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Research Article

On the Oscillation of Even-Order Nonlinear Differential Equations with Mixed Neutral Terms

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The oscillation of even-order nonlinear differential equations (NLDiffEqs) with mixed nonlinear neutral terms (MNLNTs) is investigated in this work. New oscillation criteria are obtained which improve, extend, and simplify the existing ones in other previous works. Some examples are also given to illustrate the validity and potentiality of our results.

1. Introduction

Recently, numerous research studies have been carried out concerning the oscillatory behavior of the differential equations with a linear neutral term. Some previous notable studies include the investigation of even-order quasilinear neutral functional differential equations' oscillation (DEqsOs) [1] (see also [2–4]), 3rd-order neutral delay dynamic equations on time scales [5], 2nd-order nonlinear neutral delay differential equation solutions' asymptotic behavior [6] (see also [7]), and 2nd-order superlinear Emden-Fowler neutral DEqsOs [8]. On one hand, higher-order neutral delay DEqsOs was studied in [9]. On the other hand, even-order of DEqsOs and nonlinear neutral DEqsOs with variable coefficients were investigated in [10, 11], respectively. A neutral functional delay differential equation was investigated in the sense of fractional calculus [12] (for more information about the applications of fractional calculus, refer to [13]).

However, differential equations' oscillation with nonlinear neutral terms has been rarely studied in literature. For the case of differential equations with a sublinear neutral term [14–16], Grace et al. [17] proposed differential equations with both sublinear and super-linear neutral terms, where a second-order half-linear differential equation of the following form was investigated:

$$\left(r(t)\left[y^{(n-1)}(t)\right]^{\alpha}\right)' + q(t)x^{\gamma}(\tau_1(t)) = 0, \tag{1}$$

where $n > 0$ is an even integer, and

$$y(t) = x(t) + p_1(t)x^{\beta}(\tau_2(t)) - p_2(t)x^{\delta}(\tau_2(t)). \tag{2}$$

From Equations (1) and (2), the following are assumed:

(i) α, β, γ, and δ are the ratios of two positive odd integers with $\alpha \geq 1$

(ii) $p_1, p_2, q : [t_0, \infty) \longrightarrow \mathbb{R}^+$ are continuous functions

(iii) $\tau_k : [t_0, \infty) \longrightarrow \mathbb{R}$ are continuous functions; $\tau_k(t) \leq t$ and $\tau_k(t) \longrightarrow \infty$ as $t \longrightarrow \infty$ for $k = 1, 2$

(iv) $h(t) = \tau_2^{-1}(\tau_1(t)) \leq t$, and $h(t) \longrightarrow \infty$ as $t \longrightarrow \infty$

Let us suppose that

$$A^*(t, t_0) := \int_{t_0}^{t} (t-s)^{(n-2)} A(s, t_0) ds \longrightarrow \infty \text{as } t \longrightarrow \infty, \quad (3)$$

for which

$$A(t, t_0) := \int_{t_0}^{t} r^{-1/\alpha}(s) ds \longrightarrow \infty \text{ as } t \longrightarrow \infty. \quad (4)$$

A continuous function x satisfying Equation (1) on $[t_*, \infty)$, $t_* \geq t_0$, is said to be a solution of Equation (1) on $[t_*, \infty)$ where $y(t)$ is defined in (2). We only consider those solutions x of (1) which satisfy

$$\sup \{|x(t)|: t \geq t^*\} > 0 \text{ for all } t^* \geq t_*. \quad (5)$$

A solution x of (1) is said to be *oscillatory* if there exists a sequence $\{\xi_n\}$ such that $x(\xi_n) = 0$ and

$$\lim_{n \longrightarrow \infty} \xi_n = \infty. \quad (6)$$

Otherwise, it is called *nonoscillatory*. Equation (1) is said to be an oscillatory (or nonoscillatory) equation if all its solutions are oscillatory (or nonoscillatory).

According to the best of our knowledge, the higher-order differential equations with nonlinear neutral terms have not been studied yet in any other research work. Inspired by the above studies, the oscillation of the proposed differential equations in (1) is investigated in this paper. New oscillation results for Equation (1) are obtained by comparing with the first-order delay differential equations whose oscillatory characters are well-known via an integral criterion. All results in this work are totally new, and more general oscillation results can be obtained by extending our obtained results to more general differential equations with both sublinear and super-linear neutral terms. As a result, a special research interest is hopefully stimulated from our work for possible general investigation of various neutral differential equations' classes, particularly the ones with sublinear and/or superlinear neutral terms.

