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Research Article

Fixed Point and Endpoint Theories for Two Hybrid Fractional Differential Inclusions with Operators Depending on an Increasing Function

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The main concentration of the present research is to explore several theoretical criteria for proving the existence results for the suggested boundary problem. In fact, for the first time, we formulate a new hybrid fractional differential inclusion in the φ-Caputo settings depending on an increasing function φ subject to separated mixed φ-hybrid-integro-derivative boundary conditions. In addition to this, we discuss a special case of the proposed φ-inclusion problem in the non-φ-hybrid structure with the help of the endpoint notion. To confirm the consistency of our findings, two specific numerical examples are provided which simulate both φ-hybrid and non-φ-hybrid cases.

1. Introduction

Arbitrary order calculus theory is considered as an important topic of research for all mathematicians, researchers, engineers, and scientists due to the applicability of mentioned theory in several contexts in engineering and applied science and its flexibility to model different systems and phenomena having memory effects (see, e.g., [1–3] and reference therein). Several arbitrary order derivatives have been introduced in the past decade, and the most common of them are Riemann-Liouville, Caputo, and Hadamard derivatives. Hence, arbitrary order boundary value problems (BVPs) can be formulated in the framework of different operators. In the meantime, some recent research investigations have been conducted with the aid of these operators to establish the relevant analytical results for proposed BVPs. For instance, Alzabut et al. [4] investigated the oscillatory behavior of a kind of fractional differential equations (FDEs) supplemented with damping and forcing terms by terms of generalized proportional operators. In [5], Baleanu et al. modeled an applied instrument in engineering in the context of a hybrid Caputo FBVP and studied its existence theory. Also, the same authors [6] established similar results by means of Caputo and Riemann-Liouville conformable derivation and integration operators. In 2019, Matar et al. [7] devoted their focus on solvability of nonlinear systems of FDEs via nonlocal initial value problems by terms of fixed point methods and after that, Mohammadi et al. [8] utilized another fractional operator entitled Caputo–Hadamard for modeling a hybrid FBVP with Hadamard integral boundary conditions. Zhou et al. [9] presented a fractional antiperiodic model of Langevin equation and investigated qualitative aspects of its solutions with the aid of techniques appeared in functional analysis. Similarly, one can find some papers on applications of fractional operators [10–13].

In 2017, a generalization of the Caputo fractional operator known as φ-Caputo derivative (φ-CF) was presented by Almeida [14] in which its kernel is with respect to a given increasing function φ. One of the most important advantages of the φ-CF derivative operator is its ability to produce all previous fractional derivatives, and also, it involves the semigroup property. As a result, φ-CF derivative is known as an extended structure of arbitrary order derivative operators.

To get acquainted with some previous research works done based on φ-CF operators so far, we refer to a paper published by Wahash et al. [15]. In that paper, Wahash et al. designed a generalized φ-fractional differential equation with a simple integral condition as

$$\begin{cases} {}^{\mathscr{C}}\mathfrak{D}_{0^+}^{\sigma^*;\varphi} w(z) = \hbar_*(z, w(z)), \\ w(0) = \nu + d\displaystyle\int_0^1 \xi(q)u(q)dq, \end{cases} \tag{1}$$

where $z \in [0,1]$, $\sigma^* \in (0,1)$, $\nu \in \mathbb{R}_+$, and $d \in \mathbb{R}^{\geq 0}$, and also, $\hbar_* : [0,1] \times \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ stands for a continuous function along with $\xi \in L^1_{\mathbb{R}_+}([0,1])$. The lower-upper solution is a technique implemented in that article by Wahash et al. in which they utilized a fixed point method on cones. Further, lower-upper control maps are provided with respect to the nonlinear term without a certain monotonicity criterion [15]. Similarly, by using the newly introduced φ-CF operator and its generalizations, several articles have been published like as [16] in which Almeida et al. considered a FDE via a Caputo derivative with respect to a kernel function and reviewed some applications of them. Derbazi et al. [17] used such a generalized operator to investigate a nonlinear initial value problem via monotone iterative method. Samet et al. derived some Lyapunov-type inequalities in relation to an antiperiodic FBVP involving φ-Caputo operator [18]. The analysis of the stability to an φ-Hilfer impulsive FDE is another instance of applications of such generalized operators which was studied by Sousa et al. in [19]. In 2020, Tariboon et al. [20] turned to establishment of existence theorems to sequential generalized inclusion FBVP, and then, Thabet et al. [21] achieved to similar findings on a new structure of the pantograph inclusion FBVP. In a higher level, Vivek et al. [22] defined generalized φ-operators in the context of partial operators and analyzed a PDE in the φ-Caputo settings.

With regard to ideas of aforesaid research works, we consider the following φ-hybrid fractional differential inclusion in the sense of Caputo represented as

$${}^{\mathscr{C}}\mathfrak{D}_a^{\sigma^*;\varphi}\left(\frac{\varpi^*(z)}{\mathfrak{N}_*(z,\varpi^*(z))}\right) \in \tilde{\mathfrak{O}}(z,\varpi^*(z)), \tag{2}$$

supplemented with separated mixed φ-hybrid-integro-derivative boundary conditions

$$\begin{cases} \tilde{m}_1\left(\dfrac{\varpi^*(z)}{\mathfrak{N}_*(z,\varpi^*(z))}\right)\Bigg|_{z=a} = s_1^* + \tilde{m}_2\left(\dfrac{\varpi^*(z)}{\mathfrak{N}_*(z,\varpi^*(z))}\right)\Bigg|_{z=a}, \\ \tilde{m}_1 {}^{\mathscr{RL}}\mathscr{I}_a^{\mu^*;\varphi\,\mathscr{C}}\mathfrak{D}_a^{\tilde{\mu};\varphi}\left(\dfrac{\varpi^*(z)}{\mathfrak{N}_*(z,\varpi^*(z))}\right)\Bigg|_{z=T} = s_2^* + \tilde{m}_2 {}^{\mathscr{RL}}\mathscr{I}_a^{\mu^*;\varphi\,\mathscr{C}}\mathfrak{D}_a^{1;\varphi}\left(\dfrac{\varpi^*(z)}{\mathfrak{N}_*(z,\varpi^*(z))}\right)\Bigg|_{z=T}, \end{cases} \tag{3}$$

where $z \in [a,T]$ with $a \geq 0$, $\sigma^* \in (1,2)$, $\tilde{\mu} \in (0,1)$, $\mu^* > 0$, $\tilde{m}_1, \tilde{m}_2 \in \mathbb{R}^{\neq 0}$, and $s_1^*, s_2^* \in \mathbb{R}^+$. Two notations ${}^{\mathscr{C}}\mathfrak{D}_a^{(\cdot);\varphi}$ and ${}^{\mathscr{RL}}\mathscr{I}_a^{(\cdot);\varphi}$ stand for the φ-CF derivative and the φ-Riemann-Liouville integral (φ-RLF), respectively. Also, notice that ${}^{\mathscr{C}}\mathfrak{D}_a^{1;\varphi} = (1/\varphi'(z))(d/dz)$. Besides, $\mathfrak{N}_* : [a,T] \times \mathbb{R} \longrightarrow \mathbb{R}$ is assumed to be a nonzero continuous single-valued operator, and $\tilde{\mathfrak{O}} : [0,1] \times \mathbb{R} \longrightarrow \mathscr{P}(\mathbb{R})$ is assumed to be a set-valued operator equipped with some required properties. Notice that by putting $\mathfrak{N}_*(z,\varpi^*(z)) = 1$, the given φ-hybrid Caputo fractional differential inclusion BVP (2) and (3) is transformed into a non-φ-hybrid separated inclusion BVP presented by

$$\begin{cases} {}^{\mathscr{C}}\mathfrak{D}_a^{\sigma^*;\varphi}\varpi^*(z) \in \tilde{\mathfrak{O}}(z,\varpi^*(z)), \quad (z \in [a,T]), \\ \tilde{m}_1\varpi^*(a) = s_1^* + \tilde{m}_2\varpi^*(a), \\ \tilde{m}_1 {}^{\mathscr{RL}}\mathscr{I}_a^{\mu^*;\varphi\,\mathscr{C}}\mathfrak{D}_a^{\tilde{\mu};\varphi}\varpi^*(T) = s_2^* + \tilde{m}_2 {}^{\mathscr{RL}}\mathscr{I}_a^{\mu^*;\varphi\,\mathscr{C}}\mathfrak{D}_a^{1;\varphi}\varpi^*(T). \end{cases} \tag{4}$$

Note that by taking into account the authors' knowledge, there are no research manuscripts on φ-CF operators involving mixed φ-hybrid-integro-derivative boundary conditions simultaneously. In addition, this given structure is formulated in a unique and general form in which we can consider some standard special cases studied before. Here, we derive some analytical criteria to prove the existence results for the proposed novel φ-hybrid fractional differential inclusion in the φ-Caputo settings (2) equipped with separated mixed φ-hybrid-integro-derivative boundary conditions (3). The applied approach to achieve desired purposes is based on Dhage's fixed point result. In addition, we discuss the special case of the proposed φ-inclusion problem in the non-φ-hybrid version with the aid of the endpoint notion. We organize the present manuscript as the following construction. In Section 2, we briefly collect auxiliary preliminaries on the φ-fractional operators and some required notions on the multifunctions and related properties. In Section 3, the existence criteria of solutions for both proposed φ-hybrid and non-φ-hybrid BVPs (2)–(4) are derived by two different analytical methods. To confirm the applicability of our analytical findings, two simulative numerical examples are formulated in Section 4 which cover both φ-hybrid and non-φ-hybrid cases.

2. Auxiliary Preliminaries

By continuing the path ahead, we assemble and recall several auxiliary and fundamental notions in the direction of our theoretical methods implemented in this paper. The concept of RLF integral for $\varpi^* : [0,+\infty) \longrightarrow \mathbb{R}$ of order $\sigma^* > 0$ is defined as

$${}^{\mathscr{RL}}\mathscr{I}_0^{\sigma^*}\varpi^*(z) = \int_0^z \frac{(z-q)^{\sigma^*-1}}{\Gamma(\sigma^*)}\varpi^*(q)dq, \tag{5}$$

provided that the integral has finite value [23, 24]. In this position, let us take $n-1<\sigma^*<n$ in which $n=[\sigma^*]+1$. Regarding a continuous function $\varpi^*:[0,+\infty)\longrightarrow\mathbb{R}$, the RLF derivative of order ρ^* is defined as

$$ {}^{\mathscr{RL}}\mathfrak{D}_0^{\sigma^*}\varpi^*(z)=\left(\frac{d}{dz}\right)^n\int_0^z\frac{(z-q)^{n-\sigma^*-1}}{\Gamma(n-\sigma^*)}\varpi^*(q)\,dq, \tag{6}$$

provided that the integral has finite value [23, 24]. In the next step, for an absolutely continuous n-times differentiable real-valued function ϖ^* on $[0,+\infty)$, the derivative in the Caputo settings of order σ^* is defined as

$$ {}^{\mathscr{C}}\mathfrak{D}_0^{\sigma^*}\varpi^*(z)=\int_0^z\frac{(z-q)^{n-\sigma^*-1}}{\Gamma(n-\sigma^*)}\varpi^{*^{(n)}}(q)\,dq, \tag{7}$$

such that it is finite-valued [23, 24]. Now, let $\varphi\in\mathscr{C}^n([a,b])$ be increasing with $\varphi'(z)>0,\forall z\in[a,b]$. Then, an integral in the sense of φ-Riemann-Liouville for an integrable $\varpi^*:[a,b]\longrightarrow\mathbb{R}$ of order σ^* depending on increasing function φ is defined as

$$ {}^{\mathscr{RL}}\mathscr{I}_a^{\sigma^*;\varphi}\varpi^*(z)=\frac{1}{\Gamma(\sigma^*)}\int_a^z\varphi'(q)(\varphi(z)-\varphi(q))^{\sigma^*-1}\varpi^*(q)dq, \tag{8}$$

provided that the RHS of above equality involves the finite value [25, 26]. It is clear that if we take $\varphi(z)=z$, then $\varphi'(z)=1$, and thus by inserting them into (8), we see that the φ-RLF integral is converted to the standard RLF integral given by (5). For a continuous function $\varpi^*:[0,+\infty)\longrightarrow\mathbb{R}$, a derivative in the sense of φ-RL of order σ^* is given by

$$ {}^{\mathscr{RL}}\mathfrak{D}_a^{\sigma^*;\varphi}\varpi^*(z)=\frac{1}{\Gamma(n-\sigma^*)}\left(\frac{1}{\varphi'(z)}\frac{d}{dz}\right)^n\cdot\int_a^z\varphi'(q)(\varphi(z)-\varphi(q))^{n-\sigma^*-1}\varpi^*(q)\,dq, \tag{9}$$

provided that the RHS of above equality exists [25, 26]. If $\varphi(z)=z$, then the φ-RLF derivative (9) is converted to the standard RLF derivative (6). Motivated by such operators, Almeida gave a φ-version of the CF derivative as follows:

$$ {}^{\mathscr{C}}\mathfrak{D}_a^{\sigma^*;\varphi}\varpi^*(z)=\frac{1}{\Gamma(n-\sigma^*)}\int_a^z\varphi'(q)(\varphi(z)-\varphi(q))^{n-\sigma^*-1}\cdot\left(\frac{1}{\varphi'(q)}\frac{d}{dq}\right)^n\varpi^*(q)\,dq, \tag{10}$$

provided that the RHS of above equality exists [14]. If $\varphi(z)=z$, then the φ-CF derivative (10) is converted to the standard CF derivative (7). Some useful properties of the φ-CF and φ-RLF operators can be seen in the following.

