Hindawi
Journal of Function Spaces
Volume 2021, Article ID 4414545, 11 pages
https://doi.org/10.1155/2021/4414545

*Research Article*

# Existence, Decay, and Blow-Up of Solutions for a Higher-Order Kirchhoff-Type Equation with Delay Term

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Received 19 April 2021; Accepted 18 August 2021; Published 2 September 2021

Academic Editor: Gangadharan Murugusundaramoorthy



This article deals with the study of the higher-order Kirchhoff-type equation with delay term in a bounded domain with initial boundary conditions, where firstly, we prove the global existence result of the solution. Then, we discuss the decay of solutions by using Nakao's technique and denote polynomially and exponentially. Furthermore, the blow-up result is established for negative initial energy under appropriate conditions.

## 1. Introduction

In this paper, we establish the higher-order Kirchhoff-type equation with delay term as follows:

$$\begin{cases} u_{tt} + \left(\int_\Omega \left|A^{m/2}u\right|^2 dx\right)^q A^m u + \mu_1 |\mu_t(x,t)|^{r-1}\mu_t(x,t) & \\ +\mu_2 |\mu_t(x,t-\tau)|^{r-1} u_t(x,t-\tau) = |u|^{p-1}u, & (x,t) \in \Omega \times (0,T), \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x) & x \in \Omega \\ u_t(x,t-\tau) = f_0(x,t-\tau) & x \in \Omega, t \in (0,\tau), \\ \dfrac{\partial^i u}{\partial \nu^i} = 0, \ i = 0, 1, \cdots, m-1 & x \in \partial\Omega, \end{cases} \quad (1)$$

where $A = -\Delta$, $m \geq 1$is a natural number, $q, r \geq 0$ are real numbers, $p > 1$ is a real number and is a bounded domain with smooth boundary $\partial\Omega$ in $R^n$, $n = 1; 2; 3$; $\nu$ is the outer normal. $\tau > 0$ denotes time delay, and $_1$ and $_2$ are positive real numbers. The functions $(u_0, u_1, f_0)$ are the initial data belong to a suitable space.

The problem (1) is a general form of a model introduced by Kirchhoff [1]. To be more precise, Kirchhoff recommended a model denoted by the equation for $f = g = 0$,

$$ph \frac{\partial^2 u}{\partial t^2} + \delta \frac{\partial u}{\partial t} + g\left(\frac{\partial u}{\partial t}\right) = \left\{\rho_0 + \frac{Eh}{2L}\int_0^L \left(\frac{\partial u}{\partial x}\right)^2 dx\right\} \frac{\partial^2 u}{\partial x^2} + f(u), \quad (2)$$

for $0 < x < L, t \geq 0$, where $u(x,t)$ is the lateral displacement, $\rho$ is the mass density, $h$ is the cross-section area, $E$ is the Young modulus, $L$ is the length, $\rho_0$ is the initial axial tension, and $f, g$ are the external forces. Furthermore, (2) is called a degenerate equation when $\rho_0 = 0$ and nondegenerate one when $\rho_0 > 0$.

Time delays often appear in many various problems, such as thermal, economic phenomena, biological, chemical, and physical. Recently, the partial di/erential equations with time delay have become an active area, (see [2, 3] and references therein). Datko et al. [4] indicated that a small delay in a boundary control is a source of instability. An arbitrarily small delay may destabilize a system which is uniformly asymptotically stable without delay unless additional conditions or control terms have been used in many cases [5]. Additional control terms will be necessary to stabilize hyperbolic systems including delay terms, (see [6–8] and references therein). In [6], Nicaise and Pignotti studied the equation as follows:

$$u_{tt}(x,t) - \Delta u(x,t) + a_0 u_t(x,t) + a u_t(x,t-\tau) = 0, \quad (3)$$

where $a_0$ and $a$ are positive real parameters. The authors obtained that, under the condition $0 \le \alpha \le a_0$, the system is exponentially stable. In the case $\alpha \ge \alpha_0$, they obtained a sequence of delays that shows the solution is instable. In [8], Xu et al. obtained the same result similar to the [6] for the one space dimension by adopting the spectral analysis approach. In [9], Nicaise et al. studied the wave equation in one space dimension in the case of time-varying delay. In that work, the authors showed that an exponential stability result under the condition

$$\alpha \le \sqrt{1 - d\alpha_0}, \quad (4)$$

where $d$ is a constant such that

$$\tau'(t) \le d < 1, \ \forall t > 0. \quad (5)$$

In recent years, some other authors investigate hyperbolic type equation with delay term (see [10–16]).

Without delay term $(\mu_2 |u_t(x,t\text{-}\tau)|^{r\text{-}1} u_t(x,t-\tau))$, in 2004, Li [17] studied the higher-order Kirchho/-type equation as follows:

$$\mu_{tt} + \left(\int_\Omega \left|D^{m/2}u\right|^2 dx\right)^q (-\Delta)^m u + u_t|u_t|^r = |u|^p u, \quad (6)$$

where $m > 1$ is a positive integer, and $q, p, r > 0$ is a positive constant. The author obtained that the solution exists globally if $p \le r$, while if $p > \max\{r, 2q\}$. He also established the blow-up result for $E(0) < 0$. Later, in 2007, Messaoudi and Houari [18] obtained the blow-up of solutions with $E(0) > 0$ of the equation (6). Then, Piskin and Polat [19] considered global existence and decay estimates utilizing Nakao's inequality of the equation (6).

Without delay term, when $m = 1$ and $q = 0$, equation (1) takes the form of a semilinear hyperbolic equation as follows:

$$u_{tt} - \Delta u + u_t|u_t|^{r-1} = |u|^{p-1}u. \quad (7)$$

Georgiev and Todorova [20] obtained the blow-up of solutions for $E(0) < 0$ if $1 < r < p$ $(1 < p < n/(n-2)$ for $n \ge 3$, $p > 1$ for $n < 3)$ of the equation (7). Under the condition of positive upper bounded initial energy, Vitillaro [21] proved the same results of equation (7). Also, Ohta [22, 23] studied related problems for the blow-up results of the equation (7).

Messaoudi [24] studied the following equation

$$u_{tt} + \Delta^2 u + |u_t|^{r2} u_t = |u|^{p2} u \quad (8)$$

and obtained an existence result for the equation (8) and proved that the solution continues to exists globally if $r \ge p$; however, if $r < p$ and the initial energy is negative, the solution blows up in finite time. Chen [25] established that the solution of (8) blows up with $E(0) > 0$. In the presence of strong damping term $(-\Delta u_t)$, Piskin and Polat [26] obtained the decay estimates by using Nakao's inequality of equation (8).