This article consists of the following sections: our main results are investigated in Section 2. Two illustrative examples are given in Section 3. Then, a short conclusion of our work is provided in Section 4.

2. Main Results

Some oscillation criteria for Equation (1) are studied when $\beta < 1$ and $\delta > 1$.

To obtain our results, the following lemma is needed:

Lemma 1 ([5]). *Let $\mathcal{X}$ and $\mathcal{Y}$ be two nonnegative real numbers. Then, the following inequality is obtained:*

$$\mathcal{X}^\lambda + (\lambda - 1)\mathcal{Y}^\lambda - \lambda \mathcal{X}\mathcal{Y}^{\lambda-1} \begin{cases} \geq 0 \text{ for } & \lambda > 1, \\ \leq 0 \text{ for } & 0 < \lambda < 1, \end{cases} \quad (7)$$

where equality holds if and only if $\mathcal{X} = \mathcal{Y}$.

In what follows, we let

$$\begin{aligned} g_1(t) &:= (1-\beta)\beta^{\beta/(1-\beta)} p^{\beta/(\beta-1)}(t) p_1^{1/(1-\beta)}(t), \\ g_2(t) &:= (\delta - 1)\delta^{\delta/(1-\delta)} p^{\delta/(\delta-1)}(t) p_2^{1/(1-\delta)}(t), \\ Q(t) &:= q(t)[p_2(h(t))]^{-\gamma/\delta}, \end{aligned} \quad (8)$$

for $t \geq t_1$ for some $t_1 \geq t_0$, where $p : [t_0,\infty) \longrightarrow (0,\infty)$ is a continuous function.

Theorem 2. *Let $\beta < 1$ and $\delta > 1$, conditions (i)-(iv), and (3) hold, and let $p \in C([t_0,\infty), (0,\infty))$ such that*

$$p_2(t) \neq 0 \text{ is bounded and } \lim_{t \longrightarrow \infty} [g_1(t) + g_2(t)] = 0, \quad (9)$$

and the equation

$$z'(t) + Cq(t)A^\gamma(\tau_1(t)) z^{\gamma/\alpha}(\tau_1(t)) = 0, \quad (10)$$

is oscillatory for all constant $C > 0$. Let us assume that there exist constants μ_i, $i = 1, 2, 3$, and $\varphi \in (0, 1)$ such that

$$1 \leq \mu_1 \leq \mu_2 \leq \mu_3, \quad (11)$$

$$\mu_3 h(t) \leq t, \quad (12)$$

and the equations

$$\begin{aligned} Z'(t) + Q(t)\left\{\frac{(\mu_2 - \mu_1)^{n-2}}{(n-2)!} h^{n-2}(t) A(\mu_3 h(t), \mu_2 h(t))\right\}^{\gamma/\delta} Z^{\gamma/(\alpha\delta)}(\mu_3 h(t)) &= 0, \\ X'(t) + Q(t)\left\{\frac{\varphi(\mu_2 - \mu_1)}{(n-2)!} h^{n-1}(t)\right\}^{\gamma/\delta} X^{\gamma/\alpha\delta}(\mu_2 h(t)) &= 0, \end{aligned} \quad (13)$$

are oscillatory, and

$$\int_{t_0}^{\infty} Q(s)[A^*(h(s), t_0)]^{\gamma/\delta} ds = \infty. \quad (14)$$

Then, every solution $x(t)$ of Equation (1) is oscillatory, or

$$\lim_{t \longrightarrow \infty} x(t) = \infty. \quad (15)$$

Proof. Without loss of generality, the solution $x(t)$ of Equation (1) is assumed to be positive and $x(\tau_1(t)) > 0$ for $t \geq t_1$ for some $t_1 \geq t_0$ (i.e., a nonoscillatory solution). From Equation (1), we have the following: $x(\tau_2(t)) > 0$and

$$\left(r(t)\left[y^{(n-1)}(t)\right]^{\alpha}\right)' = -q(t)x^{\gamma}(\tau_1(t)) \leq 0. \tag{16}$$

