Research Article

Extension of Nunokawa Lemma for Functions with Fixed Second Coefficient and Its Applications

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In this paper, we study some properties of analytic functions with fixed initial coefficients. The methodology of differential subordination is used for modification and improvements of several well-known results for subclasses of univalent functions by restricting the functions with fixed initial coefficients. Actually, by extending the Nunokawa lemma for fixed initial coefficient functions, we obtain some novel results on subclasses of univalent functions, such as differential inequalities for univalence or starlikeness of analytic functions. Also, we provide some new sufficient conditions for strongly starlike functions. The results of this paper extend and improve the previously known results by considering functions with fixed second coefficients.

1. Introduction and Preliminaries

Let $\mathcal{H}$ be the class of analytic functions in the unit disc $\mathbb{U} = \{z : |z| < 1\}$. For $a \in \mathbb{C}$ and $n \in \mathbb{N}$, let us define two well-known classes of analytic functions as follows:

$$\begin{aligned}\mathcal{H}[a, n] &= \{f \in \mathcal{H} : f(z) = a + a_n z^n + \dots, z \in \mathbb{U}\}, \\ \mathcal{A}_n &= \{f \in \mathcal{H} : f(z) = z + a_{n+1} z^{n+1}, z \in \mathbb{U}\}.\end{aligned}\quad (1)$$

We denote by $\mathcal{A} = \mathcal{A}_1$ and $\mathcal{S} \subset \mathcal{A}$, as the class of univalent functions. Also, we denote by $\mathcal{S}^*(\alpha)$, the set of starlike function of order α ($\alpha < 1$), as follows:

$$\mathcal{S}^*(\alpha) = \left\{f \in \mathcal{A} : \Re \frac{zf'(z)}{f(z)} > \alpha, z \in \mathbb{U}\right\}, \quad (2)$$

which is introduced by Robertson in [1]. Let $\mathcal{S}^*(0) = \mathcal{S}^*$ be the classes of starlike functions. Brannan and Kirwan in [2] introduced $\mathcal{SS}^*(\lambda)$ the class of strongly starlike function of order λ , $0 < \lambda < 1$, by

$$\mathcal{SS}^*(\lambda) = \left\{f \in \mathcal{A} : \left|\arg \frac{zf'(z)}{f(z)}\right| < \lambda, z \in \mathbb{U}\right\}. \quad (3)$$

Also, Takahashi and Nunokawa in [3] defined the following subclasses of $\mathcal{A}$:

$$\mathcal{SS}^*(\alpha, \lambda) = \left\{f \in \mathcal{A} : -\alpha \frac{\pi}{2} < \arg \frac{zf'(z)}{f(z)} < \lambda \frac{\pi}{2}, z \in \mathbb{U}\right\}. \quad (4)$$

It is easy to see that $\mathcal{SS}^*(\min\{\alpha, \lambda\}) \subset \mathcal{SS}^*(\alpha, \lambda) \subset \mathcal{SS}^*(\max\{\alpha, \lambda\})$. Also, if $\alpha = \lambda$, then $\mathcal{SS}^*(\alpha, \lambda) \subset \mathcal{S}^* \subset \mathcal{S}$. Then, they are the subclasses of univalent functions in the unit disc $\mathbb{U}$.

Recently, Ali et al. [4] have extended the theory of second-order differential subordination for functions with a fixed initial coefficient. We denote by $\mathcal{H}_\beta[a, n]$ the class of analytic functions with a fixed initial coefficient as follows:

$$\mathcal{H}_\beta[a, n] = \{p \in \mathcal{H} : p(z) = a + \beta z^n + a_{n+1} z^{n+1} + \dots\}. \quad (5)$$

Further, let

$$\mathcal{A}_{n,b} = \{f \in \mathcal{H} : f(z) = z + bz^{n+1} + \dots, z \in \mathbb{U}\}, \quad (6)$$

and also, let

$$\mathcal{S}_{n,\beta}^*(\alpha, \lambda) = \{f \in \mathcal{S} : f(z) = z + \beta z^{n+1} + \dots, z \in \mathbb{U}\}, \quad (7)$$

where $b \in \mathbb{C}$ is fixed. We denote by $\mathcal{A}_b = \mathcal{A}_{1,b}$, and we assume that β and b are positive real numbers.