When $m = 1$ and without delay term, equation (1) takes the form the following Kirchhoff-type equation:

$$u_{tt}\text{-}\left(\int_\Omega |D_u|^2 dx\right)^\gamma \Delta_u + u_t|u_t|^r = |u|^p u. \quad (9)$$

Many authors had studied existence and blow-up results at night time for equation (9) (see [27–30]). Ono [30] proved the blow-up results if $p > \max\{r, 2\gamma\}$ $(p < 2/(n-4)$ for $n \ge 5$, $p > 0$ for $n \le 4)$ and $E(0) < 0$ for equation (9). Later, Benaissa and Messaoudi [31] obtained the similar result for the generalized Kirchhoff-type equation as follows:

$$u_{tt} - M\left(\int_\Omega e^{\phi(x)}|\nabla u|^2 dx\right) e^{-\phi(x)} \operatorname{div}\left(e^{\phi(x)}\nabla u\right) + \alpha|u_t|^{r-2}u_t = b|u|^{p-2}u, \quad (10)$$

where $M : R^+ \longrightarrow R^+$ and $\phi(x)$ are bounded functions. Then, Wu [32], verified the same result of the general Kirchhoff-type equation

$$u_{tt} - M\left(\|\nabla u\|_2^2\right)\Delta u + |u_t|^{r-2}u_t = |u|^{p-2}u, \quad (11)$$

with the positive upper bounded initial energy. In 2013, Ye [33] considered the global existence results by constructing a stable set in $H_0^1(\Omega)$ and showed the decay by using a lemma of Komornik for the nonlinear Kirchhoff-type equation (11) with dissipative term. Moreover, Ye [34] obtained the global existence results by constructing a stable set in $H_0^m(\Omega)$ and showed the energy decay by using a lemma of $V$. Komornik for a nonlinear higher-order Kirchhoff-type equation with dissipative term is as follows:

$$u_{tt} + \left\|A^{1/2}u\right\|^{2p} Au + a|u_t|^{r-2}u_t = b|u|^{p-2}u, \quad (12)$$

where $A = (-\Delta)^m$, $m > 1$ is a positive integer.

Gao et al. [35] considered the Kirchhoff-type equation without delay term as follows:

$$u_{tt} + M\left(\|D^m u\|_2^2\right)(-\Delta)^m u + |u_t|^{r-2}u_t = |u|^{p-2}u. \quad (13)$$

The authors obtained the blow-up of solutions for $E(0) > 0$ under appropriate conditions for equation (13).

In [36–40], some authors studied abstract evolution equations as follows:

$$[P(u_t)]_t A(t,u) + Q(t,u_t) = F(u) \quad (14)$$

on suitable Banach space, and they proved some global nonexistence of solutions. Some other authors studied related problems (see [41–45]).

Motivated by the above works, we deal with the existence, decay, and blow-up results for the higher-order Kirchhoff type equation (1) with delay term and source term. There is no research, to our best knowledge, related to the higher-order Kirchhoff-type $((\int_\Omega |A^{m/2}u|^2 dx)^q A^m u)$ equation (1) with delay $(u_2|u_t(x,t-\tau)|^{r-1}u_t(x,t-\tau))$ and source $(|u|^{p-1}u)$ terms; hence, our work is the generalization of the above studies.

This work consists of five sections in addition to the introduction: Firstly, in Sect. 2, we recall some lemmas and assumptions. Then, in Section 3, we get the global existence of solutions. Moreover, in Section 4, we establish the decay results by using Nakao's tecnique. Finally, in Section 5, we obtain the blow-up of solutions for negative initial energy.

## 2. Preliminaries

In this part, we present some lemmas and assumptions for the proof of our result. Let $H^m(\Omega)$ denote the Sobolev space with the norm

$$\|u\|_{H^m(\Omega)} = \left(\sum_{|\alpha|\le m} \|D^\alpha u\|^2_{L^2(\Omega)}\right)^{1/2}., \tag{15}$$

$H_0^m(\Omega)$ denotes the closure in $H^m(\Omega)$ of $C_0^\infty(\Omega)$. For simplicity of notation, we denote by $\|\cdot\|_p$ the Lebesgue space $L^p(\Omega)$ norm, $\|\cdot\|$ denotes $L^2(\Omega)$ norm, and we write equivalent norm $\|\nabla\cdot\|$instead of $H_0^1(\Omega)$ norm $\|\cdot\|H_0^1(\Omega)$. We denote by $C_i (i=1,2,\cdots,n)$ various positive constants which may be different at different occurrences.

**Lemma 1** (see [46, 47] Sobolev-Poincarė inequality). *If $2 \le p \le (2n/[n-2m]^+)(2\le p<\infty$ if $n=2m)$, then for some $C_*$,*

*$\|u\|_p \le C_* \|(-\Delta)^{m/2}u\|$ for $u \in H_0^m(\Omega)$,*

*where we put $[\alpha]^+ = \max\{0,\alpha\}$, $1/[\alpha]^+ = \infty$ if $[\alpha]^+ = 0$.*

**Lemma 2** (see [48]). *Let $\phi(t)$ be nonincreasing and nonnegative function defined on $[0,T]$, $T>1$ and satisfies*

$$\phi^{1+\alpha}(t) \le w_0(\phi(t)-\phi(t+1)),\ t\in[0,T] \tag{16}$$

*for $w_0$ is a positive constant, and $\alpha$ is a nonnegative constant. Then, we have for each $t \in [0,T]$,*

$$\begin{cases} \phi(t) \le \phi(0)e^{-w_1[t-1]^+}, & \alpha = 0, \\ \phi(t) \le \left(\phi(0)^{-\alpha} + w_0^{-1}\alpha[t-1]^+\right)^{-1/\alpha}, & \alpha = 0, \end{cases} \tag{17}$$

*where $[t-1]^+ = \max\{t-1,0\}$, and $w_1 = \ln(w_0/w_0 - 1)$.*

*We make the assumptions on parameters r, p, and m as follows:*

*(A1)*

$$\begin{cases} 1<p<\infty, & n\le 2m, \\ 1<p<\dfrac{n}{n-2m}, & n>2m, \end{cases} \tag{18}$$