Hence, $r(t)[y^{(n-1)}(t)]^{\alpha}$ is nonincreasing with a constant sign. Namely, $y^{(n-1)}(t) > 0$ or $y^{(n-1)}(t) < 0$ for $t \geq t_2$ for some $t_2 \geq t_1$, so the following four cases are examined separately:

(a) $y(t) > 0$ and $y^{(n-1)}(t) < 0$

(b) $y(t) > 0$ and $y^{(n-1)}(t) > 0$

(c) $y(t) < 0$ and $y^{(n-1)}(t) > 0$

(d) $y(t) < 0$ and $y^{(n-1)}(t) < 0$

Let us first consider the case (a). Since $y^{(n-1)}(t) < 0$ for $t \geq t_2$, we obtain the following:

$$r(t)\left[y^{(n-1)}(t)\right]^{\alpha} \leq -c, \tag{17}$$

for some positive constant c, i.e.,

$$y^{(n-1)}(t) \leq \left(-\frac{c}{r(t)}\right)^{1/\alpha}, \tag{18}$$

for $t \geq t_2$. Integrating the last inequality $(n-1)$-times and by condition (3), we conclude that

$$\lim_{t\longrightarrow\infty} y^{(n-1)}(t) = -\infty, \tag{19}$$

which is a contradiction.

Next, let us consider the case (b). It is obvious that

$$\begin{aligned} y(t) = x(t) &+ \left[p(t)x(\tau_2(t)) - p_2(t)x^{\delta}(\tau_2(t))\right] \\ &+ \left[p_1(t)x^{\beta}(\tau_2(t)) - p(t)x(\tau_2(t))\right]. \end{aligned} \tag{20}$$

From Equation ((2)) of $y(t)$, i.e., we obtain the following:

$$\begin{aligned} x(t) = y(t) &- \left[p(t)x(\tau_2(t)) - p_2(t)x^{\delta}(\tau_2(t))\right] \\ &- \left[p_1(t)x^{\beta}(\tau_2(t)) - p(t)x(\tau_2(t))\right]. \end{aligned} \tag{21}$$

If we apply the first inequality in (7) with $\lambda = \delta > 1$, $\mathscr{X} = p_2^{1/\delta}(t)x(\tau_2(t))$, and

$$\mathscr{Y} = \left[\frac{1}{\delta}p(t)p_2^{-1/\delta}(t)\right]^{1/(\delta-1)}, \tag{22}$$

then we have

$$\begin{aligned} &p(t)x(\tau_2(t)) - p_2(t)x^{\delta}(\tau_2(t)) \\ &\quad\leq (\delta-1)\delta^{\delta/(1-\delta)}p^{\delta/(\delta-1)}(t)p_2^{1/(1-\delta)}(t) \\ &\quad=: g_2(t). \end{aligned} \tag{23}$$

In a similar manner, by applying the second inequality in (7) with $\lambda = \beta < 1$, $\mathscr{X} = p_1^{1/\beta}(t)x(\tau_2(t))$, and

$$\mathscr{Y} = \left[\frac{1}{\beta}p(t)p_1^{-1/\beta}(t)\right]^{1/(\beta-1)}, \tag{24}$$

we obtain the following:

$$\begin{aligned} &p_1(t)x^{\beta}(\tau_2(t)) - p(t)x(\tau_2(t)) \\ &\quad\leq (1-\beta)\beta^{\beta/(1-\beta)}p^{\beta/(\beta-1)}(t)p_1^{1/(1-\beta)}(t) \\ &\quad=: g_1(t). \end{aligned} \tag{25}$$

By using (21) and (23), (25) turns out that

$$x(t) \geq y(t) - g_1(t) - g_2(t) = \left\{1 - \frac{g_1(t)+g_2(t)}{y(t)}\right\}y(t). \tag{26}$$

Since $y(t)$ in nondecreasing, we have the following: $y(t) \geq c_0$ for some $c_0 > 0$. Hence, (26) turns that

$$x(t) \geq \left\{1 - \frac{g_1(t)+g_2(t)}{c_0}\right\}y(t). \tag{27}$$

Now, we see

$$x(t) \geq c_1 y(t), \tag{28}$$

from (9) and (27) for some $c_1 \in (0,1)$. (28) implies that Equation (1) turns to be

$$\left(r(t)\left[y^{(n-1)}(t)\right]^{\alpha}\right)' + c_1^{\gamma}q(t)y^{\gamma}(\tau_1(t)) \leq 0. \tag{29}$$