The importance of the second coefficient of analytic functions was shown in the monograph [5], for example, coefficient estimates in two well-known growth and distortion theorems for functions in the class $\mathcal{S}$. On the other hand, one of the significant tools in geometric function theory is the theory of differential subordination due to Miller and Mocanu [6]. Recently, Ali et al. [4] improved the theory of differential subordination by this assumption that the second coefficient of analytic function is fixed. For some applications of this improvement, see [7, 8]. In addition, Nunokawa has proved a theorem known as Nunokawa lemmas [9] which have had many applications in the geometric function theory. So, it is natural to ask if it is possible to extend this lemma for analytic functions with a fixed second coefficient.

In this paper, we extend this lemma, and then, it will be applied to obtain several new results by restricting the functions to have a fixed second coefficient. It is a remarkable fact when we restrict ourselves to the functions with a fixed second coefficient; then, it will expand the validity domain of the results. As a byproduct of this idea, one can also get results for other class mapping. Even though, in this paper, we are not going into details of all such applications, we remark that other applications will appear in other papers of the authors.

In [10], Silverman examined the class $\mathcal{G}_b$, consisting of normalized functions $f \in \mathcal{S}$ that satisfy the condition

$$\left| \frac{1 + zf''(z)/f'(z)}{zf'(z)/f(z)} - 1 \right| < b(z \in \mathbb{U}), \quad (8)$$

for some positive b . He has proved that $\mathcal{G}_b \subseteq \mathcal{S}^*(2/(1 + \sqrt{1 + 8b}))$.

For a fixed coefficient β , let

$$\mathcal{G}_b^{n,\beta} = \{f \in \mathcal{G}_b : f(z) = z + \beta z^{n+1} + \dots, z \in \mathbb{U}\}. \quad (9)$$

For $f(z)$ and $g(z)$ in $\mathcal{H}$, we say that $f(z)$ is subordinate to $g(z)$ if there exists a function $w(z) \in \mathcal{H}$ with $w(0) = 0$, $|w(z)| < 1$ ($z \in \mathbb{U}$), such that $f(z) = g(w(z))$ ($z \in \mathbb{U}$) denote by $f(z) \prec g(z)$. When $g(z)$ is univalent in $\mathbb{U}$, then we can see

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0), f(\mathbb{U}) \subseteq g(\mathbb{U}). \quad (10)$$

This article is arranged as follows. In Section 2, we extend the lemma due to Nunokawa [9] for fixed second coefficient functions and then will bring its nice applications for finding sufficient conditions for strongly starlike functions. In Section 3, by considering the special cases of this extension, we will provide some sufficient conditions for starlikeness and

univalence of analytic functions. Also, we discuss about preserving starlikeness of the general integral operator $I_{\delta,\gamma}$. Finally, in Theorem 26, we state a new result which is improvement of some well-known results in the literature.

We just need a definition and a fundamental lemma due to Ali et al. [4] to get the results.

Definition 1 ([6], Definition 1, p. 158). Let Q be the class of functions q that are analytic and injective in $\mathbb{U} \setminus E(q)$, where

$$E(q) := \left\{ \zeta \in \partial\mathbb{U} : \lim_{z \rightarrow \zeta} q(z) = \infty \right\}, \quad (11)$$

and are such that $q'(\zeta) \neq 0$ for $\zeta \in \partial\mathbb{U} \setminus E(q)$.

Lemma 2 (see [4]). Let $q \in Q$ with $q(0) = a$, and $p \in \mathcal{H}_c[a, n]$ with $p(z) \equiv a$. If there exist a point $z_0 \in \mathbb{U}$ such that $p(z_0) \in q(\partial\mathbb{U})$ and $p(\{z : |z| < |z_0|\}) \subset q(\mathbb{U})$, then

$$\begin{aligned} z_0 p'(z_0) &= m \zeta_0 q'(\zeta_0), \\ \Re \left(1 + \frac{z_0 p''(z_0)}{p'(z_0)} \right) &\geq m \Re \left(1 + \frac{\zeta_0 q''(\zeta_0)}{q'(\zeta_0)} \right), \end{aligned} \quad (12)$$

where $q^{-1}(p(z_0)) = \zeta_0 = e^{i\theta_0}$ (θ is real) and

$$m \geq n + \frac{|q'(0)| - |c||z_0|^n}{|q'(0)| + |c||z_0|^n}. \quad (13)$$

2. Main Results

The Nunokawa lemma is a very useful tool in the theory of differential subordination that was proved for the first time by Nunokawa [9]. At first, we mention extension of this lemma with its proof because we need it in the next theorems.