Lemma 1 [14, 24]. *Let $\sigma^*,\varrho^*,\beta^*>0$ and $\varphi\in\mathscr{C}^n([a,b])$ be increasing with $\varphi'(z)>0,\forall z\in[a,b]$. Then,*

(i1) $ {}^{\mathscr{RL}}\mathscr{I}_a^{\sigma^*;\varphi}({}^{\mathscr{RL}}\mathscr{I}_a^{\varrho^*;\varphi}\varpi^*)(z)=({}^{\mathscr{RL}}\mathscr{I}_a^{\sigma^*+\varrho^*;\varphi}\varpi^*)(z)$

(i2) $ {}^{\mathscr{RL}}\mathscr{I}_a^{\sigma^*;\varphi}(\varphi(z)-\varphi(a))^{\beta^*}(y)=(\Gamma(\beta^*+1)/\Gamma(\sigma^*+\beta^*+1))(\varphi(y)-\varphi(a))^{\sigma^*+\beta^*}$

(i3) $ {}^{\mathscr{C}}\mathfrak{D}_a^{\sigma^*;\varphi}(\varphi(z)-\varphi(a))^{\beta^*}(y)=(\Gamma(\beta^*+1)/\Gamma(\beta^*-\sigma^*+1))(\varphi(y)-\varphi(a))^{\beta^*-\sigma^*},(\beta^*>\sigma^*)$

(i4) $ {}^{\mathscr{RL}}\mathfrak{D}_a^{\sigma^*;\varphi}({}^{\mathscr{RL}}\mathscr{I}_a^{\rho^*;\varphi}\varpi^*)(z)=({}^{\mathscr{RL}}\mathscr{I}_a^{\rho^*-\sigma^*;\varphi}\varpi^*)(z),(\sigma^*<\rho^*)$

For instance, we plot the graph of φ-RLF integral and φ-CF derivative of $\varpi(z)=(z-1)^{6.5}$ for $\varphi(z)=2z+3/2$ in Figure 1.

Lemma 2 [14]. *Let $n-1<\sigma^*<n$. Then, for each $\varpi^*\in\mathscr{C}^{n-1}([a,b])$,*

$$ {}^{\mathscr{RL}}\mathscr{I}_a^{(\sigma^*;\varphi)}\left({}^{\mathscr{C}}\mathfrak{D}_a^{(\sigma^*;\varphi)}\varpi^*\right)(z)=\varpi^*(z)-\sum_{j=0}^{n-1}\frac{(\delta_\varphi)^j\varpi^*(a)}{j!}(\varphi(z)-\varphi(a))^j,\cdot\left(\delta_\varphi=\frac{1}{\varphi'(z)}\frac{d}{dz}\right). \tag{11}$$

In accordance with above lemma, the authors proved that the series solution for given homogeneous differential equation $({}^{\mathscr{C}}\mathfrak{D}_a^{\sigma^*;\varphi}\varpi^*)(z)=0$ has such a form

$$\varpi^*(z)=\sum_{j=0}^{n-1}\tilde{k}_j^*(\varphi(z)-\varphi(a))^j=\tilde{k}_0^*+\tilde{k}_1^*(\varphi(z)-\varphi(a))+\tilde{k}_2^*(\varphi(z)-\varphi(a))^2+\cdots+\tilde{k}_{n-1}^*(\varphi(z)-\varphi(a))^{n-1}, \tag{12}$$

where $n-1<\sigma^*<n$ and $\tilde{k}_0^*,\tilde{k}_1^*,\cdots,\tilde{k}_{n-1}^*\in\mathbb{R}$ [14].

We consider the normed space by notation $(\mathfrak{W},\|\cdot\|_{\mathfrak{W}})$. Also, we introduce the notations $\mathscr{P}(\mathfrak{W})$, $\mathscr{P}_{\text{bnd}}(\mathfrak{W})$, $\mathscr{P}_{\text{cls}}(\mathfrak{W})$, $\mathscr{P}_{\text{cmp}}(\mathfrak{W})$, and $\mathscr{P}_{\text{cvx}}(\mathfrak{W})$ for the category of all nonempty subsets, all bounded subsets, all closed subsets, all compact subsets, and all convex subsets of $\mathfrak{W}$, respectively. In the subsequent path, a metric function attributed to Pompeiu-Hausdorff $\mathbb{PH}_{d_{\mathfrak{W}}}:\mathscr{P}(\mathfrak{W})\times\mathscr{P}(\mathfrak{W})\longrightarrow\mathbb{R}\cup\{\infty\}$ is defined by

$$\mathbb{PH}_{d_{\mathfrak{W}}}(\mathscr{E}_1,\mathscr{E}_2)=\max\left\{\sup_{e_1\in\mathscr{E}_1}d_{\mathfrak{W}}(e_1,\mathscr{E}_2),\sup_{e_2\in\mathscr{E}_2}d_{\mathfrak{W}}(\mathscr{E}_1,e_2)\right\}, \tag{13}$$

so that $d_{\mathfrak{W}}(\mathscr{E}_1,e_2)=\inf_{e_1\in\mathscr{E}_1}d_{\mathfrak{W}}(e_1,e_2)$ and $d_{\mathfrak{W}}(e_1,\mathscr{E}_2)=\inf_{e_2\in\mathscr{E}_2}d_{\mathfrak{W}}(e_1,e_2)$ [27]. We say that $\tilde{\mathfrak{O}}:\mathfrak{W}\longrightarrow\mathscr{P}_{cls}(\mathfrak{W})$ is Lipschitzian with constant $\hat{c}>0$ if $\mathbb{PH}_{d_{\mathfrak{W}}}(\tilde{\mathfrak{O}}(\varpi_1^*),\tilde{\mathfrak{O}}(\varpi_2^*))\le\hat{c}\,d_{\mathfrak{W}}(\varpi_1^*,\varpi_2^*),\forall\varpi_1^*,\varpi_2^*\in\mathfrak{W}$. Also, $\tilde{\mathfrak{O}}$ is a contraction if $\hat{c}\in[0,1)$ [27].

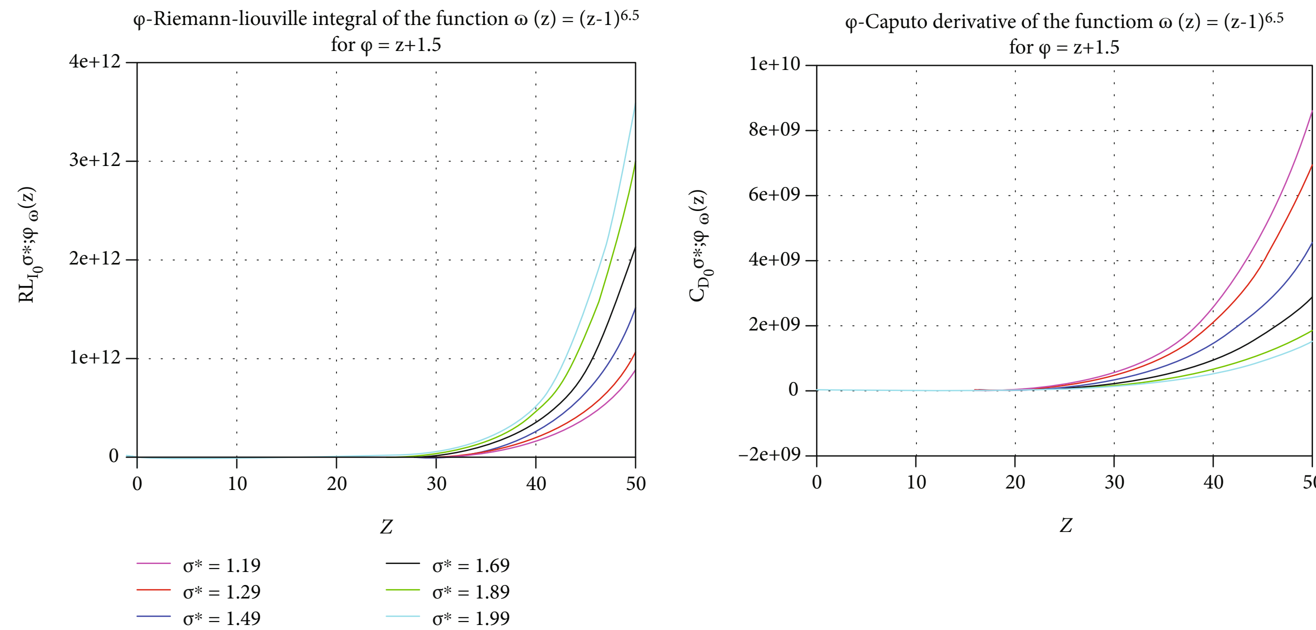

Figure 1: The RLF-integral and CF-derivative of $\varpi(z) = (z-1)^{6.5}$ for $\varphi(z) = 2z + 3/2$.

We represent the collection of all existing selections of $\tilde{\mathfrak{O}}$ at point $\varpi^* \in \mathscr{C}_{\mathbb{R}}([0,1])$ by

$$(\mathbb{SEL})_{\tilde{\mathfrak{O}},\varpi^*} := \left\{\widehat{\kappa} \in \mathscr{L}^1_{\mathbb{R}}([0,1]) \colon \widehat{\kappa}(z) \in \tilde{\mathfrak{O}}(z, \varpi^*(z))\right\}, \quad (14)$$

for almost all $z \in [0,1]$ [27, 28]. We note that $\varpi^* \in \mathfrak{W}$ is an endpoint for given set-valued operator $\tilde{\mathfrak{O}} : \mathfrak{W} \longrightarrow \mathscr{P}(\mathfrak{W})$ whenever we have $\tilde{\mathfrak{O}}(\varpi^*) = \{\varpi^*\}$ [29]. Also, the mapping $\tilde{\mathfrak{O}}$ possesses an approximate endpoint property (APXEndP-property) whenever

$$\inf_{\varpi_1^* \in \mathfrak{W}} \sup_{\varpi_2^* \in \tilde{\mathfrak{O}}(\varpi_1^*)} d_{\mathfrak{W}}(\varpi_1^*, \varpi_2^*) = 0, \quad (15)$$

[29]. We need next results.

Theorem 3 (Closed graph theorem [30]). *Let $\mathfrak{W}$ be a separable Banach space, $\tilde{\mathfrak{O}} : [0,1] \times \mathfrak{W} \longrightarrow \mathscr{P}_{cmp,cvx}(\mathfrak{W})$ be $\mathscr{L}^1$-Carathéodory and $\coprod : \mathscr{L}^1_{\mathfrak{W}}([0,1]) \longrightarrow \mathscr{C}_{\mathfrak{W}}([0,1])$ be a linear continuous map. Then, $\coprod\circ(\mathbb{SEL})_{\tilde{\mathfrak{O}}} : \mathscr{C}_{\mathfrak{W}}([0,1]) \longrightarrow \mathscr{P}_{cmp,cvx}(\mathscr{C}_{\mathfrak{W}}([0,1]))$ is another operator in $\mathscr{C}_{\mathfrak{W}}([0,1]) \times \mathscr{C}_{\mathfrak{W}}([0,1])$ with action $\varpi^* \mapsto (\coprod\circ(\mathbb{SEL})_{\tilde{\mathfrak{O}}})(\varpi^*) = \coprod((\mathbb{SEL})_{\tilde{\mathfrak{O}},\varpi^*})$ having closed graph property.*

Theorem 4 (Dhage's theorem [31]). *Consider the Banach algebra $\mathfrak{W}$, and the operators $\mathbb{A}_1^* : \mathfrak{W} \longrightarrow \mathfrak{W}$ and $\mathbb{A}_2^* : \mathfrak{W} \longrightarrow \mathscr{P}_{cmp,cvx}(\mathfrak{W})$ satisfying the following:*

(i) $\mathbb{A}_1^*$ is Lipschitzian (with $l^* > 0$)

(ii) $\mathbb{A}_2^*$ is compact upper semicontinuous

(iii) $2l^*\widehat{\mathbb{O}} < 1$ with $\widehat{\mathbb{O}} = \|\mathbb{A}_2^*(\mathfrak{W})\|$

Then, either (1i)$\mathscr{O}^* = \{\varpi^* \in \mathfrak{W} \mid \alpha_0\varpi^* \in \mathbb{A}_1^*\varpi^*\mathbb{A}_2^*\varpi^*, \alpha_0 > 1\}$ is unbounded, or (2i) a solution, belonging to $\mathfrak{W}$, exists for which $\varpi^* \in \mathbb{A}_1^*\varpi^*\mathbb{A}_2^*\varpi^*$.

Theorem 5 (Endpoint theorem [29]). *Suppose that $(\mathfrak{W}, d_{\mathfrak{W}})$ be complete and $\psi : \mathbb{R}^{\geq 0} \longrightarrow \mathbb{R}^{\geq 0}$ admits the upper semicontinuity via $\psi(z) < z$ and $\liminf_{z\longrightarrow\infty}(z - \psi(z)) > 0, \forall z > 0$. Besides, we assume that $\tilde{\mathfrak{O}} : \mathfrak{W} \longrightarrow \mathscr{P}_{cls,bnd}(\mathfrak{W})$ is such that $\mathbb{PH}_{d_{\mathfrak{W}}}(\tilde{\mathfrak{O}}\varpi_1^*, \tilde{\mathfrak{O}}\varpi_2^*) \leq \psi(d_{\mathfrak{W}}(\varpi_1^*, \varpi_2^*))$ for each $\varpi_1^*, \varpi_2^* \in \mathfrak{W}$. Then, an endpoint (uniquely) exists for $\tilde{\mathfrak{O}}$ iff $\tilde{\mathfrak{O}}$ involves the APXEndP-property.*

3. New Existence Criteria

In two previous sections, we assembled some auxiliary and useful notions to achieve our main goals. Now in the following, we first establish a required lemma to derive the main existence results. To do this, we need to consider a sup−norm given by $\|\varpi^*\|_{\mathfrak{W}} = \sup_{z\in[0,1]}|\varpi^*(z)|$ on the space $\mathfrak{W} = \{\varpi^*(z) \colon \varpi^*(z) \in \mathscr{C}_{\mathbb{R}}([0,1])\}$. In this case, the Banach space $(\mathfrak{W}, \|\cdot\|_{\mathfrak{W}})$ along with the multiplication action defined as $(\varpi_1^* \cdot \varpi_2^*)(z) = \varpi_1^*(z)\varpi_2^*(z)$ is a Banach algebra for all $\varpi_1^*, \varpi_2^* \in \mathfrak{W}$.