There exists a constant $\theta_0 \in (0,1)$ such that

$$y(\tau_1(t)) \geq \frac{\theta_0}{(n-1)!}\tau_1^{n-1}(t)y^{(n-1)}(\tau_1(t)), \tag{30}$$

for $t \geq t_1$ (see [16, 18, 19]). By setting $w(t) = r(t)[y^{(n-1)}(t)]^{\alpha}$, we obtain the following:

$$y(\tau_1(t)) \geq \frac{\theta_0}{(n-1)!}\tau_1^{n-1}(t)r^{-1/\alpha}(\tau_1(t))w^{1/\alpha}(\tau_1(t)). \tag{31}$$

By using (31), (29) turns that

$$w'(t) \leq -K\left(\tau_1^{n-1}(t)r^{-1/\alpha}(\tau_1(t))\right)^{\gamma}q(t)w^{\gamma/\alpha}(\tau_1(t)), \tag{32}$$

where

$$K = \left(\frac{c_1 \theta_0}{(n-1)!} \right)^{\gamma}. \tag{33}$$

From Corollary 1 in [20], it can easily be concluded that there exists a positive solution $w(t)$ of Equation (10) with $\lim_{t\longrightarrow\infty} w(t) = 0$, which contradicts the fact that Equation (10) is oscillatory.

Now, let us consider the cases when $y(t) < 0$ for $t \ge t_2$. Suppose that

$$\begin{aligned} v(t) &= -y(t) = -x(t) - p_1(t)x^{\beta}(\tau_2(t)) + p_2(t)x^{\delta}(\tau_2(t)) \\ &\le p_2(t)x^{\delta}(\tau_2(t)), \end{aligned} \tag{34}$$

which implies

$$x(\tau_2(t)) \ge \left[\frac{v(t)}{p_2(t)}\right]^{1/\delta}, \tag{35}$$

or

$$x(t) \ge \left[\frac{v\left(\tau_2^{-1}(t)\right)}{p_2\left(\tau_2^{-1}(t)\right)}\right]^{1/\delta}. \tag{36}$$

On the other hand, we obtain the following:

$$\begin{aligned} \left(r(t)\left[v^{(n-1)}(t)\right]^{\alpha}\right)' &= q(t)x^{\gamma}(\tau_1(t)) \ge q(t)\left[\frac{v\left(\tau_2^{-1}(\tau_1(t))\right)}{p_2\left(\tau_2^{-1}(\tau_1(t))\right)}\right]^{\gamma/\delta} \\ &= Q(t)v^{\gamma/\delta}(h(t)). \end{aligned} \tag{37}$$

Now, let us consider the case (c). Clearly, we see that $v^{(n-1)}(t) \le 0$ and either $v'(t) < 0$ or $v'(t) > 0$ for $t \ge t_1$. First, we assume that $v'(t) < 0$ for $t \ge t_1$. It is easy to see that

$$v(\mu_3 h(t)) \ge \frac{(\mu_2 - \mu_1)^{n-2}}{(n-2)!} h^{n-2}(t) v^{(n-2)}(\mu_2 h(t)), \tag{38}$$

(refer to [18]). Now, we may express

$$\begin{aligned} v^{(n-2)}(u_1) - v^{(n-2)}(u_2) &= -\int_{u_1}^{u_2} r^{-1/\alpha}(s)\left[r(s)\left[z^{(n-1)}(s)\right]^{\alpha}\right]^{1/\alpha} ds \\ &\ge A(u_2, u_1)\left[-r^{-1/\alpha}(u_2)v^{(n-1)}(u_2)\right], \end{aligned} \tag{39}$$

for $t_1 \le u_1 \le u_2$. By taking $u_1 = \mu_2 h(t)$ and $u_2 = \mu_3 h(t)$ for $t \ge t_1$ in inequality (39), we see that

$$v^{(n-2)}(\mu_2 h(t)) \ge A(\mu_3 h(t), \mu_2 h(t))\left[-r^{-1/\alpha}(\mu_3 h(t))v^{(n-1)}(\mu_3 h(t))\right]. \tag{40}$$