Theorem 3. Let $p \in \mathcal{H}_\beta[1, n]$ and $p(z) \neq 0$ in $\mathbb{U}$. If there exist $z_1 \in \mathbb{U}$ and $z_2 \in \mathbb{U}$ such that $|z_1| = |z_2| = r$ and for $z \in \{z : |z| < r\}$

$$-\frac{\pi\alpha}{2} = \arg p(z_1) < \arg p(z) < \arg p(z_2) = \frac{\pi\lambda}{2}, \quad (14)$$

with $0 < \alpha \leq 1$, $0 < \lambda \leq 1$, and β is a real number with $0 \leq \beta \leq (\lambda + \alpha) \sin \pi\alpha/(\alpha + \lambda)$. Then, we have

$$\begin{aligned} \frac{z_1 p'(z_1)}{p(z_1)} &= -i \frac{\lambda + \alpha}{2} m_1, \\ \frac{z_2 p'(z_2)}{p(z_2)} &= i \frac{\lambda + \alpha}{2} m_2, \end{aligned} \quad (15)$$

where

$$\begin{aligned} m_1 &\geq \left(n + \frac{\sin \pi \alpha / (\lambda + \alpha) - \beta / (\lambda + \alpha)}{\sin \pi \alpha / (\lambda + \alpha) + \beta / (\lambda + \alpha)} \right) \tan \frac{\pi \lambda}{2(\lambda + \alpha)}, \\ m_2 &\geq \left(n + \frac{\sin \pi \alpha / (\lambda + \alpha) - \beta / (\lambda + \alpha)}{\sin \pi \alpha / (\lambda + \alpha) + \beta / (\lambda + \alpha)} \right) \tan \frac{\pi \alpha}{2(\lambda + \alpha)}. \end{aligned} \quad (16)$$

Proof. Let us define

$$q(z) = \left(\frac{1 - e^{-2\pi i(\alpha/(\lambda+\alpha))} z}{1 - z} \right)^{(\lambda+\alpha)/2}, \quad (17)$$

with $0 < \alpha \leq 1$, $0 < \lambda \leq 1$. It is easy to check that q is analytic in $\mathbb{U}$ with $q(0) = 1$ and

$$-\frac{\pi \alpha}{2} < \arg q(\mathbb{U}) < \frac{\pi \lambda}{2}. \quad (18)$$

Also, $q \in Q$ with $E(q) = 1$. According to properties of q and assumption (14), we have $p(z_1) \in q(\partial \mathbb{U})$, $p(z_2) \in q(\partial \mathbb{U})$, and $p(\{z : |z| < |z_0|\}) \subset q(\mathbb{U})$. Set

$$\begin{aligned} p_1(z) &= \exp \left\{ -i\pi \frac{\lambda - \alpha}{2(\lambda + \alpha)} \right\} \{p(z)\}^{2/(\lambda + \alpha)}, \\ q_1(z) &= \exp \left\{ -i\pi \frac{\lambda - \alpha}{2(\lambda + \alpha)} \right\} \frac{1 - e^{-2\pi i(\alpha/(\lambda + \alpha))} z}{1 - z}. \end{aligned} \quad (19)$$

Then, it can be verified that $q_1(\mathbb{U})$ is the right half plane $\Re\{w\} > 0$, $p_1(z_1) = -ix_1$ with $x_1 > 0$, $p_2(z_2) = ix_2$ with $x_2 > 0$ and $p_1(\{z : |z| < |z_0|\}) \subset q_1(\mathbb{U})$. We notice that

$$\exp \left(-i\pi \frac{\lambda - \alpha}{2(\lambda + \alpha)} \right) = -i \exp \left(i\pi \frac{\alpha}{\lambda + \alpha} \right) = i \exp \left(i\pi \frac{\lambda}{\lambda + \alpha} \right), \quad (20)$$

and so, by considering $c = \exp p(i\pi\alpha/(\lambda + \alpha))$, we can rewrite the functions p_1 and q_1 as

$$\begin{aligned} p_1(z) &= -ic \{p(z)\}^{2/(\lambda + \alpha)}, \\ q_1(z) &= -i \frac{c - \bar{c}z}{1 - z}. \end{aligned} \quad (21)$$

Now, taking the derivative of q_1 and calculating the inverse of q_1 yields

$$q_1'(z) = \frac{2\Im mc}{(1 - z)^2}, \quad q_1^{-1}(z) = \frac{iz - c}{iz - \bar{c}}. \quad (22)$$