*(A2)*

$$\begin{cases} 1<r<\infty, & n\le 2m, \\ 1<r<\dfrac{n+2m}{n-2m}, & n>2m. \end{cases} \tag{19}$$

## 3. Global Existence

In this part, we consider the global existence results of the problem (1). Firstly, we introduce the new function $z$ similar to the [7],

$$z(x,k,t) = u_t(x,t-\tau k), x\in\Omega, k\in(0,1). \tag{20}$$

Thus, we have

$$\tau z_t(x,k,t) + z_k(x,k,t) = 0, \text{in}\Omega\times(0,1)\times(0,\infty). \tag{21}$$

Hence, problem (1) can be transformed as follows:

$$\begin{cases} u_{tt} + \left(\int_\Omega |A^{m/2}u|^2 dx\right)^q A^m u + \mu_1|u_t(x,t)|^{r-1}u_t(x,t) + \mu_2|z(x,1,t)|^{r-1}z(x,1,t) = |u|^{p-1}u & (x,t)\in\Omega\times(0,T), \\ \tau x_t(x,k,t) + z_k(x,k,t) = 0 & \text{in } \Omega\times(0,1)\times(0,\infty), \\ z(x,k,0) = f_0(x,-\tau k) & x\in\Omega, \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x) & x\in\Omega, \\ \dfrac{\partial^i u}{\partial v^i} = 0, i = 0,1,\cdots,m-1 & x\in\partial\Omega. \end{cases} \tag{22}$$

We define the energy functional for any regular solution of (22) as follows:

$$E(t)=\frac{1}{2}\|u_t\|^2+\frac{1}{2(q+1)}\left\|A^{m/2}u\right\|^{2(q+1)}-\frac{1}{p+1}\|u\|_{p+1}^{p+1}+\frac{\varsigma}{r+1}\int_\Omega\int_0^1 z^{r+1}(x,k,s)dkdx, \tag{23}$$

such that

$$\tau r|\mu_2|<\varsigma<\tau((r+1)\mu_1-|\mu_2|). \tag{24}$$

Also, have

$$J(t)=J(u(t))=\frac{1}{2(q+1)}\left\|A^{m/2}u\right\|^{2(q+1)}-\frac{1}{p+1}\|u\|_{p+1}^{p+1}+\frac{\varsigma}{r+1}\int_\Omega\int_0^1 z^{r+1}(x,k,s)dkdx, \tag{25}$$

$$I(t)=I(u(t))=\left\|A^{m/2}u\right\|^{2(q+1)}-\|u\|_{p+1}^{p+1}+\varsigma\int_\Omega\int_0^1 z^{r+1}(x,k,s)dkdx. \tag{26}$$

We easily see that

$$E(t)=J(t)+\frac{1}{2}\|u_t\|^2. \tag{27}$$

Furthermore, we define

$$\mathscr{W}=\left\{u: u\in H_0^m(\Omega)\cap H^{2m}(\Omega), I(u)>0\right\}\cup\{0\}. \tag{28}$$

Next, lemma gives that the energy functional $E(t)$ is a nonincreasing.

**Lemma 3.** *Assume that $(u,z)$ is the solution of (22), then for $t\geq 0$,*

$$E'(t)=-\left(\mu_1-\frac{\varsigma}{\tau(r-1)}-\frac{\mu_2}{r+1}\right)\|u_t(t)\|_{r+1}^{r+1}-\left(\frac{\varsigma}{\tau(r+1)}-\frac{\mu_2 r}{r+1}\right)\int_\Omega z^{r+1}(x,1,t)dx\leq 0. \tag{29}$$

*Proof.* We multiply the first equation in (22) by $u_t$, integrate over, and use integration by parts, and we obtain

$$\frac{d}{dt}\left[\frac{1}{2}\|u_t(t)\|^2+\frac{1}{2(q+1)}\left\|A^{m/2}u\right\|^{2(q+1)}-\frac{1}{p+1}\|u\|_{p+1}^{p+1}\right]+\mu_1\|u_t(t)\|_{r+1}^{r+1}\int_\Omega\mu_2|z(x,1,t)|^{r-1}z(x,1,t)u_t(x,t)dx=0. \tag{30}$$

Integrating (30) over $(0,t)$, we get

$$\left[\frac{1}{2}\|u_t(t)\|^2+\frac{1}{2(q+1)}\left\|A^{m/2}u\right\|^{2(q+1)}-\frac{1}{p+1}\|u\|_{p+1}^{p+1}\right]+\int_0^t\mu_1\|u_s(s)\|_{r+1}^{r+1}ds+\mu_2\int_0^t\int_\Omega|z(x,1,s)|^{r-1}z(x,1,s)u_s(x,s)dxds=\frac{1}{2}\|u_1\|^2. \tag{31}$$

We multiply the second equation in (22) by $\varsigma|z|^{r-1}z$ and integrate the result over $\Omega\times(0,1)\times(0,t)$, and we get

$$\begin{aligned}\frac{\varsigma}{r+1}\frac{d}{dt}\int_0^t\int_\Omega\int_0^1|z(x,k,t)|^{r-1}(x,k,t)z_t(x,k,t)dkdxds&=-\frac{\varsigma}{\tau(r+1)}\int_0^t\int_\Omega\int_0^1\frac{\partial}{\partial k}|z(x,k,t)|^{r+1}dkdxds\\&=-\frac{\varsigma}{\tau(r+1)}\int_0^t\int_\Omega\left[|z(x,1,t)|^{r+1}-|z(x,0,t)|^{r+1}\right]dxds\\&=-\frac{\varsigma}{\tau(r+1)}\int_0^t\int_\Omega|z(x,1,t)|^{r+1}dxds+\frac{\varsigma}{\tau(r+1)}\int_0^t\|u_t(t)\|_{r+1}^{r+1}ds.\end{aligned} \tag{32}$$

By combining (31) and (32), we arrive at

$$E(t)+\left(\mu_1-\frac{\varsigma}{\tau(r+1)}\right)\int_0^t\|u_s(s)\|_{r+1}^{r+1}ds+\frac{\varsigma}{\tau(r+1)}\int_0^t\int_\Omega|z(x,1,s)|^{r+1}dxds+\mu_2\int_0^t\int_\Omega|z(x,1,s)|^{r-1}z(x,1,s)u_s(x,s)dxds=E(0). \tag{33}$$