Lemma 6. *Let $h_* \in \mathfrak{W}$, $a \geq 0$, $\sigma^* \in (1,2)$, $\tilde{\mu} \in (0,1)$, $\mu^* > 0$, $\tilde{m}_1, \tilde{m}_2 \in \mathbb{R}^{\neq 0}$, and $s_1^*, s_2^* \in \mathbb{R}^+$. An element $\varpi_0^* \in \mathfrak{W}$ is a solution for given φ-hybrid fractional equation*

$${}^{\mathscr{C}}\mathfrak{D}_a^{\sigma^*;\varphi}\left(\frac{\varpi^*(z)}{\mathfrak{N}_*(z, \varpi^*(z))}\right) = h_*(z), (z \in [a, T], \sigma^* \in (1,2)), \quad (16)$$

supplemented with separated mixed φ-integro-derivative boundary conditions

$$\begin{cases} \tilde{m}_1\left(\frac{\varpi^*(z)}{\mathfrak{N}_*(z,\varpi^*(z))}\right)\Big|_{z=a} = s_1^* + \tilde{m}_2\left(\frac{\varpi^*(z)}{\mathfrak{N}_*(z,\varpi^*(z))}\right)\Big|_{z=a}, \\ \tilde{m}_1 {}^{\mathscr{RL}}\mathscr{I}_a^{\mu^*;\varphi\mathscr{C}}\mathfrak{D}_a^{\tilde{\mu};\varphi}\left(\frac{\varpi^*(z)}{\mathfrak{N}_*(z,\varpi^*(z))}\right)\Big|_{z=T} = s_2^* + \tilde{m}_2 {}^{\mathscr{RL}}\mathscr{I}_a^{\mu^*;\varphi\mathscr{C}}\mathfrak{D}_a^{1;\varphi}\left(\frac{\varpi^*(z)}{\mathfrak{N}_*(z,\varpi^*(z))}\right)\Big|_{z=T}, \end{cases} \tag{17}$$

which is given by the following:

$$\begin{aligned} \varpi^*(z) = \mathfrak{N}_*(z,\varpi^*(z))\Bigg[&\frac{s_1^*}{\tilde{m}_1-\tilde{m}_2} + \frac{1}{\Gamma(\sigma^*)}\int_a^z \varphi'(q)(\varphi(z)-\varphi(q))^{\sigma^*-1}h_*(q)dq \\ &+ \frac{s_2^*}{\tilde{m}^*}(\varphi(z)-\varphi(a)) - \frac{\tilde{m}_1(\varphi(z)-\varphi(a))}{\tilde{m}^*\Gamma(\sigma^*+\mu^*-\tilde{\mu})} \\ &\cdot \int_a^T \varphi'(q)(\varphi(T)-\varphi(q))^{\sigma^*+\mu^*-\tilde{\mu}-1}h_*(q)dq \\ &+ \frac{\tilde{m}_2(\varphi(z)-\varphi(a))}{\tilde{m}^*\Gamma(\sigma^*+\mu^*-1)}\int_a^T \varphi'(q)(\varphi(T)-\varphi(q))^{\sigma^*+\mu^*-2}h_*(q)dq\Bigg], \end{aligned} \tag{18}$$

so that $\tilde{m}^*$ is a positive real constant given as

$$\tilde{m}^* := \frac{\tilde{m}_1(\varphi(T)-\varphi(a))^{1+\mu^*-\tilde{\mu}}}{\Gamma(2+\mu^*-\tilde{\mu})} - \frac{\tilde{m}_2(\varphi(T)-\varphi(a))^{\mu^*}}{\Gamma(1+\mu^*)} \neq 0. \tag{19}$$

Proof. At first, the element ϖ_0^* is assumed to be a solution for the hybrid φ-Caputo differential Equation (16). Then, there exist $\tilde{k}_0^*, \tilde{k}_1^* \in \mathbb{R}$ such that

$$\frac{\varpi_0^*(z)}{\mathfrak{N}_*(z,\varpi_0^*(z))} = {}^{\mathscr{RL}}\mathscr{I}_a^{\sigma^*;\varphi}h_*(z) + \tilde{k}_0^* + \tilde{k}_1^*(\varphi(z)-\varphi(a)), \tag{20}$$

or more precisely, we have

$$\varpi_0^*(z) = \mathfrak{N}_*(z,\varpi_0^*(z))\left[\int_a^z \frac{\varphi'(q)(\varphi(z)-\varphi(q))^{\sigma^*-1}}{\Gamma(\sigma^*)}h_*(q)\,\mathrm{d}q + \tilde{k}_0^* + \tilde{k}_1^*(\varphi(z)-\varphi(a))\right]. \tag{21}$$

In view of the notion of fractional derivative in the φ-Caputo framework, we get the following relations for $\tilde{\mu} \in (0,1)$:

$${}^{\mathscr{C}}\mathfrak{D}_a^{1;\varphi}\left(\frac{\varpi_0^*(z)}{\mathfrak{N}_*(z,\varpi_0^*(z))}\right) = \int_a^z \frac{\varphi'(q)(\varphi(z)-\varphi(q))^{\sigma^*-2}}{\Gamma(\sigma^*-1)}h_*(q)\,dq + \tilde{k}_1^*, \tag{22}$$

$$\begin{aligned} {}^{\mathscr{C}}\mathfrak{D}_a^{\tilde{\mu};\varphi}\left(\frac{\varpi_0^*(z)}{\mathfrak{N}_*(z,\varpi_0^*(z))}\right) = &\int_a^z \frac{\varphi'(q)(\varphi(z)-\varphi(q))^{\sigma^*-\tilde{\mu}-1}}{\Gamma(\sigma^*-\tilde{\mu})}h_*(q)dq \\ &+ \tilde{k}_1^*\frac{(\varphi(z)-\varphi(a))^{1-\tilde{\mu}}}{\Gamma(2-\tilde{\mu})}. \end{aligned} \tag{23}$$

In the following, by taking integral of order $\mu^* > 0$ in the φ-Riemann-Liouville settings on both sides of (22) and (23), we obtain

$$\begin{aligned} {}^{\mathscr{RL}}\mathscr{I}_a^{\mu^*;\varphi\mathscr{C}}\mathfrak{D}_a^{1;\varphi}\left(\frac{\varpi_0^*(z)}{\mathfrak{N}_*(z,\varpi_0^*(z))}\right) = &\int_a^z \frac{\varphi'(q)(\varphi(z)-\varphi(q))^{\sigma^*+\mu^*-2}}{\Gamma(\sigma^*+\mu^*-1)}h_*(q)\,dq \\ &+ \tilde{k}_1^*\frac{(\varphi(z)-\varphi(a))^{\mu^*}}{\Gamma(\mu^*+1)}, \\ {}^{\mathscr{RL}}\mathscr{I}_a^{(\mu^*;\varphi)C}\mathfrak{D}_a^{(\tilde{\mu};\varphi)}\left(\frac{\varpi_0^*(z)}{\mathfrak{N}_*(z,\varpi_0^*(z))}\right) = &\int_a^z \frac{\varphi'(q)(\varphi(z)-\varphi(q))^{\sigma^*+\mu^*-\tilde{\mu}-1}}{\Gamma(\sigma^*+\mu^*-\tilde{\mu})}h_*(q)dq \\ &+ \tilde{k}_1^*\frac{(\varphi(z)-\varphi-(a))^{1+\mu^*-\tilde{\mu}}}{\Gamma(\mu^*-\tilde{\mu}+2)}. \end{aligned} \tag{24}$$

In this step, by considering the first boundary condition in (17), we find that $(\tilde{m}_1-\tilde{m}_2)\tilde{k}_0^* = s_1^*$ and so

$$\tilde{k}_0^* = \frac{s_1^*}{\tilde{m}_1-\tilde{m}_2}. \tag{25}$$

In addition, the second integro-derivative boundary condition given in (17) yields

$$\begin{aligned} \tilde{k}_1^* = \frac{s_2^*}{\tilde{m}^*} - &\frac{\tilde{m}_1}{\tilde{m}^*\Gamma(\sigma^*+\mu^*-\tilde{\mu})}\int_a^T \varphi'(q)(\varphi(T)-\varphi(q))^{\sigma^*+\mu^*-\tilde{\mu}-1}h_*(q)dq \\ &+ \frac{\tilde{m}_2}{\tilde{m}^*\Gamma(\sigma^*+\mu^*-1)}\int_a^T \varphi'(q)(\varphi(T)-\varphi(q))^{\sigma^*+\mu^*-2}h_*(q)dq. \end{aligned} \tag{26}$$

In the last step, if we insert the values $\tilde{k}_0^*$ and $\tilde{k}_1^*$ obtained in (25) and (26) into (21), then we get

$$\begin{aligned} \varpi_0^*(z) = \mathfrak{N}_*(z,\varpi_0^*(z))\Bigg[&\frac{s_1^*}{\tilde{m}_1-\tilde{m}_2} + \frac{1}{\Gamma(\sigma^*)}\int_a^z \varphi'(q)(\varphi(z)-\varphi(q))^{\sigma^*-1}h_*(q)\,dq \\ &+ \frac{s_2^*}{\tilde{m}^*}(\varphi(z)-\varphi(a)) - \frac{\tilde{m}_1(\varphi(z)-\varphi(a))}{\tilde{m}^*\Gamma(\sigma^*+\mu^*-\tilde{\mu})} \\ &\cdot \int_a^T \varphi'(q)(\varphi(T)-\varphi(q))^{\sigma^*+\mu^*-\tilde{\mu}-1}h_*(q)dq + \frac{\tilde{m}_2(\varphi(z)-o(a))}{\tilde{m}^*\Gamma(\sigma^*+\mu^*-1)} \\ &\cdot \int_a^T \varphi'(q)(\varphi(T)-\varphi(q))^{\sigma^*+\mu^*-2}h_*(q)dq\Bigg]. \end{aligned} \tag{27}$$

The resultant integral equation confirms that ϖ_0^* satisfies the mentioned φ-integral Equation (18), and the proof is completed. □

Now, by considering Lemma 6, we can present the following definition.

Definition 7. An absolutely continuous function $\varpi^* : [a,T] \longrightarrow \mathbb{R}$ is called a solution function for the φ-hybrid inclusion BVP in the sense of φ-Caputo (2) and (3) if there is $\hat{\kappa} \in \mathscr{L}^1([a,T],\mathbb{R})$ with $\hat{\kappa}(z) \in \tilde{\mathfrak{O}}(z,\varpi^*(z))$ for almost all $z \in [a,T]$ which satisfies separated mixed φ-integro-derivative boundary conditions

$$\begin{cases} \tilde{m}_1\left(\frac{\varpi^*(z)}{\mathfrak{N}_*(z,\varpi^*(z))}\right)\Big|_{z=a} = s_1^* + \tilde{m}_2\left(\frac{\varpi^*(z)}{\mathfrak{N}_*(z,\varpi^*(z))}\right)\Big|_{z=a}, \\ \tilde{m}_1{}^{\mathscr{RL}}\mathscr{I}_a^{\mu^*;\varphi\mathscr{C}}\mathfrak{D}_a^{\tilde{\mu};\varphi}\left(\frac{\varpi^*(z)}{\mathfrak{N}_*(z,\varpi^*(z))}\right)\Big|_{z=T} = s_2^* + \tilde{m}_2{}^{\mathscr{RL}}\mathscr{I}_a^{\mu^*;\varphi\mathscr{C}}\mathfrak{D}_a^{1;\varphi}\left(\frac{\varpi^*(z)}{\mathfrak{N}_*(z,\varpi^*(z))}\right)\Big|_{z=T}, \end{cases} \quad (28)$$

and also

$$\begin{aligned} \varpi^*(z) = \mathfrak{N}_*(z,\varpi^*(z)) \Bigg[& \frac{s_1^*}{\tilde{m}_1 - \tilde{m}_2} + \frac{1}{\Gamma(\sigma^*)}\int_a^z \varphi'(q)(\varphi(z)-\varphi(q))^{\sigma^*-1}\widehat{\kappa}(q)dq \\ & + \frac{s_2^*}{\tilde{m}^*}(\varphi(z)-\varphi(a)) - \frac{\tilde{m}_1(\varphi(z)-\varphi(a))}{\tilde{m}^*\Gamma(\sigma^*+\mu^*-\tilde{\mu})} \\ & \cdot \int_a^T \varphi'(q)(\varphi(T)-\varphi(q))^{\sigma^*+\mu^*-\tilde{\mu}-1}\widehat{\kappa}(q)dq + \frac{\tilde{m}_2(\varphi(z)-\varphi(a))}{\tilde{m}^*\Gamma(\sigma^*+\mu^*-1)} \\ & \cdot \int_a^T \varphi'(q)(\varphi(T)-\varphi(q))^{\sigma^*+\mu^*-2}\widehat{\kappa}(q)dq \Bigg], \end{aligned} \quad (29)$$

for each $z \in [a, T]$.

Now, we are in a position that we can prove the first existence result about the hybrid φ-Caputo inclusion BVP (2) and (3).