Since $p_1 \in \mathcal{H}_{c_1}[a, n]$ with

$$c_1 = \frac{2 \exp(-i\pi(\alpha - \lambda)/2(\alpha + \lambda))}{\alpha + \lambda} \beta, \quad a = \exp \left(-i \frac{\pi(\alpha - \lambda)}{2(\alpha + \lambda)} \right), \quad (23)$$

and so, from Lemma 2, we deduce that there exist two complex numbers ζ_1 and ζ_2 on $\partial \mathbb{U}$ with $p_1(z_1) = q_1(\zeta_1)$ and $p_1(z_2) = q_1(\zeta_2)$ such that

$$z_1 p_1'(z_1) = k_1 \zeta_1 q_1'(\zeta_1), \quad z_2 p_1'(z_2) = k_2 \zeta_2 q_1'(\zeta_2), \quad (24)$$

where

$$k_1 \geq n + \frac{|q_1'(0)| - |c_1||z_1|^n}{|q_1'(0)| + |c_1||z_1|^n}, \quad k_2 \geq n + \frac{|q_1'(0)| - |c_1||z_2|^n}{|q_1'(0)| + |c_1||z_2|^n}. \quad (25)$$

But

$$\zeta_1 = q_1^{-1}(p_1(z_1)) = \frac{x_1 - c}{x_1 - \bar{c}}; \quad (26)$$

hence,

$$\begin{aligned} \frac{z_1 p_1'(z_1)}{p_1(z_1)} &= \frac{\lambda + \alpha}{2} \frac{z_1 p_1'(z_1)}{p_1(z_1)} = k_1 \frac{\lambda + \alpha}{2} \zeta_1 \frac{q_1'(\zeta_1)}{p_1(z_1)} \\ &= k_1 \frac{\lambda + \alpha}{2} \frac{x_1 - c}{x_1 - \bar{c}} \times \frac{1}{-ix_1} \times \frac{2\Im mc}{(1 - (x_1 - c)/(x_1 - \bar{c}))^2} \\ &= -ik_1 \left(\frac{\alpha + \lambda}{2} \right) \frac{x_1^2 + 2x_1 \cos \pi \lambda / (\alpha + \lambda) + 1}{2x_1 \sin \pi \lambda / (\alpha + \lambda)}. \end{aligned} \quad (27)$$

But the function

$$f(x) = \frac{x_1^2 + 2x_1 \cos \pi \lambda / (\alpha + \lambda) + 1}{2x_1 \sin \pi \lambda / (\alpha + \lambda)} \quad (28)$$

takes its minimum at the point $x = 1$, and we have $f(1) = \tan \pi \lambda / 2(\alpha + \lambda)$. In view of $q_1'(0) = 2 \sin \alpha \pi / (\alpha + \lambda)$ and (25), we obtain

$$\begin{aligned} m_1 &= k_1 f(x) > k_1 f(1) \\ &= \left(n + \frac{\sin \pi \alpha / (\lambda + \alpha) - \beta / (\lambda + \alpha)}{\sin \pi \alpha / (\lambda + \alpha) + \beta / (\lambda + \alpha)} \right) \tan \frac{\pi \lambda}{2(\lambda + \alpha)}; \end{aligned} \quad (29)$$

thus, we showed that

$$\begin{aligned} \frac{z_1 p'(z_1)}{p(z_1)} &= -i \frac{\lambda + \alpha}{2} m_1, m_1 \\ &\geq \left(n + \frac{\sin \pi \alpha / (\lambda + \alpha) - \beta / (\lambda + \alpha)}{\sin \pi \alpha / (\lambda + \alpha) + \beta / (\lambda + \alpha)} \right) \tan \frac{\pi \lambda}{2(\lambda + \alpha)}. \end{aligned} \quad (30)$$

In a similar way, by noting that $p(z_2) = ix_2$ with $x_2 > 0$ and letting

$$\zeta_2 = q^{-1}(ix_2) = \frac{x_2 + c}{x_2 + \bar{c}}, \quad (31)$$

we can see that

$$\begin{aligned} \frac{z_2 p'(z_2)}{p(z_2)} &= \frac{\lambda + \alpha}{2} \times \frac{z_2 p'_1(z_2)}{p_1(z_2)} = k_2 \frac{\lambda + \alpha}{2} \zeta_2 \frac{q'_1(\zeta_2)}{p_1(z_2)} \\ &= k_2 \frac{\lambda + \alpha}{2} \times \frac{x_2 + c}{x_2 + \bar{c}} \times \frac{1}{ix_2} \times \frac{2\Im m c}{(1 - (x_2 + c)/(x_2 + \bar{c}))^2} \\ &= ik_2 \left(\frac{\alpha + \lambda}{2} \right) \frac{x_2^2 + 2x_2 \cos \pi \alpha / (\alpha + \lambda) + 1}{2x_2 \sin \pi \alpha / (\alpha + \lambda)}. \end{aligned} \quad (32)$$