Utilizing the Young inequality on the fourth term of the left hand side of (33), we conclude that

$$E(t)+\left(\mu_1-\frac{\varsigma}{\tau(r+1)}-\frac{\mu_2}{r+1}\right)\int_0^t\|u_s(s)\|_{r+1}^{r+1}ds+\left(\frac{\varsigma}{\tau(r+1)}-\frac{\mu_2 r}{r+1}\right)\int_0^t\int_\Omega|z(x,1,s)|^{r+1}dxds=E(0). \tag{34}$$

Deriving the (34), we have the desired result. Hence, the proof is completed. □

*Remark 4.* From the condition (24), we obtain

$$c_1=\left(\mu_1-\frac{\varsigma}{\tau(r+1)}-\frac{\mu_2}{r+1}\right)>0, c_2=\left(\frac{\varsigma}{\tau(r+1)}-\frac{\mu_2 r}{r+1}\right)>0. \tag{35}$$

**Lemma 5.** *Assume that (19) and $p > 2q + 1$ hold. Let $u_0 \in \mathscr{W}$ and $u_1 \in H_0^m(\Omega)$, such that*

$$\beta = C_* \left( \frac{2(q+1)(p+1)}{p-2q-1} E(0) \right)^{p-2q-1/2(q+1)} < 1, \tag{36}$$

*then $u \in \mathscr{W}$ for each $t \geq 0$.*

*Proof.* It follows the continuity of $u(t)$, since $I(0) > 0$, such that

$$I(t) > 0, \tag{37}$$

for some interval near $t = 0$. Assume that $T_m > 0$ is a maximal time, when (26) holds on $[0, T_m]$.

By (25) and (26), we obtain

$$\begin{aligned} J(t) &= \frac{1}{p+1} I(t) + \frac{p-2q-1}{2(q+1)(p+1)} \left\| A^{m/2} u \right\|^{2(q+1)} \\ &\quad + \frac{(p-r)}{(r+1)(p+1)} \left( \varsigma \int_\Omega \int_0^1 z^{r+1}(x,k,t) dk dx \right) \\ &\geq \frac{p-2q-1}{2(q+1)(p+1)} \left\| A^{m/2} u \right\|^{2(q+1)}. \end{aligned} \tag{38}$$

From (23), (38), and Lemma 3, we have

$$\begin{aligned} \left\| A^{m/2} u \right\|^{2(q+1)} &\leq \frac{2(q+1)(p+1)}{p-2q-1} J(t) \leq \frac{2(q+1)(p+1)}{p-2q-1} E(t) \\ &\leq \frac{2(q+1)(p+1)}{p-2q-1} E(0). \end{aligned} \tag{39}$$

Using Lemma 1 and (39), we get

$$\begin{aligned} \|u\|_{p+1}^{p+1} &\leq C_* \left\| A^{m/2} u \right\|^{p+1} = C_* \left\| A^{m/2} u \right\|^{p-2q-1} \left\| A^{m/2} u \right\|^{2(q+1)} \\ &\leq C_* \left( \frac{2(q+1)(p+1)}{p-2q-1} E(0) \right)^{p-2q-1/2(q+1)} \left\| A^{m/2} u \right\|^{2(q+1)} \\ &= \beta \left\| A^{m/2} u \right\|^{2(q+1)} < \left\| A^{m/2} u \right\|^{2(q+1)} \text{ on } t \in [0, T_m]. \end{aligned} \tag{40}$$

Thus, from (26), we arrive at $I(t) > 0$ for all $t \in [0, T_m]$. $T_m$ is extended to $T$, by repeating the procedure. Hence, the proof is completed. □

**Lemma 6.** *Suppose that the assumptions of Lemma 5 hold. Then, there exists $\eta_1 = 1 - \beta$, such that*

$$\|u\|_{p+1}^{p+1} \leq (1 - \eta_1) \left\| A^{m/2} u \right\|^{2(q+1)}. \tag{41}$$

*Proof.* By (40), we obtain

$$\|u\|_{p+1}^{p+1} \leq \beta \left\| A^{m/2} u \right\|^{2(q+1)}. \tag{42}$$

Let $\eta_1 = 1 - \beta$; therefore, we obtain the result. □

*Remark 7.* By Lemma 6, we conclude that

$$\left\| A^{m/2} u \right\|^{2(q+1)} \leq \frac{1}{\eta_1} I(t). \tag{43}$$

**Theorem 8.** *Assume that the assumptions (A2), $\mu_2 < \mu_1$, and $p > 2q + 1$ hold. Let $u_0 \in \mathscr{W}$ satisfying (36) and $f_0 \in L^2(\Omega \times (0, 1))$ be given. Then, the solution of problem (22) is global.*

*Proof.* It is sufficient to show that $\|u_t\|^2 + \|A^{m/2} u\|^{2(q+1)}$ is bounded independently of $t$. To obtain this, by using (23) and (26), we have

$$\begin{aligned} E(0) \geq E(t) &= \frac{1}{2} \|u_t\|^2 + \frac{1}{2(q+1)} \left\| A^{m/2} u \right\|^{2(q+1)} - \frac{1}{p+1} \|u\|_{p+1}^{p+1} \\ &\quad + \frac{\varsigma}{r+1} \int_\Omega \int_0^1 z^{r+1}(x,k,x) dk dx = \frac{1}{2} \|u_t\|^2 \\ &\quad + \frac{p-2q-1}{2(q+1)(p+1)} \left\| A^{m/2} u \right\|^{2(q+1)} \\ &\quad + \frac{(p-r)}{(r+1)(p+1)} \left( \varsigma \int_\Omega \int_0^1 z^{r+1}(x,k,t) dk dx \right) \\ &\quad + \frac{1}{p+1} I(t) \geq \frac{1}{2} \|u_t\|^2 + \frac{p-2q-1}{2(q+1)(p+1)} \left\| A^{m/2} u \right\|^{2(q+1)}, \end{aligned} \tag{44}$$

since $I(t) \geq 0$. Thus,

$$\|u_t\|^2 + \left\| A^{m/2} u \right\|^{2(q+1)} \leq C E(0), \tag{45}$$

where $C = \max \{2, (2(q+1)(p+1)/p-2q-1)\}$. Therefore, we obtain the global existence of solutions. Therefore, we completed the proof. □

## 4. Decay of Solution

In this part, we obtain the decay of solutions of the problem (22) by using Nakao's technique.