Theorem 8. *Assume that $\tilde{\mathfrak{O}} : [a, T] \times \mathfrak{W} \longrightarrow \mathscr{P}_{cmp,cvx}(\mathfrak{W})$ is a set-valued operator and a function $\mathfrak{N}_* : [a, T] \times \mathfrak{W} \longrightarrow \mathfrak{W} \setminus \{0\}$ is continuous. In addition, let*

($\mathfrak{C}_1$) a bounded function $\chi : [a, T] \longrightarrow \mathbb{R}^+$ exists such that for each $\varpi_1^*, \varpi_2^* \in \mathfrak{W}$ and $z \in [a, T]$

$$|\mathfrak{N}_*(z,\varpi_1^*(z)) - \mathfrak{N}_*(z,\varpi_2^*(z))| \le \chi(z)|\varpi_1^*(z) - \varpi_2^*(z)| \quad (30)$$

($\mathfrak{C}_2$) $\tilde{\mathfrak{O}}$ is $\mathscr{L}^1$-Caratheodory

($\mathfrak{C}_3$) a function $Y(z) \in \mathscr{L}^1([a, T], \mathbb{R}^+)$ exists such that

$$\left\|\tilde{\mathfrak{O}}(z,\varpi^*)\right\| = \sup\left\{|\widehat{\kappa}| : \widehat{\kappa} \in \tilde{\mathfrak{O}}(z,\varpi^*(z))\right\} \le Y(z), \quad (31)$$

for all $\varpi^* \in \mathfrak{W}$ and for almost all $z \in [a, T]$

($\mathfrak{C}_4$) a real number $\widehat{\varepsilon} \in \mathbb{R}$ exists so that

$$\widehat{\varepsilon} > \frac{\mathscr{F}^*\Pi^*\|Y\|_{\mathscr{L}^1}}{1 - \chi^*\Pi^*\|Y\|_{\mathscr{L}^1}}, \quad (32)$$

where $\mathscr{F}^* = \sup_{z\in[a,T]}|\tilde{\mathfrak{N}}_*(z,0)|$, $\|Y\|_{\mathscr{L}^1} = \int_a^T |Y(q)|dq$, $\chi^* = \sup_{z\in[a,T]}|\chi(z)|$, and

$$\begin{aligned} \Pi^* = & \frac{s_1^*}{|\tilde{m}_1 - \tilde{m}_2|} + \frac{(\varphi(T)-\varphi(a))^{\sigma^*}}{\Gamma(\sigma^*+1)} + \frac{s_2^*}{|\tilde{m}^*|}(\varphi(T)-\varphi(a)) \\ & + \frac{|\tilde{m}_1|(\varphi(T)-\varphi(a))^{\sigma^*+\mu^*-\tilde{\mu}+1}}{|\tilde{m}^*|\Gamma(\sigma^*+\mu^*-\tilde{\mu}+1)} + \frac{|\tilde{m}_2|(\varphi(T)-\varphi(a))^{\sigma^*+\mu^*}}{|\tilde{m}^*|\Gamma(\sigma^*+\mu^*)} \end{aligned} \quad (33)$$

If $\chi^*\Pi^*\|Y\|_{\mathscr{L}^1} < 1/2$, then the φ-hybrid inclusion BVP (2) and (3) has at least a solution.

Proof. For each $\varpi^* \in \mathfrak{W}$, the collection of all existing selections of $\tilde{\mathfrak{O}}$ is defined as

$$(\mathbb{SEL})_{\tilde{\mathfrak{O}},\varpi^*} := \left\{\widehat{\kappa} \in \mathscr{L}_{\mathbb{R}}^1([a,T]) : \widehat{\kappa}(z) \in \tilde{\mathfrak{O}}(z,\varpi^*(z))\right\}, \quad (34)$$

for every $\varpi^* \in \mathfrak{W}$ and for almost all $z \in [a, T]$. Define a set-valued map $\mathfrak{K} : \mathfrak{W} \longrightarrow \mathscr{P}(\mathfrak{W})$ by

$$\mathfrak{K}(\varpi^*) = \left\{\zeta^* \in \mathfrak{W} : \zeta^*(z) = \psi^*(z)\right\}, \quad (35)$$

where

$$\begin{aligned} \psi^*(z) = \mathfrak{N}_*(z,\varpi^*(z)) \Bigg[& \frac{s_1^*}{\tilde{m}_1 - \tilde{m}_2} + \frac{1}{\Gamma(\sigma^*)}\int_a^z \varphi'(q)(\varphi(z)-\varphi(q))^{\sigma^*-1}\widehat{\kappa}(q)dq \\ & + \frac{s_2^*}{\tilde{m}^*}(\varphi(z)-\varphi(a)) - \frac{\tilde{m}_1(\varphi(z)-\varphi(a))}{\tilde{m}^*\Gamma(\sigma^*+\mu^*-\tilde{\mu})} \\ & \cdot \int_a^T \varphi'(q)(\varphi(T)-\varphi(q))^{\sigma^*+\mu^*-\tilde{\mu}-1}\widehat{\kappa}(q)dq + \frac{\tilde{m}_2(\varphi(z)-\varphi(a))}{\tilde{m}^*\Gamma(\sigma^*+\mu^*-1)} \\ & \cdot \int_a^T \varphi'(q)(\varphi(T)-\varphi(q))^{\sigma^*+\mu^*-2}\widehat{\kappa}(q)dq \Bigg], \end{aligned} \quad (36)$$

for some $\widehat{\kappa} \in (\mathbb{SEL})_{\tilde{\mathfrak{O}},\varpi^*}$ and for almost all $z \in [a, T]$. It is obvious that the function ψ_0^* is a solution to the φ-hybrid BVP (2) and (3) if ψ_0^* is a fixed point of $\mathfrak{K}$. Now, define $\mathbb{A}_1^* : \mathfrak{W} \longrightarrow \mathfrak{W}$ by $(\mathbb{A}_1^*\varpi^*)(z) = \mathfrak{N}_*(z,\varpi^*(z))$ and $\mathbb{A}_2^* : \mathfrak{W} \longrightarrow \mathscr{P}(\mathfrak{W})$ by

$$(\mathbb{A}_2^*\varpi^*)(z) = \left\{h^* \in \mathfrak{W} : h^*(z) = \xi^*(z)\right\}, \quad (37)$$

where

$$\begin{aligned} \xi^*(z) = & \frac{s_1^*}{\tilde{m}_1 - \tilde{m}_2} + \frac{1}{\Gamma(\sigma^*)}\int_a^z \varphi'(q)(\varphi(z)-\varphi(q))^{\sigma^*-1}\widehat{\kappa}(q)dq \\ & + \frac{s_2^*}{\tilde{m}^*}(\varphi(z)-\varphi(a)) - \frac{\tilde{m}_1(\varphi(z)-\varphi(a))}{\tilde{m}^*\Gamma(\sigma^*+\mu^*-\tilde{\mu})} \\ & \cdot \int_a^T \varphi'(q)(\varphi(T)-\varphi(q))^{\sigma^*+\mu^*-\tilde{\mu}-1}\widehat{\kappa}(q)dq \\ & + \frac{\tilde{m}_2(\varphi(z)-\varphi(a))}{\tilde{m}^*\Gamma(\sigma^*+\mu^*-1)}\int_a^T \varphi'(q)(\varphi(T)-\varphi(q))^{\sigma^*+\mu^*-2}\widehat{\kappa}(q)dq, \end{aligned} \quad (38)$$

for some $\widehat{\kappa} \in (\mathbb{SEL})_{\tilde{\mathfrak{O}},\varpi^*}$ and for almost all $z \in [a, T]$. This implies $\mathfrak{K}(\varpi^*) = (\mathbb{A}_1^*\varpi^*)(\mathbb{A}_2^*\varpi^*)$. We show that both operators $\mathbb{A}_1^*$ and $\mathbb{A}_2^*$ satisfy Theorem 4. We at first prove that $\mathbb{A}_1^*$ is Lipschitzian. Let $\varpi_1^*, \varpi_2^* \in \mathfrak{W}$. We have

$$\begin{aligned} |(\mathbb{A}_1^*\varpi_1^*(z)) - (\mathbb{A}_1^*\varpi_2^*(z))| & = |\mathfrak{N}_*(z,\varpi_1^*(z)) - \mathfrak{N}_*(z,\varpi_2^*(z))| \\ & \le \chi(z)|\varpi_1^*(z) - \varpi_2^*(z)|, \end{aligned} \quad (39)$$

for all $z \in [a, T]$. Therefore, for all $\varpi_1^*, \varpi_2^* \in \mathfrak{W}$, we get

$$\|\mathbb{A}_1^*\varpi_1^* - \mathbb{A}_1^*\varpi_2^*\|_{\mathfrak{W}} \le \chi^*\|\varpi_1^* - \varpi_2^*\|\mathfrak{W}. \quad (40)$$

Hence, $\mathbb{A}_1^*$ is Lipschitz with constant $l^* = \chi^* > 0$. In the current moment, we check the convexity of $\mathbb{A}_2^*$. For this, let $\varpi_1^*, \varpi_2^* \in \mathbb{A}_2^*\varpi^*$. Choose $\widehat{\kappa}_1, \widehat{\kappa}_2 \in (\mathbb{SEL})_{\tilde{\mathfrak{O}},\varpi^*}$ such that

$$\begin{aligned}\varpi_i^*(z) &= \frac{s_1^*}{\tilde{m}_1 - \tilde{m}_2} + \frac{1}{\Gamma(\sigma^*)}\int_a^z \varphi'(q)(\varphi(z)-\varphi(q))^{\sigma^*-1}\widehat{\kappa}_i(q)dq \\ &+ \frac{s_2^*}{\tilde{m}^*}(\varphi(z)-\varphi(a)) - \frac{\tilde{m}_1(\varphi(z)-\varphi(a))}{\tilde{m}^*\Gamma(\sigma^*+\mu^*-\tilde{\mu})} \\ &\cdot \int_a^T \varphi'(q)(\varphi(T)-\varphi(q))^{\sigma^*+\mu^*-\tilde{\mu}-1}\widehat{\kappa}_i(q)dq + \frac{\tilde{m}_2(\varphi(z)-\varphi(a))}{\tilde{m}^*\Gamma(\sigma^*+\mu^*-1)} \\ &\cdot \int_a^T \varphi'(q)(\varphi(T)-\varphi(q))^{\sigma^*+\mu^*-2}\widehat{\kappa}_i(q)dq, (i=1,2),\end{aligned} \tag{41}$$

for almost all $z \in [a, T]$. Let $0 < \eta < 1$. Then,

$$\begin{aligned}\eta\varpi_1^*(z) + (1-\eta)\varpi_2^*(z) &= \frac{s_1^*}{\tilde{m}_1 - \tilde{m}_2} + \frac{s_2^*}{\tilde{m}^*}(\varphi(z)-\varphi(a)) + \frac{1}{\Gamma(\sigma^*)} \\ &\cdot \int_a^z \varphi'(q)(\varphi(z)-\varphi(q))^{\sigma^*-1}[\eta\widehat{\kappa}_1(q) + (1-\eta)\widehat{\kappa}_2(q)]dq \\ &- \frac{\tilde{m}_1(\varphi(z)-\varphi(a))}{\tilde{m}^*\Gamma(\sigma^*+\mu^*-\tilde{\mu})}\int_a^T \varphi'(q)(\varphi(T)-\varphi(q))^{\sigma^*+\mu^*-\tilde{\mu}-1} \\ &\cdot [\eta\widehat{\kappa}_1(q) + (1-\eta)\widehat{\kappa}_2(q)]dq + \frac{\tilde{m}_2(\varphi(z)-\varphi(a))}{\tilde{m}^*\Gamma(\sigma^*+\mu^*-1)} \\ &\cdot \int_a^T \varphi'(q)(\varphi(T)-\varphi(q))^{\sigma^*+\mu^*-2}[\eta\widehat{\kappa}_1(q) + (1-\eta)\widehat{\kappa}_2(q)]dq,\end{aligned} \tag{42}$$

for almost all $z \in [a, T]$. With due attention to the convexity of $\tilde{\mathfrak{O}}$, $(\mathbb{SEL})_{\tilde{\mathfrak{O}},\varpi^*}$ has convex values. This implies that

$$\eta\widehat{\kappa}_1(z) + (1-\eta)\widehat{\kappa}_2(z) \in (\mathbb{SEL})_{\tilde{\mathfrak{O}},\varpi^*}, \tag{43}$$

for almost all $z \in [a, T]$. Therefore, $\mathbb{A}_2^*\varpi^*$ is convex for each $\varpi^* \in \mathfrak{W}$. Next, we claim that $\mathbb{A}_2^*$ is completely continuous. To confirm this claim, we verify that the set $\mathbb{A}_2^*(\mathfrak{W})$ is equicontinuous and uniformly bounded. Firstly, we prove that $\mathbb{A}_2^*$ corresponds bounded sets to bounded sets contained in $\mathfrak{W}$. For $\alpha^* \in \mathbb{R}^+$, define the bounded ball $\mathfrak{B}_{\alpha^*} = \{\varpi^* \in \mathfrak{W} : \|\varpi^*\|_{\mathfrak{W}} \le \alpha^*\}$. For every $\varpi^* \in \mathfrak{B}_{\alpha^*}$ and $\xi^* \in \mathbb{A}_2^*\varpi^*$, there exists a function $\widehat{\kappa} \in (\mathbb{SEL})_{\tilde{\mathfrak{O}},\varpi^*}$ such that