Now, doing analogous above, we find that

$$m_2 \geq \left(n + \frac{\sin \pi \alpha / (\lambda + \alpha) - \beta / (\lambda + \alpha)}{\sin \pi \alpha / (\lambda + \alpha) + \beta / (\lambda + \alpha)} \right) \tan \frac{\pi \alpha}{2(\lambda + \alpha)}. \quad (33)$$

Thus, we showed that

$$\begin{aligned} \frac{z_2 p'(z_2)}{p(z_2)} &= -i \frac{\lambda + \alpha}{2} m_2, m_2 \\ &\geq \left(n + \frac{\sin \pi \alpha / (\lambda + \alpha) - \beta / (\lambda + \alpha)}{\sin \pi \alpha / (\lambda + \alpha) + \beta / (\lambda + \alpha)} \right) \tan \frac{\pi \alpha}{2(\lambda + \alpha)}, \end{aligned} \quad (34)$$

which is (16); therefore, the proof is complete. $\square$

Remark 4. Nunokawa [9] provided a similar result to Theorem 3 with this assumption that $p \in \mathcal{H}[1, n]$. But, if we fix the second initial coefficient, then we may have some reform as seen above.

Theorem 5. Let $0 < \lambda \leq 1$, $0 < \alpha \leq 1$ and $0 \leq \beta \leq (\lambda + \alpha)/n$ $\sin \pi \alpha / (\alpha + \lambda)$. If $f \in \mathcal{G}_b^{n, \beta}$ with

$$b = \delta \min \left\{ m_1 \frac{\tilde{a}_1^{1-\delta} + 2\tilde{a}_1^{-\delta} \cos \pi \lambda / 2\delta + \tilde{a}_1^{-1-\delta}}{2 \sin \pi \lambda / 2\delta}, m_2 \frac{\tilde{a}_2^{1-\delta} + 2\tilde{a}_2^{-\delta} \cos \pi \alpha / 2\delta + \tilde{a}_2^{-1-\delta}}{2 \sin \pi \alpha / 2\delta} \right\}, \quad (35)$$

where $\delta = (\lambda + \alpha)/2$ and

$$\begin{aligned} m_1 &= \left(n + \frac{\sin \pi \alpha / 2\delta - n\beta / 2\delta}{\sin \pi \alpha / 2\delta + n\beta / 2\delta} \right) \tan \frac{\pi \lambda}{4\delta}, \\ m_2 &= \left(n + \frac{\sin \pi \alpha / 2\delta - n\beta / 2\delta}{\sin \pi \alpha / 2\delta + n\beta / 2\delta} \right) \tan \frac{\pi \alpha}{4\delta}, \\ \tilde{a}_1 &= \frac{\sqrt{1 - \delta^2 \sin^2 \pi \alpha / 2\delta} - \delta \cos \pi \alpha / 2\delta}{1 - \delta}, \\ \tilde{a}_2 &= \frac{\sqrt{1 - \delta^2 \sin^2 \pi \lambda / 2\delta} - \delta \cos \pi \lambda / 2\delta}{1 - \delta}, \end{aligned} \quad (36)$$

then $f \in SS_{n, \beta}^*(\alpha, \lambda)$.

Proof. Suppose that $f \in \mathcal{G}_b^{n, \beta}$. Let us define

$$p(z) = \frac{zf'(z)}{f(z)}, p_1(z) = c\{p(z)\}^{2/(\lambda + \alpha)}, \quad (37)$$

with

$$c = \exp \left\{ -i\pi \frac{\lambda - \alpha}{2(\lambda + \alpha)} \right\}. \quad (38)$$

Then, it is clear that $p \in \mathcal{H}_{n\beta}[1, n]$, $p_1 \in \mathcal{H}_{c_1}[a, n]$ with

$$\begin{aligned} c_1 &= \frac{2 \exp(-i\pi(\alpha - \lambda)/2(\alpha + \lambda))}{\alpha + \lambda} n\beta, \\ a &= \exp \left(-i \frac{\pi(\alpha - \lambda)}{2(\alpha + \lambda)} \right). \end{aligned} \quad (39)$$

Also, it is easy to see that

$$\frac{zp'(z)}{p^2(z)} = \frac{1 + zf''(z)/f'(z)}{zf'(z)/f(z)} - 1. \quad (40)$$