**Theorem 9.** *Assume that the assumption (A2) and (36) hold. Let $u_0 \in \mathscr{W}$, $f_0 \in L^2(\Omega \times (0, 1))$, be given. Hence, we have following decay estimates:*

$$E(t) \leq \begin{cases} E(0) e^{-w_1 [t-1]^+}, & \text{if } r = 1, \\ \left( E(0)^{-\alpha} + C_7^{-1} \alpha [t-1]^+ \right)^{-1/\alpha}, & \text{if } r > 1, \end{cases} \tag{46}$$

*where $w_1, \alpha$ and $C_7$ are positive constants which will be defined later.*

*Proof.* We integrate (29) over $[t, t+1]$, $t > 0$, to get

$$E(t) - E(t+1) = [D(t)]^{r+1}, \tag{47}$$

where

$$[D(t)]^{r+1} = c_1\int_t^{t+1} \|u_t\|_{r+1}^{r+1} ds + c_2\int_t^{t+1}\int_\Omega z^{r+1}(x,1,s)dxds. \tag{48}$$

From (48) and Hölder inequality, we see that

$$\int_t^{t+1}\int_\Omega |u_t|^2 dxdt + \int_t^{t+1}\int_\Omega |z(x,1,s)|^2 dxds \le x(\Omega)[D(t)]^2, \tag{49}$$

where $c(\Omega) = vol(\Omega)$.Therefore, by (49), there exists $t_1 \in [t, t+1/4]$ and $t_2 \in [t+3/4, t+1]$, so that

$$\|u_t(t_i)\|^2 + \|z(x,1,t_i)\|^2 \le c(\Omega)[D(t)]^2, i=1,2. \tag{50}$$

We multiply the first equation in (22) by $u$ and integrate over $\Omega \times [t_1, t_2]$. Use integration by parts, Hölder's inequality, adding, and subtracting the term $\int_{t_1}^{t_2}\int_\Omega\int_0^1 \varsigma z^{r+1}(x,k,t)dk\, dxdt$, we have

$$\begin{aligned}\int_{t_1}^{t_2} I(t)dt \le{}& \|u_t(t_1)\|_2\|u(t_1)\|_2 + \|u_t(t_2)\|_2\|u(t_2)\|_2 \\ &+ \int_{t_1}^{t_2}\|u_t\|^2 dt + \int_{t_1}^{t_2}\int_0^1 \varsigma\int_\Omega z^{r+1}(x,k,t)dxdkdt \\ &- \mu_1\int_{t_1}^{t_2}\int_\Omega |u_t|^{r-1}|u_t|udxdt \\ &- \mu_2\int_{t_1}^{t_2}\int_\Omega |z(x,1,t)|^{r-1}z(x,1,t)udxdt.\end{aligned} \tag{51}$$

Now, we estimate the right hand side for (51).
From (39), (50), and Lemma 1, we obtain

$$\|u_t(t_i)\|_2\|u(t_i)\|_2 \le C_1 D(t)\sup_{t_1\le s\le t_2} E^{1/2}(s), \tag{52}$$

where $C_1 = 2C_*((2(q+1)(p+1)/p-2q-1)E(0))^{1/2(q+1)}$.
By using (32) that

$$\begin{aligned}\int_{t_1}^{t_2}\int_0^1\int_\Omega z^{r+1}(x,k,t)dxdkdt &\le \frac{1}{2\tau}\int_{t_1}^{t_2}\int_{t_1}^{t_2}\|u_t(s)\|_{r+1}^{r+1}dsdv \\ &\le \left(\int_{t_1}^{t_2} dv\right)\left(\int_{t_1}^{t_2}\|u_t(s)\|_{r+1}^{r+1}ds\right) \le (t_2-t_1)[D(t)]^{r+1}.\end{aligned} \tag{53}$$

Utilizing Hölder inequality, we get

$$\int_{t_1}^{t_2}\int_\Omega |u_t|^{r-1}|u_t|udxdt \le \int_{t_1}^{t_2}\|u_t(t)\|_{r+1}^r\|u(t)\|_{r+1}. \tag{54}$$

Utilizing the Sobolev-Poincare inequality and (39), we have

$$\begin{aligned}\left|\int_{t_1}^{t_2}\|u_t(t)\|_{r+1}^r\|u(t)\|_{r+1}dt\right| &\le C_*\int_{t_1}^{t_2}\|u_t(t)\|_{r+1}^r\left\|A^{m/2}u\right\|dt \\ &\le C_*\left(\frac{2(q+1)(p+1)}{p-2q-1}E(0)\right)^{1/2(q+1)}\int_{t_1}^{t_2}\|u_t(t)\|_{r+1}^r E^{1/2}(s)dt \\ &\le C_*\left(\frac{2(q+1)(p+1)}{p-2q-1}E(0)\right)^{1/2(q+1)}\sup_{t_1,\le s\le t_2} E^{1/2}(s)\int_{t_1}^{t_2}\|u_t(t)\|_{r+1}^r dt \\ &\le C_*\left(\frac{2(q+1)(p+1)}{p-2q-1}E(0)\right)^{1/2(q+1)}\sup_{t_1,\le s\le t_2} E^{1/2}(s)[D(t)]^r.\end{aligned} \tag{55}$$

Also, we obtain

$$\begin{aligned}\int_{t_1}^{t_2}\int_\Omega |z(x,1,t)|^{r-1}z(x,1,t)udxdt &\le C_*\int_{t_1}^{t_2}\|z(x,1,t)\|_{r+1}^r\|u\|_{r+1}dt \\ &\le C_*\left(\frac{2(q+1)(p+1)}{p-2q-1}E(0)\right)^{1/2(q+1)}\int_{t_1}^{t_2}\|z(x,1,t)\|_{r+1}^r E^{1/2}(s)dt \\ &\le C_*\left(\frac{2(q+1)(p+1)}{p-2q-1}E(0)\right)^{1/2(q+1)}\sup_{t_1\le s\le t_2} E^{1/2}(s)\int_{t_1}^{t_2}\|z(x,1,t)\|_{r+1}^r dt \\ &\le C_*\left(\frac{2(q+1)(p+1)}{p-2q-1}E(0)\right)^{1/2(q+1)}\sup_{t_1\le s\le t_2} E^{1/2}(s)[D(t)]^r.\end{aligned} \tag{56}$$

Then, from (51)-(56), we get

$$\begin{aligned}\int_{t_1}^{t_2} I(t)dt \le C_2\Bigg[&\sup_{t_1\le s\le t_2} E^{1/2}(s)D(t) + [D(t)]^2 + (t_2-t_1)[D(t)]^{r+1} \\ &+ 2C_*\left(\frac{2(q+1)(p+1)}{p-2q-1}E(0)\right)^{1/2(q+1)}\sup_{t_1\le s\le t_2} E^{1/2}(s)[D(t)]^r\Bigg].\end{aligned} \tag{57}$$

Moreover, by (23), (26), and Remark 7, we have

$$E(t) \le \frac{1}{2}\|u_t\|^2 + C_3 I(t), \tag{58}$$

where $C_3 = (1/\eta_1)(p-2q-1/2(q+1)(p+1)) + (1/p+1)$

.