$$\begin{aligned}\xi^*(z) &= \frac{s_1^*}{\tilde{m}_1 - \tilde{m}_2} + \frac{1}{\Gamma(\sigma^*)}\int_a^z \varphi'(q)(\varphi(z)-\varphi(q))^{\sigma^*-1}\widehat{\kappa}(q)dq \\ &+ \frac{s_2^*}{\tilde{m}^*}(\varphi(z)-\varphi(a)) - \frac{\tilde{m}_1(\varphi(z)-\varphi(a))}{\tilde{m}^*\Gamma(\sigma^*+\mu^*-\tilde{\mu})} \\ &\cdot \int_a^T \varphi'(q)(\varphi(T)-\varphi(q))^{\sigma^*+\mu^*-\tilde{\mu}-1}\widehat{\kappa}(q)dq \\ &+ \left.\frac{\tilde{m}_2(\varphi(z)-\varphi(a))}{\tilde{m}^*\Gamma(\sigma^*+\mu^*-1)}\int_a^T \varphi'(q)(\varphi(T)-\varphi(q))^{\sigma^*+\mu^*-2}\widehat{\kappa}(q)dq\right],\end{aligned} \tag{44}$$

for almost all $z \in [a, T]$. Then,

$$\begin{aligned}|\xi^*(z)| &\le \frac{1}{\Gamma(\sigma^*)}\int_a^z \varphi'(q)(\varphi(z)-\varphi(q))^{\sigma^*-1}|\widehat{\kappa}(q)|dq + \frac{s_2^*}{|\tilde{m}^*|}(\varphi(z)-\varphi(a)) \\ &+ \frac{|\tilde{m}_1|(\varphi(z)-\varphi(a))}{|\tilde{m}^*|\Gamma(\sigma^*+\mu^*-\tilde{\mu})}\int_a^T \varphi'(q)(\varphi(T)-\varphi(q))^{\sigma^*+\mu^*-\tilde{\mu}-1}|\widehat{\kappa}(q)|dq \\ &+ \frac{|\tilde{m}_2|(\varphi(z)-\varphi(a))}{|\tilde{m}^*|\Gamma(\sigma^*+\mu^*-1)}\int_a^T \varphi'(q)(\varphi(T)-\varphi(q))^{\sigma^*+\mu^*-2}|\widehat{\kappa}(q)|dq \\ &+ \frac{s_1^*}{|\tilde{m}_1 - \tilde{m}_2|} \le \frac{1}{\Gamma(\sigma^*)}\int_a^z \varphi'(q)(\varphi(z)-\varphi(q))^{\sigma^*-1}Y(q)dq \\ &+ \frac{s_2^*}{|\tilde{m}^*|}(\varphi(z)-\varphi(a)) + \frac{|\tilde{m}_1|(\varphi(z)-\varphi(a))}{|\tilde{m}^*|\Gamma(\sigma^*+\mu^*-\tilde{\mu})} \\ &\cdot \int_a^T \varphi'(q)(\varphi(T)-\varphi(q))^{\sigma^*+\mu^*-\tilde{\mu}-1}Y(q)dq + \frac{|\tilde{m}_2|(\varphi(z)-\varphi(a))}{|\tilde{m}^*|\Gamma(\sigma^*+\mu^*-1)} \\ &\cdot \int_a^T \varphi'(q)(\varphi(T)-\varphi(q))^{\sigma^*+\mu^*-2}Y(q)dq + \frac{s_1^*}{|\tilde{m}_1 - \tilde{m}_2|} \\ &\le \left[\frac{(\varphi(T)-\varphi(a))^{\sigma^*}}{\Gamma(\sigma^*+1)} + \frac{s_2^*}{|\tilde{m}^*|}(\varphi(T)-\varphi(a)) + \frac{|\tilde{m}_1|(\varphi(T)-\varphi(a))^{\sigma^*+\mu^*-\tilde{\mu}+1}}{|\tilde{m}^*|\Gamma(\sigma^*+\mu^*-\tilde{\mu}+1)}\right. \\ &+ \left.\frac{|\tilde{m}_2|(\varphi(T)-\varphi(a))^{\sigma^*+\mu^*}}{|\tilde{m}^*|\Gamma(\sigma^*+\mu^*)} + \frac{s_1^*}{|\tilde{m}_1 - \tilde{m}_2|}\right]\|Y\|_{\mathscr{L}^1} = \Pi^*\|Y\|_{\mathscr{L}^1},\end{aligned} \tag{45}$$

where Π^* is given in (33). Thus, $\|\xi^*\| \le \Pi^*\|Y\|_{\mathscr{L}^1}$, and so the set $\mathbb{A}_2^*(\mathfrak{W})$ is uniformly bounded. Now, we want to prove that $\mathbb{A}_2^*$ corresponds bounded sets to equicontinuous sets. Take $\varpi^* \in \mathfrak{B}_{\alpha^*}$, $\xi^* \in \mathbb{A}_2^*\varpi^*$ and choose $\widehat{\kappa} \in (\mathbb{SEL})_{\tilde{\mathfrak{O}},\varpi^*}$ so that

$$\begin{aligned}\xi^*(z) &= \frac{1}{\Gamma(\sigma^*)}\int_a^z \varphi'(q)(\varphi(z)-\varphi(q))^{\sigma^*-1}\widehat{\kappa}(q)\,dq + \frac{s_2^*}{\tilde{m}^*}(\varphi(z)-\varphi(a)) \\ &- \frac{\tilde{m}_1(\varphi(z)-\varphi(a))}{\tilde{m}^*\Gamma(\sigma^*+\mu^*-\tilde{\mu})}\int_a^T \varphi'(q)(\varphi(T)-\varphi(q))^{\sigma^*+\mu^*-\tilde{\mu}-1}\widehat{\kappa}(q)dq \\ &+ \frac{\tilde{m}_2(\varphi(z)-\varphi(a))}{\tilde{m}^*\Gamma(\sigma^*+\mu^*-1)}\int_a^T \varphi'(q)(\varphi(T)-\varphi(q))^{\sigma^*+\mu^*-2}\widehat{\kappa}(q)dq \\ &+ \frac{s_1^*}{\tilde{m}_1 - \tilde{m}_2},\end{aligned} \tag{46}$$

for almost all $z \in [a, T]$. Let $z_1, z_2 \in [a, T]$ with $z_1 < z_2$, Then,

$$\begin{aligned}|\xi^*(z_2) - \xi^*(z_1)| &\le \frac{1}{\Gamma(\sigma^*)}\int_a^{z_1} \varphi'(q)\left[(\varphi(z_2)-\varphi(q))^{\sigma^*-1} - (\varphi(z_1)-\varphi(q))^{\sigma^*-1}\right] \\ &\cdot |\widehat{\kappa}(q)|dq + \frac{1}{\Gamma(\sigma^*)}\int_{z_1}^{z_2} \varphi'(q)(\varphi(z_2)-\varphi(q))^{\sigma^*-1}|\widehat{\kappa}(q)|dq \\ &+ \frac{s_2^*}{|\tilde{m}^*|}[\varphi(z_2)-\varphi(z_1)] + \frac{|\tilde{m}_1|[\varphi(z_2)-\varphi(z_1)]}{|\tilde{m}^*|\Gamma(\sigma^*+\mu^*-\tilde{\mu})} \\ &\cdot \int_a^T \varphi'(q)(\varphi(T)-\varphi(q))^{\sigma^*+\mu^*-\tilde{\mu}-1}|\widehat{\kappa}(q)|dq + \frac{|\tilde{m}_2|[\varphi(z_2)-\varphi(z_1)]}{|\tilde{m}^*|\Gamma(\sigma^*+\mu^*-1)} \\ &\cdot \int_a^T \varphi'(q)(\varphi(T)-\varphi(q))^{\sigma^*+\mu^*-2}|\widehat{\kappa}(q)|dq \le \frac{1}{\Gamma(\sigma^*)}\int_a^{z_1} \varphi'(q) \\ &\cdot \left[(\varphi(z_2)-\varphi(q))^{\sigma^*-1} - (\varphi(z_1)-\varphi(q))^{\sigma^*-1}\right]Y(q)dq + \frac{1}{\Gamma(\sigma^*)} \\ &\cdot \int_{z_1}^{z_2} \varphi'(q)(\varphi(z_2)-\varphi(q))^{\sigma^*-1}Y(q)dq + \frac{s_2^*}{|\tilde{m}^*|}[\varphi(z_2)-\varphi(z_1)] \\ &+ \frac{|\tilde{m}_1|[\varphi(z_2)-\varphi(z_1)]}{|\tilde{m}^*|\Gamma(\sigma^*+\mu^*-\tilde{\mu})}\int_a^T \varphi'(q)(\varphi(T)-\varphi(q))^{\sigma^*+\mu^*-\tilde{\mu}-1}Y(q)dq \\ &+ \frac{|\tilde{m}_2|[\varphi(z_2)-\varphi(z_1)]}{|\tilde{m}^*|\Gamma(\sigma^*+\mu^*-1)}\int_a^T \varphi'(q)(\varphi(T)-\varphi(q))^{\sigma^*+\mu^*-2}Y(q)dq.\end{aligned} \tag{47}$$

The right-hand side of the latter inequalities tends to zero (independent of $\varpi^* \in \mathfrak{B}_{\alpha^*}$) as z_1 tends to z_2. Application of Arzela–Ascoli theorem gives the complete continuity of $\mathbb{A}_2^*$. We here discuss that $\mathbb{A}_2^*$ has a closed graph, and this finding implies that $\mathbb{A}_2^*$ is upper semicontinuous. To achieve this aim, let $\varpi_n^* \in \mathfrak{B}_{\alpha^*}$ and $\xi_n^* \in (\mathbb{A}_2^* \varpi_n^*)$ with $\varpi_n^* \longrightarrow \varpi^{**}$ and $\xi_n^* \longrightarrow \tilde{\xi}^*$. We claim that $\tilde{\xi}^* \in (\mathbb{A}_2^* \varpi^{**})$. For every $n \geq 1$ and $\xi_n^* \in (\mathbb{A}_2^* \varpi_n^*)$, choose $\widehat{\kappa}_n \in (\mathbb{SEL})_{\tilde{\mathfrak{D}},\varpi_n^*}$ such that

$$\begin{aligned}\xi_n^*(z) = {} & \frac{1}{\Gamma(\sigma^*)}\int_a^z \varphi'(q)(\varphi(z)-\varphi(q))^{\sigma^*-1}\widehat{\kappa}_n(q)\mathrm{d}q \\ & + \frac{s_2^*}{\tilde{m}^*}(\varphi(z)-\varphi(a)) - \frac{\tilde{m}_1(\varphi(z)-\varphi(a))}{\tilde{m}^*\Gamma(\sigma^*+\mu^*-\tilde{\mu})} \\ & \cdot \int_a^T \varphi'(q)(\varphi(T)-\varphi(q))^{\sigma^*+\mu^*-\tilde{\mu}-1}\widehat{\kappa}_n(q)dq \\ & + \frac{\tilde{m}_2(\varphi(z)-\varphi(a))}{\tilde{m}^*\Gamma(\sigma^*+\mu^*-1)}\int_a^T \varphi'(q)(\varphi(T)-\varphi(q))^{\sigma^*+\mu^*-2}\widehat{\kappa}_n(q)dq \\ & + \frac{s_1^*}{\tilde{m}_1-\tilde{m}_2},\end{aligned} \tag{48}$$

for almost all $z \in [a, T]$. It is suffices to find that there is a member $\kappa\wedge^* \in (\mathbb{SEL})_{\tilde{\mathfrak{D}},\varpi^{**}}$ so that

$$\begin{aligned}\tilde{\xi}^*(z) = {} & \frac{1}{\Gamma(\sigma^*)}\int_a^z \varphi'(q)(\varphi(z)-\varphi(q))^{\sigma^*-1}\kappa\wedge^*(q)dq \\ & + \frac{s_2^*}{\tilde{m}^*}(\varphi(z)-\varphi(a)) - \frac{\tilde{m}_1(\varphi(z)-\varphi(a))}{\tilde{m}^*\Gamma(\sigma^*+\mu^*-\tilde{\mu})} \\ & \cdot \int_a^T \varphi'(q)(\varphi(T)-\varphi(q))^{\sigma^*+\mu^*-\tilde{\mu}-1}\kappa\wedge^*(q)dq \\ & + \frac{\tilde{m}_2(\varphi(z)-\varphi(a))}{\tilde{m}^*\Gamma(\sigma^*+\mu^*-1)}\int_a^T \varphi'(q)(\varphi(T)-\varphi(q))^{\sigma^*+\mu^*-2}\kappa\wedge^*(q)dq \\ & + \frac{s_1^*}{\tilde{m}_1-\tilde{m}_2},\end{aligned} \tag{49}$$

for almost all $z \in [a, T]$. Define a linear continuous operator $\coprod : \mathscr{L}^1([a, T], \mathbb{R}) \longrightarrow \mathfrak{W} = \mathscr{C}([a, T], \mathbb{R})$ as

$$\begin{aligned}\coprod\widehat{\kappa}(z) = \varpi^*(z) = {} & \frac{1}{\Gamma(\sigma^*)}\int_a^z \varphi'(q)(\varphi(z)-\varphi(q))^{\sigma^*-1}\widehat{\kappa}(q)dq \\ & + \frac{s_2^*}{\tilde{m}^*}(\varphi(z)-\varphi(a)) - \frac{\tilde{m}_1(\varphi(z)-\varphi(a))}{\tilde{m}^*\Gamma(\sigma^*+\mu^*-\tilde{\mu})} \\ & \cdot \int_a^T \varphi'(q)(\varphi(T)-\varphi(q))^{\sigma^*+\mu^*-\tilde{\mu}-1}\widehat{\kappa}(q)dq \\ & + \frac{\tilde{m}_2(\varphi(z)-\varphi(a))}{\tilde{m}^*\Gamma(\sigma^*+\mu^*-1)}\int_a^T \varphi'(q)(\varphi(T)-\varphi(q))^{\sigma^*+\mu^*-2}\widehat{\kappa}(q)dq \\ & + \frac{s_1^*}{\tilde{m}_1-\tilde{m}_2},\end{aligned} \tag{50}$$

for almost all $z \in [a, T]$. Hence,

$$\begin{aligned}\left\|\xi_n^*(z)-\tilde{\xi}^*(z)\right\| = {} & \left\|\frac{1}{\Gamma(\sigma^*)}\int_a^z \varphi'(q)(\varphi(z)-\varphi(q))^{\sigma^*-1}\right. \\ & \cdot (\widehat{\kappa}_n(q)-\kappa\wedge^*(q))dq + \frac{s_2^*}{\tilde{m}^*}(\varphi(z)-\varphi(a)) \\ & - \frac{\tilde{m}_1(\varphi(z)-\varphi(a))}{\tilde{m}^*\Gamma(\sigma^*+\mu^*-\tilde{\mu})}\int_a^T \varphi'(q)(\varphi(T)-\varphi(q))^{\sigma^*+\mu^*-\tilde{\mu}-1} \\ & \cdot (\widehat{\kappa}_n(q)-\kappa\wedge^*(q))dq + \frac{\tilde{m}_2(\varphi(z)-\varphi(a))}{\tilde{m}^*\Gamma(\sigma^*+\mu^*-1)}\int_a^T \varphi' \\ & \left.\cdot (q)(\varphi(T)-\varphi(q))^{\sigma^*+\mu^*-2}(\widehat{\kappa}_n(q)-\kappa\wedge^*(q))dq\right\| \longrightarrow 0.\end{aligned} \tag{51}$$