If $f \notin SS_{n, \beta}^*(\alpha, \lambda)$, then $p_1(\mathbb{U})$ is not contained in the right half plane $\Re w > 0$; hence, there exists a point $z_1 \in \mathbb{U}$ such that $p_1(\{z : |z| < |z_1|\})$ is contained in the right half plane $\Re w > 0$ while, $p_1(z_1) = -ix_1$, or $p_1(z_1) = ix_2$, where $x_1, x_2 > 0$. In the first step, let $p_1(z_1) = -ix_1$, then by applying the same argument as Theorem 3, we obtain

$$\begin{aligned} \left| \frac{z_1 p'(z_1)}{p(z_1)} \right| &= \left| \frac{\lambda + \alpha}{2} \frac{z_1 p'_1(z_1)}{p_1(z_1)} \right| \\ &= \left| k_1 \left(\frac{\alpha + \lambda}{2} \right) \frac{x_1^2 + 2x_1 \cos \pi \lambda / (\alpha + \lambda) + 1}{2x_1 \sin \pi \lambda / (\alpha + \lambda)} \right|, \end{aligned} \quad (41)$$

where

$$k_1 \geq m_1 = \left(n + \frac{\sin \pi \alpha / 2\delta - n\beta / 2\delta}{\sin \pi \alpha / 2\delta + n\beta / 2\delta} \right) \tan \frac{\pi \lambda}{4\delta}. \quad (42)$$

Also, from the relation of p and p_1 , we have $|p(z_1)| = x_1^\delta$, and so, by making use of (41), it yields

$$\left| \frac{z_1 p'(z_1)}{p^2(z_1)} \right| = \left| k_1 \left(\frac{\alpha + \lambda}{2} \right) \frac{x_1^{1-\delta} + 2x_1^{-\delta} \cos \pi\lambda/(\alpha + \lambda) + x_1^{-1-\delta}}{2 \sin \pi\lambda/(\alpha + \lambda)} \right|. \quad (43)$$

If we define

$$g(x) = x_1^{1-\delta} + 2x_1^{-\delta} \cos \frac{\pi\lambda}{\alpha + \lambda} + x_1^{-1-\delta}, \quad (44)$$

then one can verify that g takes its minimum at $\tilde{a}_1$; hence, applying (42) and (43), we have

$$\left| \frac{z_1 p'(z_1)}{p^2(z_1)} \right| \geq \delta m_1 \frac{\tilde{a}_1^{1-\delta} + 2\tilde{a}_1^{-\delta} \cos \pi\lambda/2\delta + \tilde{a}_1^{-1-\delta}}{2 \sin \pi\lambda/2\delta} \geq b, \quad (45)$$

which contradicts with the assumption $f \in \mathcal{G}_b^{n,\beta}$.

If $p_1(z_0) = ix_2$, where $x_2 > 0$, using a similar argument as above leads the contradiction. Indeed, applying the previous considerations, we obtain

$$\begin{aligned} \left| \frac{z_1 p'(z_1)}{p(z_1)} \right| &= \left| \frac{\lambda + \alpha z_1 p'_1(z_1)}{2 p_1(z_1)} \right| \\ &= \left| k_2 \left(\frac{\alpha + \lambda}{2} \right) \frac{x_2^2 + 2x_2 \cos \pi\alpha/(\alpha + \lambda) + 1}{2x_2 \sin \pi\alpha/(\alpha + \lambda)} \right|, \end{aligned} \quad (46)$$

where

$$k_2 \geq m_2 = \left(n + \frac{\sin \pi\alpha/2\delta - n\beta/2\delta}{\sin \pi\alpha/2\delta + n\beta/2\delta} \right) \tan \frac{\pi\alpha}{4\delta}. \quad (47)$$

Notice that $|p(z_1)| = x_2^\delta$, and so, by making use of (46), it yields

$$\left| \frac{z_1 p'(z_1)}{p^2(z_1)} \right| = \left| k_2 \left(\frac{\alpha + \lambda}{2} \right) \frac{x_2^{1-\delta} + 2x_2^{-\delta} \cos \pi\alpha/(\alpha + \lambda) + x_2^{-1-\delta}}{2 \sin \pi\alpha/(\alpha + \lambda)} \right|. \quad (48)$$

If we define

$$h(x) = x_2^{1-\delta} + 2x_2^{-\delta} \cos \frac{\pi\alpha}{\alpha + \lambda} + x_2^{-1-\delta}, \quad (49)$$

then by some calculation, we find that h takes its minimum at $\tilde{a}_2$; hence, applying (47) and (48), we have

$$\left| \frac{z_1 p'(z_1)}{p^2(z_1)} \right| \geq \delta m_2 \frac{\tilde{a}_2^{1-\delta} + 2\tilde{a}_2^{-\delta} \cos \pi\alpha/2\delta + \tilde{a}_2^{-1-\delta}}{2 \sin \pi\alpha/2\delta} \geq b, \quad (50)$$

which contradicts with the assumption $f \in \mathcal{G}_b^{n,\beta}$. So, in both cases, we come to a contradiction, and the proof is complete. $\square$

Letting $\alpha = \lambda$ in the Theorem 5, we get the following corollary.