Integrating (58) over $[t_1, t_2]$, we get

$$\int_{t_1}^{t_2} E(t)dt \le \frac{1}{2}\int_{t_1}^{t_2}\|u_t\|^2 dt + C_3\int_{t_1}^{t_2} I(t)dt. \tag{59}$$

Hence, from (57) and (59), we obtain

$$\int_{t_1}^{t_2} E(t)dt \le \frac{1}{2} C[D(t)]^2 + C_3C_2 \left[ \sup_{t_1 \le s \le t_2} E^{1/2}(s)D(t) + [D(t)]^2 + (t_2 - t_1)[D(t)]^{r+1} + 2C_* \cdot \left( \frac{2(q+1)(p+1)}{p-2q-1} E(0) \right)^{1/2(q+1)} \sup_{t_1 \le s \le t_2} E^{1/2}(s)[D(t)]^r \right]. \quad (60)$$

Integrating $(d/dt)E(t)$ over $[t, t2]$, we conclude that

$$E(t) = E(t_2) + \int_t^{t_2} \left( \mu_1 - \frac{\varsigma}{\tau(r+1)} - \frac{\mu_2}{r+1} \right) \|u_s(s)\|_{r+1}^{r+1} ds + \int_t^{t_2} \left( \frac{\varsigma}{\tau(r+1)} - \frac{\mu_2 r}{r+1} \right) \int_\Omega |z(x,1,s)|^{r+1} dxds. \quad (61)$$

Thus, since $t_2 - t_1 \ge 1/2$, we arrive at

$$\int_{t_1}^{t_2} E(t)dt \ge (t_2 - t_1)E(t_2) \ge \frac{1}{2} E(t_2). \quad (62)$$

Hence,

$$E(t_2) \le 2 \int_{t_1}^{t_2} E(t)dt. \quad (63)$$

As a result, from (47), (60), (61), (63), and since $t_1, t_2 \in [t, t+1]$, we get

$$E(t) \le 2 \int_{t_1}^{t_2} E(t)dt + \int_t^{t+1} \left( \mu_1 - \frac{\varsigma}{\tau(r+1)} - \frac{\mu_2}{r+1} \right) \|u_s(s)\|_{r+1}^{r+1} ds + \int_t^{t+1} \left( \frac{\varsigma}{\tau(r+1)} - \frac{\mu_2 r}{r+1} \right) \int_\Omega |z(x,1,s)|^{r+1} dxds = 2 \int_{t_1}^{t_2} E(t)dt + [D(t)]^{r+1}. \quad (64)$$

Then, from (60), we have

$$E(t) \le \left( \frac{1}{2} C + C_3C_2 \right) [D(t)]^2 + C_3C_2[D(t)]^{r+1} + C_4[D(t) + [D(t)]^r]E^{1/2}(t). \quad (65)$$

Thus, utilizing Young inequality, we have

$$E(t) \le C_5 \left[ [D(t)]^2 + [D(t)]^{r+1} + [d(t)]^{2r} \right]. \quad (66)$$

□

Hence, we have the decay estimates as follows:

*Case 1.* If $r = 1$, by (66), we obtain

$$E(t) \le 3C_5[D(t)]^2 = 3C_5[E(t) - E(t-1)]. \quad (67)$$

Utilizing Lemma 2, we have

$$E(t) \le E(0)e^{-w_1[t-1]^+}, \quad (68)$$

where $w_1 = \ln(3C_5/3C_5 - 1)$.

*Case 2.* If $r > 1$, by (66), we have

$$E(t) \le C_5[D(t)]^2 \left( 1 + [D(t)]^{r-1} + [D(t)]^{2r-2} \right). \quad (69)$$

Then, by (47), since $E(t) \le E(0), \forall t \ge 0$, we see that

$$E(t) \le C_5 \left( 1 + E^{r-1/r+1}(0) + E^{2(r-1)/r+1}(0) \right) [D(t)]^2 \le C_6[D(t)]^2, t \ge 0. \quad (70)$$

Then, we get

$$E(t)^{r+1/2} \le C_7[D(t)]^{r+1} \le C_7(E(t) - E(t+1)). \quad (71)$$

Therefore, by (71) and Lemma 2, we obtain

$$E(t) \le \left( E(0)^{-\alpha} + C_7^{-1}\alpha[t-1]^+ \right)^{-1/\alpha}. \quad (72)$$

Thus, we completed the proof of Theorem 9.

## 5. Blow-Up of Solution

In this part, we get the blow-up of solutions for negative initial energy, in the case $r > 1$.

**Theorem 10.** *Let* $(u_0, u_1) \in (H^{2m}(\Omega) \cap H_0^m(\Omega)) \times H_0^m(\Omega)$ *and* $f_0 \in L^2(\Omega \times (0,1))$ *be given. Assume that* $p > \max\{2, r, 2q+1\}$ *and the assumptions (A1)-(A2) hold. Then, the solution of (22) blows up in a finite time with* $E(0) < 0$.