Application of Theorem 3 shows that $\coprod\circ(\mathbb{SEL})_{\tilde{\mathfrak{D}}}$ has a closed graph. Besides, since $\xi_n^* \in \coprod((\mathbb{SEL})_{\tilde{\mathfrak{D}},\varpi_n})$ and $\varpi_n \longrightarrow \varpi^{**}$, so there exists $\kappa\wedge^* \in (\mathbb{SEL})_{\tilde{\mathfrak{D}},\varpi^{**}}$ such that

$$\begin{aligned}\tilde{\xi}^*(z) = {} & \frac{1}{\Gamma(\sigma^*)}\int_a^z \varphi'(q)(\varphi(z)-\varphi(q))^{\sigma^*-1}\kappa\wedge^*(q)dq \\ & + \frac{s_2^*}{\tilde{m}^*}(\varphi(z)-\varphi(a)) - \frac{\tilde{m}_1(\varphi(z)-\varphi(a))}{\tilde{m}^*\Gamma(\sigma^*+\mu^*-\tilde{\mu})} \\ & \cdot \int_a^T \varphi'(q)(\varphi(T)-\varphi(q))^{\sigma^*+\mu^*-\tilde{\mu}-1}\kappa\wedge^*(q)dq \\ & + \frac{\tilde{m}_2(\varphi(z)-\varphi(a))}{\tilde{m}^*\Gamma(\sigma^*+\mu^*-1)}\int_a^T \varphi'(q)(\varphi(T)-\varphi(q))^{\sigma^*+\mu^*-2}\kappa\wedge^*(q)dq \\ & + \frac{s_1^*}{\tilde{m}_1-\tilde{m}_2},\end{aligned} \tag{52}$$

for almost all $z \in [a, T]$. Hence, $\tilde{\xi}^* \in (\mathbb{A}_2^* \varpi^{**})$, and so $\mathbb{A}_2^*$ possesses closed graph which implies that $\mathbb{A}_2^*$ is upper semicontinuous. On the other hand, because of the compactness of values of $\mathbb{A}_2^*$, it is immediately deduced that $\mathbb{A}_2^*$ is compact and upper semicontinuous. Utilizing $(\mathfrak{C}_3)$, we get

$$\begin{aligned}\widehat{\mathbb{O}} = \|\mathbb{A}_2^*(\mathfrak{W})\| = {} & \sup\{|\mathbb{A}_2^*\varpi^*| : \varpi^* \in \mathfrak{W}\} \\ = {} & \left[\frac{s_1^*}{|\tilde{m}_1-\tilde{m}_2|} + \frac{(\varphi(T)-\varphi(a))^{\sigma^*}}{\Gamma(\sigma^*+1)} + \frac{s_2^*}{|\tilde{m}^*|}(\varphi(T)-\varphi(a))\right. \\ & \left. + \frac{|\tilde{m}_1|(\varphi(T)-\varphi(a))^{\sigma^*+\mu^*-\tilde{\mu}+1}}{|\tilde{m}^*|\Gamma(\sigma^*+\mu^*-\tilde{\mu}+1)} + \frac{|\tilde{m}_2|(\varphi(T)-\varphi(a))^{\sigma^*+\mu^*}}{|\tilde{m}^*|\Gamma(\sigma^*+\mu^*)}\right] \\ & \cdot \|Y\|_{\mathscr{L}^1} = \Pi^*\|Y\|_{\mathscr{L}^1}.\end{aligned} \tag{53}$$

Put $l^* = \chi^*$. We have $\widehat{\mathbb{O}}l^* < 1/2$. Utilizing Theorem 4, we prove that one of the items (i) or (ii) is possible. First, we check that the item (i) is not the case. From Theorem 4 and the assumption $(\mathfrak{C}_4)$, consider an arbitrary member ϖ_0^* of Σ^* with $\|\varpi_0^*\| = \widehat{\varepsilon}$. Then, $\alpha_0\varpi_0^*(z) \in (\mathbb{A}_1^*\varpi_0^*)(\mathbb{A}_2^*\varpi_0^*)(z)$ for all $\alpha_0 > 1$.

Choosing a function $\widehat{\kappa} \in (\mathbb{SEL})_{\tilde{\mathfrak{D}},\varpi_0^*}$, for each $\alpha_0 > 1$, we have

$$\begin{aligned}\varpi_0^*(z) = \frac{1}{\alpha_0}\mathfrak{N}_*(z,\varpi_0^*(z))\Bigg[\frac{s_1^*}{\tilde{m}_1-\tilde{m}_2} + \frac{1}{\Gamma(\sigma^*)}\int_a^z \varphi'(q)(\varphi(z)-\varphi(q))^{\sigma^*-1}\widehat{\kappa} \\ \cdot (q)dq + \frac{s_2^*}{\tilde{m}^*}(\varphi(z)-\varphi(a)) - \frac{\tilde{m}_1(\varphi(z)-\varphi(a))}{\tilde{m}^*\Gamma(\sigma^*+\mu^*-\tilde{\mu})} \\ \cdot \int_a^T \varphi'(q)(\varphi(T)-\varphi(q))^{\sigma^*+\mu^*-\tilde{\mu}-1}\widehat{\kappa}(q)dq + \frac{\tilde{m}_2(\varphi(z)-\varphi(a))}{\tilde{m}^*\Gamma(\sigma^*+\mu^*-1)} \\ \cdot \int_a^T \varphi'(q)(\varphi(T)-\varphi(q))^{\sigma^*+\mu^*-2}\widehat{\kappa}(q)dq\Bigg],\end{aligned} \tag{54}$$

for almost all $z \in [a,T]$. Thus, one can write

$$\begin{aligned}|\varpi_0^*(z)| = \frac{1}{\alpha_0}|\mathfrak{N}_*(z,\varpi_0^*(z))|\Bigg[\frac{s_1^*}{|\tilde{m}_1-\tilde{m}_2|} + \frac{1}{\Gamma(\sigma^*)} \\ \cdot \int_a^z \varphi'(q)(\varphi(z)-\varphi(q))^{\sigma^*-1}|\widehat{\kappa}(q)|dq \\ + \frac{s_2^*}{|\tilde{m}^*|}(\varphi(z)-\varphi(a)) + \frac{|\tilde{m}_1|(\varphi(z)-\varphi(a))}{|\tilde{m}^*|\Gamma(\sigma^*+\mu^*-\tilde{\mu})} \\ \cdot \int_a^T \varphi'(q)(\varphi(T)-\varphi(q))^{\sigma^*+\mu^*-\tilde{\mu}-1}|\widehat{\kappa}(q)|dq \\ + \frac{|\tilde{m}_2|(\varphi(z)-\varphi(a))}{|\tilde{m}^*|\Gamma(\sigma^*+\mu^*-1)}\int_a^T \varphi'(q)(\varphi(T)-\varphi(q))^{\sigma^*+\mu^*-2}|\widehat{\kappa}(q)|dq\Bigg] \\ \le \frac{1}{\alpha_0}[|\mathfrak{N}_*(z,\varpi_0^*(z))-\mathfrak{N}_*(z,0)| + |\mathfrak{N}_*(z,0)|] \\ \times \Bigg[\frac{s_1^*}{|\tilde{m}_1-\tilde{m}_2|} + \frac{1}{\Gamma(\sigma^*)}\int_a^z \varphi'(q)(\varphi(z)-\varphi(q))^{\sigma^*-1}|\widehat{\kappa}(q)|dq \\ + \frac{s_2^*}{|\tilde{m}^*|}(\varphi(z)-\varphi(a)) + \frac{|\tilde{m}_1|(\varphi(z)-\varphi(a))}{|\tilde{m}^*|\Gamma(\sigma^*+\mu^*-\tilde{\mu})} \\ \int_a^T \varphi'(q)(\varphi(T)-\varphi(q))^{\sigma^*+\mu^*-\tilde{\mu}-1}|\widehat{\kappa}(q)|dq + \frac{|\tilde{m}_2|(\varphi(z)-\varphi(a))}{|\tilde{m}^*|\Gamma(\sigma^*+\mu^*-1)} \\ \int_a^T \varphi'(q)(\varphi(T)-\varphi(q))^{\sigma^*+\mu^*-2}|\widehat{\kappa}(q)|dq\Bigg] \le [\chi^*\widehat{\varepsilon} + \mathcal{F}^*]\Pi^*\|Y\|_{\mathcal{L}^1}.\end{aligned} \tag{55}$$

Hence, we get

$$\widehat{\varepsilon} \le \frac{\|Y\|_{\mathcal{L}^1}\mathcal{F}^*\Pi^*}{1-\Pi^*\chi^*\|Y\|_{\mathcal{L}^1}}, \tag{56}$$

which is a contradiction. Hence, the item (ii) indicated in Theorem 4 is valid. Thus, $\varpi^* \in \mathfrak{W}$ exists so that $\varpi^* \in (\mathbb{A}_1^*\varpi^*)(\mathbb{A}_2^*\varpi^*)$. In consequence, the operator $\mathfrak{K}$ has a fixed point. So the φ-hybrid inclusion BVP (2) and (3) has a solution, and this completes the proof. □

Definition 9. An absolutely continuous function $\varpi^* : [a,T] \longrightarrow \mathbb{R}$ is called a solution for the non-φ-hybrid inclusion BVP (4) in the sense of Caputo if there is $\widehat{\kappa} \in \mathcal{L}^1([a,T],\mathbb{R})$ with $\widehat{\kappa}(z) \in \tilde{\mathfrak{D}}(z,\varpi^*(z))$ for almost all $z \in [a,T]$ which satisfies separated mixed φ-integro-derivative boundary conditions

$$\begin{cases}\tilde{m}_1\varpi^*(a) = s_1^* + \tilde{m}_2\varpi^*(a), \\ \tilde{m}_1{}^{\mathscr{RL}}\mathcal{J}_a^{\mu^*;\varphi\mathscr{C}}\mathfrak{D}_a^{\tilde{\mu};\varphi}\varpi^*(T) = s_2^* + \tilde{m}_2{}^{\mathscr{RL}}\mathcal{J}_a^{\mu^*;\varphi\mathscr{C}}\mathfrak{D}_a^{1;\varphi}\varpi^*(T),\end{cases} \tag{57}$$

$$\begin{aligned}\varpi^*(z) = \frac{s_1^*}{\tilde{m}_1-\tilde{m}_2} + \frac{1}{\Gamma(\sigma^*)}\int_a^z \varphi'(q)(\varphi(z)-\varphi(q))^{\sigma^*-1}\widehat{\kappa}(q)\mathrm{d}q \\ + \frac{s_2^*}{\tilde{m}^*}(\varphi(z)-\varphi(a)) - \frac{\tilde{m}_1(\varphi(z)-\varphi(a))}{\tilde{m}^*\Gamma(\sigma^*+\mu^*-\tilde{\mu})} \\ \cdot \int_a^T \varphi'(q)(\varphi(T)-\varphi(q))^{\sigma^*+\mu^*-\tilde{\mu}-1}\widehat{\kappa}(q)dq \\ + \frac{\tilde{m}_2(\varphi(z)-\varphi(a))}{\tilde{m}^*\Gamma(\sigma^*+\mu^*-1)}\int_a^T \varphi'(q)(\varphi(T)-\varphi(q))^{\sigma^*+\mu^*-2}\widehat{\kappa}(q)dq,\end{aligned} \tag{58}$$

for almost all $z \in [a,T]$.

For each $\varpi^* \in \mathfrak{W}$, the collection of all existing selections of $\mathfrak{D}$ is defined as

$$(\mathbb{SEL})_{\mathfrak{D},\varpi^*} = \left\{\widehat{\kappa} \in \mathcal{L}^1([a,T]) : \widehat{\kappa}(z) \in \mathfrak{D}(z,\varpi^z)\right\}, \tag{59}$$

for almost all $z \in [a,T]$. Define $\mathfrak{F} : \mathfrak{W} \longrightarrow \mathcal{P}(\mathfrak{W})$ as

$$\mathfrak{F}(\varpi^*) = \{\vartheta \in \mathfrak{W} : \vartheta(z) = \rho(z)\}, \tag{60}$$

where

$$\begin{aligned}\varrho(z) = \frac{s_1^*}{\tilde{m}_1-\tilde{m}_2} + \frac{1}{\Gamma(\sigma^*)}\int_a^z \varphi'(q)(\varphi(z)-\varphi(q))^{\sigma^*-1}\widehat{\kappa}(q)\mathrm{d}q \\ + \frac{s_2^*}{\tilde{m}^*}(\varphi(z)-\varphi(a)) - \frac{\tilde{m}_1(\varphi(z)-\varphi(a))}{\tilde{m}^*\Gamma(\sigma^*+\mu^*-\tilde{\mu})} \\ \cdot \int_a^T \varphi'(q)(\varphi(T)-\varphi(q))^{\sigma^*+\mu^*-\tilde{\mu}-1}\widehat{\kappa}(q)dq \\ + \frac{\tilde{m}_2(\varphi(z)-\varphi(a))}{\tilde{m}^*\Gamma(\sigma^*+\mu^*-1)}\int_a^T \varphi'(q)(\varphi(T)-\varphi(q))^{\sigma^*+\mu^*-2}\widehat{\kappa} \\ \cdot (q)dq, \widehat{\kappa} \in (\mathbb{SEL})_{\mathfrak{D},\varpi^*}.\end{aligned} \tag{61}$$

By making use of endpoints for the multifunction $\mathfrak{F}$, we prove the following theorem.