Corollary 6. Let $0 < \alpha \leq 1$ and $0 \leq \beta \leq 2\alpha/n$. If $f \in \mathcal{G}_b^{n,\beta}$ with

$$b = \left(n + \frac{2\alpha - n\beta}{2\alpha + n\beta} \right) \frac{\alpha}{\sqrt{(1-\alpha)^{1-\alpha}(1+\alpha)^{1+\alpha}}}, \quad (51)$$

then $f \in SS_{n,\beta}^*(\alpha)$.

Putting $\alpha = \beta = 1/2$ and $n = 1$ in Corollary 6, we obtain the following result.

Corollary 7. If $f \in \mathcal{A}_{1,1/2}$, and

$$\left| \frac{1 + zf''(z)/f'(z)}{zf'(z)/f(z)} - 1 \right| < \frac{4\sqrt{3}}{9}, \quad (52)$$

then f is strongly starlike of order $1/2$.

In the following theorem, we mention a strong result which provides sufficient conditions to strongly starlike functions.

Theorem 8. Let $0 < \lambda < 1$, $0 < \alpha < 1$ and $0 \leq \beta \leq \lambda + \alpha/n \sin \pi\alpha/\alpha + \lambda$. If $f \in \mathcal{A}_{n,\beta}$ and

$$\frac{1 + zf''(z)/f'(z)}{zf'(z)/f(z)} - 1 \in g(\mathbb{U}) (z \in \mathbb{U}), \quad (53)$$

where

$$g(z) = b \left(\frac{2\gamma z}{1 - z^2} \right)^{1-\delta}, \quad \gamma = \exp \left[i\pi \frac{\alpha - \lambda}{2 - \lambda - \alpha} \right], \quad (54)$$

and b, δ are defined as Theorem 5. Then,

$$f \in SS_{n,\beta}^*(\alpha, \lambda). \quad (55)$$

Proof. Since

$$g(z) = b \left(\frac{2\gamma z}{1 - z^2} \right)^{1-\delta}, \quad (56)$$

we have

$$g(e^{i\theta}) = b \left(\frac{i\gamma}{\sin \theta} \right)^{1-\delta}. \quad (57)$$

Now, for $0 < \theta < \pi$, we have

$$g(e^{i\theta}) = tb \exp \left[i \frac{\pi}{2} (1 - \delta) \right] = \omega_1(t), \quad (58)$$

with $t \geq 1$, and for $\pi < \theta < 2\pi$,

$$g(e^{i\theta}) = tb \exp \left[-i \frac{\pi}{2} (1 - \alpha) \right] = \omega_2(t), \quad (59)$$

with $t \geq 1$. Hence, $g(\mathbb{U})$ is a whole complex plane except the half lines $\omega_1(t)$ and $\omega_2(t)$. Suppose that

$$p(z) = \frac{zf'(z)}{f(z)}, p_1(z) = c\{p(z)\}^{2/(\lambda+\alpha)}, \quad (60)$$

with

$$c = \exp \left[i\pi \frac{\lambda - \alpha}{2(\lambda + \alpha)} \right]. \quad (61)$$

Then, it is clear that $p \in \mathcal{H}_{n\beta}[1, n]$, $p_1 \in \mathcal{H}_{c_1}[a, n]$ with

$$\begin{aligned} c_1 &= \frac{2 \exp \left[-i\pi(\alpha - \lambda)/2(\alpha + \lambda) \right]}{\alpha + \lambda} n\beta, \\ a &= \exp \left[-i \frac{\pi(\alpha - \lambda)}{2(\alpha + \lambda)} \right]. \end{aligned} \quad (62)$$

Also, it is easy to see that

$$\frac{zp'(z)}{p^2(z)} = \frac{1 + zf''(z)/f'(z)}{zf'(z)/f(z)} - 1. \quad (63)$$