By using (40), (38) turns out to be

$$\begin{aligned} v(\mu_3 h(t)) \ge{}& \frac{(\mu_2 - \mu_1)^{n-2}}{(n-2)!} h^{n-2}(t) A(\mu_3 h(t), \mu_2 h(t)) \\ &\times \left[-r^{-1/\alpha}(\mu_3 h(t))v^{(n-1)}(\mu_3 h(t))\right]. \end{aligned} \tag{41}$$

By setting $V(t) := -r(t)[v^{(n-1)}(t)]^{\alpha}$ for $t \ge t_1$, (41) turns that

$$\begin{aligned} v(h(t)) \ge v(\mu_3 h(t)) \ge{}& \frac{(\mu_2 - \mu_1)^{n-2}}{(n-2)!} h^{n-2}(t) A(\mu_3 h(t), \mu_2 h(t)) \\ &\times \left[-V^{1/\alpha}(\mu_3 h(t))\right]. \end{aligned} \tag{42}$$

From (42) and (31), we obtain the following:

$$\begin{aligned} -V'(t) &\ge Q(t)v^{\gamma/\delta}(h(t)) \\ &\ge Q(t)\left\{\frac{(\mu_2 - \mu_1)^{n-2}}{(n-2)!} h^{n-2}(t) A(\mu_3 h(t), \mu_2 h(t))\right\}^{\gamma/\delta} \\ &\quad \times \left[V^{\gamma/(\alpha\delta)}(\mu_3 h(t))\right], \end{aligned} \tag{43}$$

which implies

$$\begin{aligned} V'(t) + Q(t)&\left\{\frac{(\mu_2 - \mu_1)^{n-2}}{(n-2)!} h^{n-2}(t) A(\mu_3 h(t), \mu_2 h(t))\right\}^{\gamma/\delta} \\ &\cdot \left[V^{\gamma/(\alpha\delta)}(\mu_3 h(t))\right] \le 0. \end{aligned} \tag{44}$$

The proof can be easily completed by following the same steps as we did for the case (a) and hence is omitted.

Next, we assume that $v'(t) > 0$ for $t \ge t_1$. Clearly, we have the following:

$$v^{(n-2)}(\mu_3 h(t)) \ge -(\mu_2 - \mu_1)h(t)v^{(n-1)}(\mu_2 h(t)). \tag{45}$$

There exists a constant $\theta_1 \in (0,1)$ such that

$$\begin{aligned} v(h(t)) &\ge \frac{\theta_1}{(n-2)!} h^{n-2}(t) v^{(n-2)}(h(t)) \\ &\ge \frac{\theta_1}{(n-2)!} h^{n-2}(t) v^{(n-2)}(\mu_1 h(t)), \end{aligned} \tag{46}$$

for $t \ge t_1$. Now, we see that

$$\begin{aligned} v(h(t)) &\ge \frac{\theta_1}{(n-2)!} h^{n-2}(t) v^{(n-2)}(\mu_1 h(t)) \\ &\ge \frac{\theta_1}{(n-2)!} h^{n-2}(t)(\mu_2 - \mu_1)h(t)\left[-v^{(n-1)}(\mu_2 h(t))\right]. \end{aligned} \tag{47}$$

The rest of the proof is similar to that of the above case and hence is omitted.

Finally, let us consider the case (d). Clearly, we have $r(t)[v'(t)]^{\alpha} > 0$ and so

$$r(t)\left[v^{(n-1)}(t)\right]^{\alpha} \geq c_2, \tag{48}$$

or that

$$v^{(n-1)}(t) \geq \left(\frac{c_2}{r(t)}\right)^{1/a}, \tag{49}$$

for some $c_2 > 0$. Thus, we obtain the following:

$$v(t) \geq c_2^{1/\alpha} A^*(t, t_2) \tag{50}$$

for $t \geq t_3 \geq t_2$. By using (50), (37) turns out

$$\left(r(t)\left[v^{(n-1)}(t)\right]^{\alpha}\right)' \geq Q(t) v^{\gamma/\delta}(h(t)) \geq Q(t)\left[c_2^{1/\alpha} A^*(h(t), t_2)\right]^{\gamma/\delta}. \tag{51}$$