*Proof.* Setting

$$H(t) = -E(t), \quad (73)$$

from Lemma 3, we obtain

$$H'(t) \ge -\left( \mu_1 - \frac{\varsigma}{\tau(r+1)} - \frac{\mu_2}{r+1} \right) \|u_t(t)\|_{r+1}^{r+1} - \left( \frac{\varsigma}{\tau(r+1)} - \frac{\mu_2 r}{r+1} \right) \int_\Omega z^{r+1}(x,1,t)dx. \quad (74)$$

Thus,

$$0 < H(0) \le H(t) \le \frac{1}{p+1} \|u\|_{p+1}^{p+1}, t > 0. \quad (75)$$

Let

$$M(t) = \|u\|_2^2. \tag{76}$$

Differentiating (76) twice, we get

$$M'(t) = 2\int_\Omega u_t u dx,$$
$$M''(t) = 2\|u_t\|^2 + 2\int_\Omega u_{tt} u dx. \tag{77}$$

Using the first equation in (22), to have

$$M''(t) = 2\|u_t\|^2 - 2\left\|A^{\frac{m}{2}}u\right\|_2^{2(q+1)} - 2\mu_1\int_\Omega |u_t(x,t)|^{r-1}uu_t(x,t)dx - 2\mu_2\int_\Omega |z(x,1,t)|^{r-1}uz(x,1,t)dx + 2\|u\|_{p+1}^{p+1}, \tag{78}$$

we add and subtract the term $2(p+1)H(t)$, and then (78) becomes the form

$$\begin{aligned} M''(t) \geq{}& (p+3)\|u_t\|^2 + 2(p+1)H(t) + \left(\frac{p+1}{q+1} - 2\right)\left\|A^{\frac{m}{2}}u\right\|_2^{2(q+1)} \\ &- 2\mu_1\int_\Omega |u_t(x,t)|^{r-1}uu_t(x,t)dx \\ &- 2\mu_2\int_\Omega |z(x,1,t)|^{r-1}uz(x,1,t)dx \\ &+ \frac{2\varsigma(p+1)}{r+1}\int_\Omega\int_0^1 z^{r+1}(x,k,s)dkdx. \end{aligned} \tag{79}$$

Now, we define

$$L(t) = H(t)^{1-\kappa} + 2\epsilon M'(t). \tag{80}$$

Differentiating (80), we obtain

$$L'(t) = (1-\kappa)H(t)^{-\kappa}H'(t) + 2\varepsilon M''(t). \tag{81}$$

Replacing (79) in (81), we arrive at

$$\begin{aligned} L'(t) \geq{}& (1-\kappa)H(t)^{-\kappa}H'(t)2\varepsilon(p+3)\|\mu_t\|^2 \\ &+ 4\varepsilon(p+1)H(t) + 2\varepsilon\left(\frac{p+1}{q+1} - 2\right)\left\|A^{\frac{m}{2}}\mu\right\|_2^{2(q+1)} \\ &- 4\varepsilon\mu_1\int_\Omega |\mu_t(x,t)|^{r-1}uu_t(x,t)dx \\ &- 4\varepsilon\mu_2\int_\Omega |z(x,1,t)|^{r-1}uz(x,1,t)dx \\ &+ \frac{4\varepsilon\varsigma(p+1)}{r+1}\int_\Omega\int_0^1 z^{r+1}(x,k,s)dkdx. \end{aligned} \tag{82}$$

From (75) and utilizing Hölder inequality, we get

$$\begin{aligned} \left|\int_\Omega |u_t)x,t|^{r-1}uu_t(x,t)dx\right| &\leq \|u_t\|_{r+1}^r\|u\|_{r+1} \\ &\leq c_1\|u\|_{r+1}^{r+1/p+1}\|u\|_{r+1}^{1-r+1/p+1}\|u\|_{r+1}^r \\ &\leq c_2\|u\|_{r+1}^{r+1/p+1}H(t)^{\frac{1}{p+1}-\frac{r+1}{(p+1)^2}}\|u_t\|_{r+1}^r. \end{aligned} \tag{83}$$

From Young's inequality and (74), we have

$$\begin{aligned} &\left|\int_\Omega |u_t(x,t)|^{r-1}uu_t(x,t)dx\right| \\ &\quad\leq c_3\left(\rho^{1/1+p}\|\mu\|_{r+1}^{r+1}H(0)^{-\bar{k}} + \rho^{-r'}H(0)^{k-\bar{k}}H'(t)H(t)^{-k}\right), \end{aligned} \tag{84}$$

where $\bar{k} = 1/p+1-r+1/(p+1)^2 > 0$, $\rho > 0$, $r' = r+1/r$, letting $0 < k < \bar{k}$. In a similar way, we obtain

$$\begin{aligned} &\left|\int_\Omega u|z(x,1,t)|^{r-1}z(x,1,t)dx\right| \\ &\quad\leq c_3\left(\rho^{\frac{1}{1+p}}\|\mu\|_{r+1}^{r+1}H(0)^{-\bar{k}} + \rho^{-r'}H(0)^{k-\bar{k}}H'(t)H(t)^{-k}\right). \end{aligned} \tag{85}$$

By using (82), (84), and (85), to have

$$\begin{aligned} L'(t) \geq{}& \left[(1-\kappa) - 4\varepsilon(\mu_1+\mu_2)H^{k-\bar{k}}(0)\rho^{-r'}\right]H(t)^{-\kappa}H'(t) \\ &- 4\varepsilon(\mu_1+\mu_2)H(0)^{-\bar{k}}\rho^{1/1+p}\|u\|_{r+1}^{r+1} + 2\varepsilon(p+3)\|u_t\|^2 \\ &+ 4\varepsilon(p+1)H(t) + 2\varepsilon\left(\frac{p+1}{q+1} - 2\right)\left\|A^{m/2}\mu\right\|_2^{2(q+1)} \\ &+ \frac{4\varepsilon\varsigma(p+1)}{r+1}\int_\Omega\int_0^1 z^{r+1}(x,k,s)dkdx, \end{aligned} \tag{86}$$

for $\varepsilon$ sufficiently small, we obtain

$$\left[(1-\kappa) - 4\varepsilon(\mu_1+\mu_2)H^{k-\bar{k}}(0)\rho^{-r'}\right] \geq 0. \tag{87}$$

Setting $s = r+1 \leq p+1$ such that

$$\|\mu\|_{r+1}^s \leq c_1\left(\left\|A^{m/2}\mu\right\| + \|\mu\|_{p+1}^{p+1}\right), \tag{88}$$

where $c = 4(\mu_1+\mu_2)H(0)^{-\bar{k}}\rho^{1/p+1}c_1$ and taking $(p+1/q+1-2) > c$. Thus, we have

$$\begin{aligned} L'(t) \geq{}& 2\varepsilon\left(\frac{p+1}{q+1} - 2 - c\right)\left\|A^{m/2}u\right\|_2^{2(q+1)} \\ &- \varepsilon c\|\mu\|_{p+1}^{p+1} + 4\varepsilon(p+1)H(t) + 2\varepsilon(p+3)\|\mu\|^2 \\ &+ 4\varepsilon\frac{\zeta(p+1)}{r+1}\int_\Omega\int_0^1 z^{r+1}(x,k,s)dkdx. \end{aligned} \tag{89}$$