Theorem 10. *Consider $\mathfrak{D} : [a,T] \times \mathfrak{W} \longrightarrow \mathcal{P}_{cp}(\mathfrak{W})$ as a set-valued operator. Let*

($\mathfrak{C}_5$) $\psi : [0,\infty) \longrightarrow [0,\infty)$ be increasing and upper semi-continuous with $\liminf_{z\longrightarrow\infty}(z-\psi(z)) > 0$ *and $z > \psi(z), \forall z > 0$*

($\mathfrak{C}_6$) the multifunction $\mathfrak{D} : [a,T] \times \mathfrak{W} \longrightarrow \mathcal{P}_{cp}(\mathfrak{W})$ be integrable and bounded so that $\mathfrak{D}(\cdot,\varpi^*) : [a,T] \longrightarrow \mathcal{P}_{cp}(\mathfrak{W})$ be measurable for each $\varpi^* \in \mathfrak{W}$

($\mathfrak{C}_7$) a function $\varrho \in C([a,T],[0,\infty))$ exists such that

$$\mathbb{PH}_{d\mathfrak{W}}\left(\mathfrak{D}(z,\varpi_1^*),\mathfrak{D}\left(z,\varpi'^*_1\right)\right) \le \varrho(z)\psi\left(\left|\varpi_1^*(z)-\varpi'^*_1(z)\right|\right)\frac{1}{\Omega^{**}}, \tag{62}$$

for almost all $z \in [a, T]$ and all $\varpi_1^*, {\varpi'}_1^* \in \mathfrak{W}$, where $\sup_{z\in[a,T]} |\varrho(z)| = \|\varrho\|$ and

$$\Omega^{**} = \left[\frac{1}{\Gamma(\sigma^*+1)} - \frac{|\tilde{m}_1|(\varphi(T)-\varphi(a))^{\sigma^*+\mu^*-\tilde{\mu}+1}}{|\tilde{m}^*|\Gamma(\sigma^*+\mu^*-\tilde{\mu}+1)} + \frac{|\tilde{m}_2|(\varphi(T)-\varphi(a))^{\sigma^*+\mu^*}}{|\tilde{m}^*|\Gamma(\sigma^*+\mu^*)}\right]\|\varrho\| \tag{63}$$

($\mathfrak{C}_8$) the operator $\mathfrak{F}$ given by (60) possesses APXEndP-property

Then, a solution exists to the non-φ-hybrid inclusion FBVP (4).

Proof. In such an argument, we try to prove the existence of endpoint to the set-valued operator $\mathfrak{F}: \mathfrak{W} \longrightarrow \mathscr{P}(\mathfrak{W})$ defined by (60). To proceed this, we first investigate that $\mathfrak{F}(\varpi^*)$ is closed for each $\varpi^* \in \mathfrak{W}$. By taking into account the hypothesis ($\mathfrak{C}_6$), $z \mapsto \mathfrak{O}(z, \varpi^*(z))$ is a closed-valued measurable multifunction for each $\varpi^* \in \mathfrak{W}$. In consequence, $\mathfrak{O}$ has a measurable selection $(\mathbb{SEL})_{\mathfrak{O},\varpi^*} \neq \emptyset$. Now, we show that $\mathfrak{F}(\varpi^*) \subseteq \mathfrak{W}$ is closed for all $\varpi^* \in \mathfrak{W}$. Consider the sequence $(\varpi_n^*)_{n\geq 1}$ contained in $\mathfrak{F}(\varpi^*)$ with $\varpi_n^* \longrightarrow \mu$. For each n, there exists $\widehat{\kappa}_n \in (\mathbb{SEL})_{\mathfrak{O},\varpi^*}$ such that

$$\begin{aligned}\varpi_n^*(z) &= \frac{s_1^*}{\tilde{m}_1-\tilde{m}_2} + \frac{1}{\Gamma(\sigma^*)}\int_a^z \varphi'(q)(\varphi(z)-\varphi(q))^{\sigma^*-1}\widehat{\kappa}_n(q)dq \\ &\quad + \frac{s_2^*}{\tilde{m}^*}(\varphi(z)-\varphi(a)) - \frac{\tilde{m}_1(\varphi(z)-\varphi(a))}{\tilde{m}^*\Gamma(\sigma^*+\mu^*-\tilde{\mu})} \\ &\quad \cdot \int_a^T \varphi'(q)(\varphi(T)-\varphi(q))^{\sigma^*+\mu^*-\tilde{\mu}-1}\widehat{\kappa}_n(q)dq \\ &\quad + \frac{\tilde{m}_2(\varphi(z)-\varphi(a))}{\tilde{m}^*\Gamma(\sigma^*+\mu^*-1)}\int_a^T \varphi'(q)(\varphi(T)-\varphi(q))^{\sigma^*+\mu^*-2}\widehat{\kappa}_n(q)dq,\end{aligned} \tag{64}$$

for almost all $z \in [a, T]$. Since $\mathfrak{O}$ is compact multifunction, we acquire a subsequence $(\widehat{\kappa}_n)_{n\geq 1}$ tending to $\widehat{\kappa} \in \mathscr{L}^1([a, T])$. Hence, we have $\widehat{\kappa} \in (\mathbb{SEL})_{\mathfrak{O},\varpi^*}$ and

$$\begin{aligned}\lim_{n\longrightarrow\infty}\varpi_n^*(z) &= \frac{s_1^*}{\tilde{m}_1-\tilde{m}_2} + \frac{1}{\Gamma(\sigma^*)}\int_a^z \varphi'(q)(\varphi(z)-\varphi(q))^{\sigma^*-1}\widehat{\kappa}(q)\mathrm{d}q \\ &\quad + \frac{s_2^*}{\tilde{m}^*}(\varphi(z)-\varphi(a)) - \frac{\tilde{m}_1(\varphi(z)-\varphi(a))}{\tilde{m}^*\Gamma(\sigma^*+\mu^*-\tilde{\mu})} \\ &\quad \cdot \int_a^T \varphi'(q)(\varphi(T)-\varphi(q))^{\sigma^*+\mu^*-\tilde{\mu}-1}\widehat{\kappa}(q)dq \\ &\quad + \frac{\tilde{m}_2(\varphi(z)-\varphi(a))}{\tilde{m}^*\Gamma(\sigma^*+\mu^*-1)}\int_a^T \varphi'(q)(\varphi(T)-\varphi(q))^{\sigma^*+\mu^*-2}\widehat{\kappa} \\ &\quad \cdot (q)dq = \varpi^*(z),\end{aligned} \tag{65}$$

for almost all $z \in [a, T]$. Hence, $\varpi^* \in \mathfrak{F}$ which indicates that $\mathfrak{F}$ is closed-valued. In addition, $\mathfrak{F}(\varpi^*)$ is bounded for each $\varpi^* \in \mathfrak{W}$ since $\mathfrak{O}$ is compact. Finally, we investigate if $\mathbb{P}\mathbb{H}_{d_{\mathfrak{W}}}(\mathfrak{F}(\varpi^*), \mathfrak{F}(\tilde{\varpi}^*)) \leq \psi(\|\varpi^* - \tilde{\varpi}^*\|)$ holds. Let $\varpi^*, \tilde{\varpi}^* \in \mathfrak{W}$, and $x_1 \in \mathfrak{F}(\tilde{\varpi}^*)$. Choose $\widehat{\kappa}_1 \in (\mathbb{SEL})_{\mathfrak{O},\tilde{\varpi}^*}$ such that

$$\begin{aligned}x_1(z) &= \frac{s_1^*}{\tilde{m}_1-\tilde{m}_2} + \frac{1}{\Gamma(\sigma^*)}\int_a^z \varphi'(q)(\varphi(z)-\varphi(q))^{\sigma^*-1}\widehat{\kappa}_1(q)dq \\ &\quad + \frac{s_2^*}{\tilde{m}^*}(\varphi(z)-\varphi(a)) - \frac{\tilde{m}_1(\varphi(z)-\varphi(a))}{\tilde{m}^*\Gamma(\sigma^*+\mu^*-\tilde{\mu})} \\ &\quad \cdot \int_a^T \varphi'(q)(\varphi(T)-\varphi(q))^{\sigma^*+\mu^*-\tilde{\mu}-1}\widehat{\kappa}_1(q)dq \\ &\quad + \frac{\tilde{m}_2(\varphi(z)-\varphi(a))}{\tilde{m}^*\Gamma(\sigma^*+\mu^*-1)}\int_a^T \varphi'(q)(\varphi(T)-\varphi(q))^{\sigma^*+\mu^*-2}\widehat{\kappa}_1(q)dq,\end{aligned} \tag{66}$$

for almost all $z \in [a, T]$. Since

$$\mathbb{P}\mathbb{H}_{d_{\mathfrak{W}}}(\mathfrak{O}(z, \varpi^*), \mathfrak{O}(z, \tilde{\varpi}^*)) \leq \varrho(z)(\psi(\varpi^*(z) - \varpi^{\sim *}(z)))\frac{1}{\Omega^{**}}, \tag{67}$$

for almost all $z \in [a, T]$, so there exists $k^* \in \mathfrak{O}(z, \varpi^*)$ such that

$$|\widehat{\kappa}_1(z) - k^*| \leq (\psi(\varpi^*(z) - \varpi^{\sim *}(z)))\frac{\varrho(z)}{\Omega^{**}}, \tag{68}$$

for almost all $z \in [a, T]$. Define the multifunction $\mathfrak{A} : [a, T] \longrightarrow \mathscr{P}(\mathfrak{W})$ given by

$$\mathfrak{A}(z) = \left\{k^* \in \mathfrak{W} : |\widehat{\kappa}_1(z) - k^*| \leq \varrho(z)(\psi(\varpi^*(z) - \varpi^{\sim *}(z)))\frac{1}{\Omega^{**}}\right\}. \tag{69}$$

Since $\widehat{\kappa}$ and $\sigma = \varrho(\psi(\varpi^* - \tilde{\varpi}^*))1/\Omega^{**}$ are measurable, thus we choose $\widehat{\kappa}_2(z) \in \mathfrak{O}(z, \varpi^*(z))$ such that

$$|\widehat{\kappa}_1(z) - \widehat{\kappa}_2(z)| \leq (\psi(\varpi^*(z) - \varpi^{\sim *}(z)))\frac{\varrho(z)}{\Omega^{**}}, \tag{70}$$

for almost all $z \in [a, T]$. Select $x_2 \in \mathfrak{F}(\varpi^*)$ such that

$$\begin{aligned}x_2(z) &= \frac{s_1^*}{\tilde{m}_1-\tilde{m}_2} + \frac{1}{\Gamma(\sigma^*)}\int_a^z \varphi'(q)(\varphi(z)-\varphi(q))^{\sigma^*-1}\widehat{\kappa}_2(q)\mathrm{d}q \\ &\quad + \frac{s_2^*}{\tilde{m}^*}(\varphi(z)-\varphi(a)) - \frac{\tilde{m}_1(\varphi(z)-\varphi(a))}{\tilde{m}^*\Gamma(\sigma^*+\mu^*-\tilde{\mu})} \\ &\quad \cdot \int_a^T \varphi'(q)(\varphi(T)-\varphi(q))^{\sigma^*+\mu^*-\tilde{\mu}-1}\widehat{\kappa}_2(q)\, dq \\ &\quad + \frac{\tilde{m}_2(\varphi(z)-\varphi(a))}{\tilde{m}^*\Gamma(\sigma^*+\mu^*-1)}\int_a^T \varphi'(q)(\varphi(T)-\varphi(q))^{\sigma^*+\mu^*-2}\widehat{\kappa}_2(q)dq,\end{aligned} \tag{71}$$

for almost all $z \in [a, T]$. Hence, we get

$$
\begin{aligned}
|x_1(z) - x_2(z)| \le & \frac{1}{\Gamma(\sigma^*)}\int_a^z \varphi'(q)(\varphi(z)-\varphi(q))^{\sigma^*-1}|\widehat{\kappa}_1(q)-\widehat{\kappa}_2(q)|dq \\
& + \frac{|\tilde{m}_1|(\varphi(z)-\varphi(a))}{|\tilde{m}^*|\Gamma(\sigma^*+\mu^*-\tilde{\mu})}\int_a^T \varphi'(q)(\varphi(T)-\varphi(q))^{\sigma^*+\mu^*-\tilde{\mu}-1} \\
& \cdot |\widehat{\kappa}_1(q)-\widehat{\kappa}_2(q)|dq + \frac{|\tilde{m}_2|(\varphi(z)-\varphi(a))}{|\tilde{m}^*|\Gamma(\sigma^*+\mu^*-1)} \\
& \cdot \int_a^T \varphi'(q)(\varphi(T)-\varphi(q))^{\sigma^*+\mu^*-2}|\widehat{\kappa}_1(q)-\widehat{\kappa}_2(q)|dq \\
\le & \frac{1}{\Gamma(\sigma^*+1)}\|\varrho\|\psi(\|\varpi^*-\tilde{\varpi}^*\|)\frac{1}{\Omega^{**}} - \frac{|\tilde{m}_1|(\varphi(T)-\varphi(a))^{\sigma^*+\mu^*-\tilde{\mu}+1}}{|\tilde{m}^*|\Gamma(\sigma^*+\mu^*-\tilde{\mu}+1)} \\
& \cdot \|\varrho\|\psi(\|\varpi^*-\tilde{\varpi}^*\|)\frac{1}{\Omega^{**}} + \frac{|\tilde{m}_2|(\varphi(T)-\varphi(a))^{\sigma^*+\mu^*}}{|\tilde{m}^*|\Gamma(\sigma^*+\mu^*)}\|\varrho\|\psi \\
& \cdot (\|\varpi^*-\tilde{\varpi}^*\|)\frac{1}{\Omega^{**}} = \left[\frac{1}{\Gamma(\sigma^*+1)} - \frac{|\tilde{m}_1|(\varphi(T)-\varphi(a))^{\sigma^*+\mu^*-\tilde{\mu}+1}}{|\tilde{m}^*|\Gamma(\sigma^*+\mu^*-\tilde{\mu}+1)}\right. \\
& \left. + \frac{|\tilde{m}_2|(\varphi(T)-\varphi(a))^{\sigma^*+\mu^*}}{|\tilde{m}^*|\Gamma(\sigma^*+\mu^*)}\right]\|\varrho\|\psi(\|\varpi^*-\tilde{\varpi}^*\|)\frac{1}{\Omega^{**}} \\
= & \Omega^{**}\psi(\|\varpi^*-\tilde{\varpi}^*\|)\frac{1}{\Omega^{**}} = \psi(\|\varpi^*-\tilde{\varpi}^*\|).
\end{aligned} \tag{72}
$$

