frac{T}{(\vartheta+1)(3\vartheta+1)(2\vartheta+1)^2} \cdot \left[\frac{c^4}{3}\left(\left|-2\sqrt{2}(4\vartheta+1) + \frac{9\vartheta^2}{\sqrt{2}}\right| - 4\left(9\vartheta^2+4\vartheta+1\right) + 2\sqrt{2}\left(1+4\vartheta+6\vartheta^2\right)\right) + \frac{8}{3}c^2\left(3\left(8\vartheta^2+4\vartheta+1\right) - \sqrt{2}\left(12\vartheta^2+8\vartheta+2\right)\right) + \frac{64\sqrt{2}}{3}(3\vartheta+1)(\vartheta+1)\right].\end{aligned} \tag{87}$$

Differentiating $G(c,\vartheta)$ partially with respect to c yields

$$G'(c,\vartheta)=\frac{T}{(\vartheta+1)(3\vartheta+1)(2\vartheta+1)^2}\left[\frac{4c^3}{3}\left(\left|-2\sqrt{2}(4\vartheta+1)+\frac{9\vartheta^2}{\sqrt{2}}\right|-4\left(9\vartheta^2+4\vartheta+1\right)+2\sqrt{2}\left(1+4\vartheta+6\vartheta^2\right)\right)+\frac{16c}{3}\left(3\left(8\vartheta^2+4\vartheta+1\right)-\sqrt{2}\left(12\vartheta^2+8\vartheta+2\right)\right)\right]. \tag{88}$$

If $G'(c,\vartheta)=0$ then its root is $c=0$. Also, we have

$$G''(c,\vartheta)=\frac{T}{(\vartheta+1)(3\vartheta+1)(2\vartheta+1)^2}\left[4c^2\left(\left|-2\sqrt{2}(4\vartheta+1)+\frac{9\vartheta^2}{\sqrt{2}}\right|-4\left(9\vartheta^2+4\vartheta+1+22\sqrt{2}\left(1+4\vartheta+6\vartheta^2\right)\right)\right)+\frac{16}{3}\left(3\left(8\vartheta^2+4\vartheta+1\right)-\sqrt{2}\left(12\vartheta^2+8\vartheta+2\right)\right)\right], \tag{89}$$

is negative for $c=0$, which means that the function $G(c,\vartheta)$ can take the maximum value at $c=0$, also which is

$$|a_2a_4-a_3^2|\le G(0,\vartheta)=\frac{1}{18(2\vartheta+1)^2}. \tag{90}$$

□

Data Availability

No data is used.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

The author Marwan Amin Kutbi expresses his thanks and grateful to King AbdulAziz University (Jeddah, Saudi Arabia) for unlimited support during this research.

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