By using the notations $a_1 = 2(p + 1/q + 1 - 2 - c)$, $a_2 = c$, $a_3 = 4(p + 1)$, and $a_4 = 2(p + 3)$, (89) takes the form

$$L'(t) \geq a_1 \varepsilon \left\| A^{m/2} \mu \right\|_2^{2(q+1)} - \varepsilon a_2 \|\mu\|_{p+1}^{p+1} + \varepsilon a_3 H(t) + \varepsilon a_4 \|\mu_t\|^2. \tag{90}$$

Similarly to the approach of Messaoudi [49], we assume that $p = 2a_5 + (p - 2a_5)$, where $a_5 < \min\ (a_1, a_2, a_3, a_4)$, and then (90) becomes the form

$$\begin{aligned} L'(t) \geq (a_1 - a_5)\varepsilon \left\| A^{m/2}\mu \right\|_2^{2(q+1)} + \varepsilon(a_5 - a_2)\|\mu\|_{p+1}^{p+1} \\ + \varepsilon(a_3 - a_5)H(t) + \varepsilon(a_4 - a_5)\|\mu_t\|^2. \end{aligned} \tag{91}$$

Then,

$$L'(t) \geq \delta\varepsilon \left[ \left\| A^{\frac{m}{2}} u \right\|_2^{2(q+1)} + \|\mu_t\|_{p+1}^{p+1} + H(t) + \|u_t\|^2 \right]. \tag{92}$$

We conclude that

$$L'(t) \geq \delta\varepsilon \left[ \|u\|_{p+1}^{p+1} + H(t) + \|u_t\|^2 \right], \tag{93}$$

where $\delta > 0$ is the minimum of the coefficients of $\|u\|_{p+1}^{p+1}$, $H(t)$, $\|u_t\|^2$. Pick out $\varepsilon$ such that

$$L(0) = H^{1-\kappa}(0) + 2\varepsilon \int_\Omega u_1 u_0 dx > 0. \tag{94}$$

As a result, we getsetting $\varpi = 1/1 - k$, and since $k < \bar{k} < 1$, we see that $1 < \varpi < 1/1 - \bar{k}$. Set

$$L(t) = H(t)^{1-\kappa} + 2\varepsilon \int_\Omega u u_t dx. \tag{95}$$

Then,

$$\begin{aligned} L(t) = H(t)^{1/\varpi} + 2\varepsilon \int_\Omega u u_t dx \leq H(t)^{1/\varpi} \\ + 2\varepsilon \int_\Omega u u_t dx + 2E_1 \left( \|u\|_{p+1} \right)^{p+1/\varpi}. \end{aligned} \tag{96}$$

Utilizing Young, Hölder's inequalities, and (96), we conclude that

$$\begin{aligned} L(t)^\varpi &\leq \left[ H(t)^{1/\varpi} + 2\varepsilon \int_\Omega u u_t dx + 2E_1 \left( \|u\|_{p+1} \right)^{p+1/\varpi} \right]^\varpi \\ &\leq 2^{\varpi-1} \left[ H(t) + \left( 2\varepsilon \int_\Omega u u_t dx + 2E_1 \left( \|u\|_{p+1} \right)^{p+1/\varpi} \right)^\varpi \right] \\ &\leq 2^{\varpi-1} \left[ H(t) + 2^{\varpi-1} \left( \left( 2\varepsilon \int_\Omega u u_t dx \right)^\varpi + 2E_1 \|u\|_{p+1}^{p+1} \right) \right] \\ &\leq 2^{\varpi-1} \left[ H(t) + 2^{\varpi-1} \left( \beta^\varpi \|u_t\|_2^\varpi \|u\|_2^\varpi + 2E_1 \|u\|_{p+1}^{p+1} \right) \right] \\ &\leq 2^{\varpi-1} \left[ H(t) + 2^{\varpi-1} \left( \beta^\varpi \|u_t\|_2^\varpi \|u\|_2^\varpi \right) + 2^{\varpi-1} \left( 2E_1 \|u\|_{p+1}^{p+1} \right) \right] \\ &\leq c_2 [H(t) + ] \|u_t\|_2^\varpi \|u\|_2^\varpi + \|u\|_{p+1}^{p+1}, \end{aligned} \tag{97}$$

where $c_2 = \max\{2^{\varpi-1}, \beta^\varpi\}$. Furthermore, for $p > 1$, utilizing Hölder and Young inequalities, we obtain

$$\begin{aligned} \|u_t\|_2^\varpi \|u\|_2^\varpi \leq c_3 \|u_t\|_2^\varpi \|u\|_{p+1}^\varpi \leq c_4 \left( \|u_t\|_2^2 + \|u\|_{p+1}^{2(1-k)/1-2k} \right), \\ \|u\|_{p+1}^{2(1-k)/1-2k} = \|u\|_{p+1}^{p+1} \|u\|_{p+1}^{2(1-k)/1-2k-(p+1)} \\ \leq c_5 H(0)^{2(1-k)/(1-2k)(p+1)-1} \|u\|_{p+1}^{p+1}. \end{aligned} \tag{98}$$

Then, (97) becomes the form

$$L(t)^\varpi \leq c_6 \left( H(t) + \|u_t\|^2 + \|u\|_{p+1}^{p+1} \right). \tag{99}$$

By combining (93) and (99), we conclude that

$$L'(t) \leq c_7 L(t)^\varpi, c_7 > 0, \varpi > 1. \tag{100}$$

Therefore, a simple integration over $(0, t)$, we have the desired result. Hence, we completed the proof. □

## 6. Conclusions

Time delays often appear in many various problems, such as, thermal, economic phenomena, biological, chemical, and physical. Recently, the partial differential equations with time delay have become an active area (see [2, 3] and references therein). In recent years, there has been published much work concerning the wave equation with constant delay or time-varying delay. However, to the best of our knowledge, there were no global existence, decay, and blow-up results for the higher-order Kirchhoff-type equation with delay term. Firstly, we have been obtained the global existence result. Later, we have been established the decay results by using Nakao's technique. Finally, we have proved the blow-up of solutions with negative initial energy for the problem (1) under the sufficient conditions in a bounded domain. In the next work, we will extend our current study to more general case of the problem (1).

## Data Availability

No data were used to support the study.

## Conflicts of Interest

The authors declare that they do not have any conflicts of interest.

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