This gives $\|x_1 - x_2\| \le \psi(\|\varpi^* - \tilde{\varpi}^*\|)$ and shows that $\mathbb{P}\mathbb{H}_{d\mathfrak{W}}(\mathfrak{F}(\varpi^*), \mathfrak{F}(\tilde{\varpi}^*)) \le \psi(\|\varpi^* - \tilde{\varpi}^*\|)$ for all $\varpi^*, \tilde{\varpi}^* \in \mathfrak{W}$. Also from $(\mathfrak{C}_8)$, we realize that $\mathfrak{F}$ has approximate endpoint property. Application of Theorem 5 gives that $\mathfrak{K}$ has a unique endpoint, i.e., there exists $\varpi^{**} \in \mathfrak{W}$ such that $\mathfrak{F}(\varpi^{**}) = \{\varpi^{**}\}$. In conclusion, ϖ^{**} is a solution to the non-φ-hybrid inclusion BVP (4). □

4. Some Examples

This section involves two different numerical simulation examples corresponding to the relevant φ-hybrid and non-φ-hybrid fractional inclusion boundary problems to guarantee the applicability of proved theorems.

Example 1. With due attention to (2) and (3), we design the Caputo φ-hybrid differential inclusion BVP as

$$
{}^{\mathscr{C}}\mathfrak{D}_0^{1.62;z+2}\left(\frac{\varpi^*(z)}{z\sin\varpi^*(z)/42+1/4}\right) \in \cdot\left[\frac{\sin\varpi^*(z)}{z(2+|\sin z|)}, \frac{|\cos\varpi^*(z)|}{4(1+|\cos\varpi^*(z)|)}+\frac{3}{5}\right], \tag{73}
$$

supplemented with separated mixed φ-integro-derivative boundary conditions

$$
\begin{cases}
0.8\left(\dfrac{\varpi^*(z)}{z\sin\varpi^*(z)/42+1/4}\right)\Bigg|_{z=0} = 1.2 + (0.4)\left(\dfrac{\varpi^*(z)}{z\sin\varpi^*(z)/42+1/4}\right)\Bigg|_{z=0}, \\
0.8\,{}^{\mathscr{RL}}\mathscr{I}_0^{1.4;z+2}\,{}^{\mathscr{C}}\mathfrak{D}_0^{0.4;z+2}\left(\dfrac{\varpi^*(z)}{z\sin\varpi^*(z)/42+1/4}\right)\Bigg|_{z=1} = 0.9 + 0.4\,{}^{\mathscr{RL}}\mathscr{I}_0^{1.4;z+2}\,{}^{\mathscr{C}}\mathfrak{D}_0^{1;z+2}\left(\dfrac{\varpi^*(z)}{z\sin\varpi^*(z)/42+1/4}\right)\Bigg|_{z=1},
\end{cases} \tag{74}
$$

where $z \in [0, 1]$, $\sigma^* = 1.62$, $\tilde{\mu} = 0.4$, $\mu^* = 1.4$, $\tilde{m}_1 = 0.8$, $\tilde{m}_2 = 0.4$, $s_1^* = 1.2$, $s_2^* = 0.9$, and $\varphi(z) = z + 2$. We define $\mathfrak{N}_* : [0, 1] \times \mathbb{R} \longrightarrow \mathbb{R}\{0\}$ by $\mathfrak{N}_*(z, \varpi^*(z)) = z\sin\varpi^*(z)/42 + 1/4$ which is nonzero and continuous. Notice that $\mathscr{F}^* = \sup_{z\in[0,1]}|\mathfrak{N}_*(z, 0)| = 1/4$. Moreover, the function $\mathfrak{N}_*$ is Lipschitz, that is, for each $\varphi_1^*, \varphi_2^* \in \mathbb{R}$, we have

$$
\begin{aligned}
|\mathfrak{N}_*(z, \varpi_1^*(z)) - \mathfrak{N}_*(z, \varpi_2^*(z))| &= \left|z\sin\frac{\varpi_1^*(z)}{42} - z\sin\frac{\varpi_2^*(z)}{42}\right| \\
&\le \frac{z}{42}|\varpi_1^*(z) - \varpi_2^*(z)|.
\end{aligned} \tag{75}
$$

If $\chi(z) = z/42$, then $\chi^* = \sup_{z\in[0,1]}|\chi(z)| = 1/42$. Now, define $\tilde{\mathfrak{S}} : [0, 1] \times \mathbb{R} \longrightarrow P(\mathbb{R})$ by

$$
\tilde{\mathfrak{S}}(z, \varpi^*(z)) = \left[\frac{\sin\varpi^*(z)}{z|\sin z| + 2z}, \frac{|\cos\varpi^*(z)|}{4(|\cos\varpi^*(z)|+1)} + \frac{3}{5}\right]. \tag{76}
$$

Since

$$
|\zeta| \le \max\left[\frac{\sin\varpi^*(z)}{z|\sin z| + 2z}, \frac{|\cos\varpi^*(z)|}{4(|\cos\varpi^*(z)|+1)} + \frac{3}{5}\right] \le 1, \tag{77}
$$

therefore, we have

$$
\left\|\tilde{\mathfrak{S}}(z, \varpi^*(z))\right\| = \sup\left\{|\widehat{\kappa}| : \widehat{\kappa} \in \tilde{\mathfrak{S}}(z, \varpi^*(z))\right\} \le Y(z) = 1. \tag{78}
$$

Then, $\|Y\|_{\mathscr{L}_1} = \int_0^1 |Y(q)|dq = \int_0^1 1 \cdot dq = 1$. By using above values, we have $\Pi^* = 20.1356266$. Also, we can find $\widehat{\varepsilon}$ with $\widehat{\varepsilon} > 9.67254$. Finally, we have $\chi^*\Pi^*\|Y\|_{\mathscr{L}^1} = 0.47949 < 1/2$. Thus, all assertions of Theorem 8 are verified. Hence, the φ-hybrid Caputo differential inclusion BVP (73) supplemented with separated mixed φ-integro-derivative boundary conditions (74) has a solution.

Example 2. With due attention to (4), we design the Caputo non-φ-hybrid differential inclusion BVP as

$$ {}^{\mathscr{C}}\mathfrak{D}_0^{1.4;2z+3/2}\varpi^*(z) \in \left[0, \frac{2|\varpi^*(z)|}{3(1/2+z)(2+|\varpi^*(z)|)}\right], \quad (79) $$

supplemented with separated mixed φ-integro-derivative boundary

$$ \begin{cases} (0.73)\varpi^*(0) = 0.7 + 0.3\varpi^*(0), \\ 0.73^{\mathscr{RL}}\mathscr{I}_0^{1.6;2z+3/2\,\mathscr{C}}\mathfrak{D}_0^{0.73;2z+3/2}\varpi^*(1) = 0.6 + 0.3^{\mathscr{RL}}\mathscr{I}_0^{1.6;2z+3/2\,\mathscr{C}}\mathfrak{D}_0^{1;2z+3/2}\varpi^*(1). \end{cases} \quad (80) $$

where ${}^{\mathscr{C}}\mathfrak{D}_0^{1.4;2z+3/2}$ denotes the φ-CF derivative of order $\sigma^* = 1.4$, $z \in [0, 1]$, $\tilde{\mu} = 0.6$, $\mu^* = 1.6$, $\tilde{m}_1 = 0.73$, $\tilde{m}_2 = 0.3$, $s_1^* = 0.7$, $s_2^* = 0.6$, and $\varphi(z) = 2z + 3/2$. Using these values, we have $\tilde{m}^* = 0.15516$. We consider the Banach space $\mathfrak{W} = \{\varpi^*(z): \varpi^*(z) \in \mathscr{C}([0, 1], \mathbb{R})\}$ equipped with $\|\varpi^*\|_{\mathfrak{W}} = \sup_{z\in[0,1]}|\varpi^*(z)|$. Now, we define a multivalued map $\tilde{\mathfrak{O}} : [0, 1] \times \mathfrak{W} \longrightarrow \mathscr{P}(\mathfrak{W})$ by

$$ \tilde{\mathfrak{O}}(z, \varpi^*(z)) = \left[0, \frac{2|\varpi^*(z)|}{36(1/2+z)(2+|\varpi^*(z)|)}\right], \quad (81) $$

for almost all $z \in [0, 1]$. We define $\psi : ;[0,\infty) \longrightarrow [0,\infty)$ by $\psi(z) = z/3, \forall z > 0$. Obviously, $\lim\inf_{z\longrightarrow\infty}(z - \psi(z)) > 0$ and $\psi(z) < z$ for all $z > 0$. Now, for each $\varpi_1^*, \varpi_2^* \in \mathfrak{W}$, we have

$$ \begin{aligned} &\mathbb{PH}_{d_{\mathfrak{W}}}\left(\tilde{\mathfrak{O}}(z, \varpi_1^*(z)), \tilde{\mathfrak{O}}(z, \varpi_2^*(z))\right) \\ &\le \frac{2}{36(1/2+z)}(|\varpi_1^*(z) - \varpi_2^*(z)|) \le \psi(|\varpi_1^*(z) - \varpi_2^*(z)|)\frac{\rho(z)}{\Omega^{**}}, \end{aligned} \quad (82) $$

where $\Omega^{**} = 0.21377$ and $\varrho \in \mathscr{C}([0, 1], [0,\infty))$ is defined as $\varrho(z) = 2/12(1/2+z)$ for all z. Then, $\|\varrho\| = \sup_{z\in[0,1]} = 1/3$. Lastly, we introduce $\mathfrak{F} : \mathfrak{W} \longrightarrow \mathscr{P}(\mathfrak{W})$ by

$$ \mathfrak{F}(\varpi^*) = \left\{\vartheta \in \mathfrak{W} : \text{there exists } \tilde{\kappa} \in (\mathbb{SEL})_{\tilde{\mathfrak{O}},\varpi^*} \text{ s.t. } \vartheta(z) = \rho(z), \forall z \in [0, 1]\right\}, \quad (83) $$

in which

$$ \begin{aligned} \rho(z) &= \frac{0.7}{0.43} + \frac{1}{\Gamma(1.4)}\int_0^z \varphi'(q)(\varphi(z) - \varphi(q))^{0.4}\hat{\kappa}(q)dq \\ &+ \frac{0.6}{0.15516}(\varphi(z) - \varphi(0)) - \frac{0.73(\varphi(z) - \varphi(0))}{0.15516\Gamma(2.4)} \\ &\cdot \int_0^T \varphi'(q)(\varphi(1) - \varphi(q))^{1.4}\hat{\kappa}(q)dq + \frac{0.3(\varphi(z) - \varphi(0))}{0.15516\Gamma(2)} \\ &\cdot \int_0^T \varphi'(q)(\varphi(1) - \varphi(q))\hat{\kappa}(q)dq. \end{aligned} \quad (84) $$

Thus, all assertions of Theorem 10 are verified. Hence, the non-φ-hybrid Caputo differential inclusion BVP (79) with separated mixed φ-integro-derivative boundary (80) has a solution.

5. Conclusion

In the current research study, we derived some theoretical criteria to prove the existence results to a new φ-hybrid fractional differential inclusion in the Caputo settings depending on the increasing function φ with separated mixed φ-hybrid-integro-derivative boundary conditions. The applied method to achieve desired purposes is based on Dhage's fixed point result. In addition, we discussed a special case of the proposed φ-inclusion problem in the non-φ-hybrid structure with the help of the endpoint notion. To confirm the applicability of our theoretical findings, two specific numerical examples are provided which simulate both φ-hybrid and non-φ-hybrid cases. Hence, this research work can motivate other researchers in this field to concentrate on various investigations of different φ-hybrid structures formulated by other fractional operators.

Data Availability

Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

Conflicts of Interest

All authors declare that they have no competing interests.

Authors' Contributions

All authors declare that the present study was realized in collaboration with equal responsibility. All authors read and approved the final version of the current manuscript.

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