The rest of the proof is trivial and hence is omitted. This completes the proof. □

Corollary 3. *Let $\beta < 1$ and $\delta > 1$, conditions (i)-(iv), and (3) hold, and let $p \in C([t_0,\infty), ((0,\infty)$ such that (9) holds. Assume that there exist real numbers μ_i, $i = 1, 2, 3$ such that (11) is satisfied. If we have condition (14), then*

$$\begin{aligned}
&\lim_{t\longrightarrow\infty} \int_{\tau_1(t)}^{t} q(s) A^{\gamma}(\tau_1(s)) ds = \infty \text{when } \gamma \leq \alpha, \\
&\liminf_{t\longrightarrow\infty} \int_{\mu_3 h(t)}^{t} Q(s)\left\{h^{n-2}(s) A(\mu_3 h(s), \mu_2 h(s))\right\}^{\gamma/\delta} ds > \frac{1}{e}\left(\frac{(n-2)!}{(\mu_2-\mu_1)^{n-2}}\right)^{\gamma/\delta} \text{ when } \gamma = \alpha\delta, \\
&\lim_{t\longrightarrow\infty} \int_{\mu_3 h(t)}^{t} Q(s)\left\{h^{n-2}(s) A(\mu_3 h(s), \mu_2 h(s))\right\}^{\gamma/\delta} ds = \infty \text{when } \gamma < \alpha\delta, \\
&\lim_{t\longrightarrow\infty} \int_{\mu_2 h(t)}^{t} \left[h^{n-1}(s)(\mu_2-\mu_1)\right]^{\gamma/\delta} Q(s) ds = \infty \text{when } \gamma \leq \alpha.
\end{aligned} \tag{52}$$

Then, Equation (1) is oscillatory.

3. Illustrative Examples

Two illustrative examples are presented in this section as follows:

Example 1. Consider the following second-order equation:

$$\begin{aligned}
&\left(e^{-t}\left(x(t) + \frac{1}{t} x^{1/3}(t/2) - x^3(t/2)\right)'\right)' \\
&\quad + \left(\frac{3}{4} - \left(\frac{5}{36t} + \frac{1}{2t^2} + \frac{2}{t^3}\right) e^{-4t/3}\right) x(t/2) = 0.
\end{aligned} \tag{53}$$

Clearly, $r(t) = e^{-t}$, $p_1(t) = p(t) = t^{-1}$, and $p_2(t) = 1$, and hence, there exists a $t_* \geq 3$ such that

$$\frac{3}{4} - \left(\frac{5}{36t} + \frac{1}{2t^2} + \frac{2}{t^3}\right) e^{-4t/3} > 0, \tag{54}$$

for $t \geq t_*$. The verification of all the conditions of Theorem 2 gives that every solution x of Equation (53) is oscillatory; otherwise, $\lim_{t\longrightarrow\infty} x(t) = \infty$. It is worth mentioning that $x_1(t) = e^t$ is such a solution of Equation (53).

Example 2. Consider the following even-order equation:

$$\begin{aligned}
&\left(e^{-t}\left(x(t) + \frac{1}{t} x^{1/3}(t/2) - x^3(t/2)\right)^{(n-1)}\right)' \\
&\quad + \left(\frac{1}{t} e^{-t/2}\right) x(t/2) = 0.
\end{aligned} \tag{55}$$

By noting that $r(t) = e^{-t}$, $p_1(t) = p(t) = t^{-1}$, $p_2(t) = 1$, and $q(t) = e^{-t/2}/t$ and letting $\mu_1 = 1/8$, $\mu_2 = 1/4$, and $\mu_3 = 3/8$, it can be easily seen that all the conditions of Corollary 3 hold, and hence, Equation (55) is oscillatory.

4. Conclusion

New results concerning the oscillation of NLDiffEq with MNLNTs have been successfully established in this paper. We have used novel technique which is based on a basic inequality and some comparison results to prove the main theorem. Demonstrating the validity and applicability of our results, two examples have been presented in this regard. It is worth mentioning that the oscillation of Equations (53) and (55) cannot be commented by previous works.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

All authors have taken equal part in this research, and they read and approve the final manuscript.

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