If $f \notin SS_{n\beta}^*(\alpha, \lambda)$, then $p_1(\mathbb{U})$ is not contained in the right half plane $\Re w > 0$; hence, there exists a point $z_1 \in \mathbb{U}$ such that $p_1(\{z : |z| < |z_1|\})$ is contained in the right half plane $\Re w > 0$ while $p_1(z_1) = -ix_1$, or $p_1(z_1) = ix_2$, where $x_1, x_2 > 0$. In the first step, let $p_1(z_1) = -ix_1$; then, using the same argument as Theorem 3, we obtain

$$\begin{aligned} \frac{z_1 p'(z_1)}{p(z_1)} &= \frac{\lambda + \alpha}{2} \frac{z_1 p'_1(z_1)}{p_1(z_1)} = -ik_1 \left(\frac{\alpha + \lambda}{2} \right) \\ &\cdot \frac{x_1^2 + 2x_1 \cos \pi\lambda/(\alpha + \lambda) + 1}{2x_1 \sin \pi\lambda/(\alpha + \lambda)}, \end{aligned} \quad (64)$$

where

$$k_1 \geq m_1 = \left(n + \frac{\sin \pi\alpha/2\delta - n\beta/2\delta}{\sin \pi\alpha/2\delta + n\beta/2\delta} \right) \tan \frac{\pi\lambda}{4\delta}. \quad (65)$$

Also, from the relation of p and p_1 , we have

$$p(z_1) = x_1^\delta \exp \left[-\frac{\pi}{2} \alpha \right], \quad (66)$$

and so, by making use of (64), it yields

$$\begin{aligned} \arg \left(\frac{z_1 p'(z_1)}{p^2(z_1)} \right) &= \exp \left[-\frac{\pi}{2} (1 - \alpha) \right], \\ \left| \frac{z_1 p'(z_1)}{p^2(z_1)} \right| &= \left| k_1 \left(\frac{\alpha + \lambda}{2} \right) \frac{x_1^{1-\delta} + 2x_1^{-\delta} \cos \pi\lambda/(\alpha + \lambda) + x_1^{-1-\delta}}{2 \sin \pi\lambda/(\alpha + \lambda)} \right| \geq b, \end{aligned} \quad (67)$$

which contradicts with the assumption.

For the case $p_1(z_1) = ix_2$, where $x_2 > 0$, using the same argument as Theorem 3, we obtain

$$\begin{aligned} \frac{z_1 p'(z_1)}{p(z_1)} &= i \frac{\lambda + \alpha}{2} \times \frac{z_1 p'_1(z_1)}{p_1(z_1)} = ik_2 \left(\frac{\alpha + \lambda}{2} \right) \\ &\cdot \frac{x_1^2 + 2x_1 \cos \pi\alpha/(\alpha + \lambda) + 1}{2x_1 \sin \pi\alpha/(\alpha + \lambda)}, \end{aligned} \quad (68)$$

where

$$k_2 \geq m_2 = \left(n + \frac{\sin \pi\alpha/2\delta - n\beta/2\delta}{\sin \pi\alpha/2\delta + n\beta/2\delta} \right) \tan \frac{\pi\alpha}{4\delta}. \quad (69)$$

Notice that

$$p(z_1) = x_2^\delta \exp \left[i \frac{\pi}{2} \lambda \right], \quad (70)$$

and so, by making use of (68), we have

$$\begin{aligned} \arg \left(\frac{z_1 p'(z_1)}{p^2(z_1)} \right) &= \exp \left[i \frac{\pi}{2} (1 - \lambda) \right], \\ \left| \frac{z_1 p'(z_1)}{p^2(z_1)} \right| &= \left| k_2 \left(\frac{\alpha + \lambda}{2} \right) \frac{x_2^{1-\delta} + 2x_2^{-\delta} \cos \pi\alpha/(\alpha + \lambda) + x_2^{-1-\delta}}{2 \sin \pi\alpha/(\alpha + \lambda)} \right| \geq b, \end{aligned} \quad (71)$$

which contradicts with the assumption. So, in both cases, we come to a contradiction, and the proof is complete. $\square$

3. Starlikeness of Analytic Functions

There are many differential inequalities in geometric function theory which is used for univalence or starlikeness of analytic functions. For example, Ozaki [11] has proved that if $f \in \mathcal{A}$ satisfies the condition $\Re \{1 + zf''(z)/f'(z)\} > -1/2$, then f is univalent. Also, the well-known result known as the Mark-strohhäcker [5] states that $f \in \mathcal{A}$, then

$$\Re \left(1 + \frac{zf''(z)}{f'(z)} \right) > 0 \implies \Re \frac{zf'(z)}{f(z)} > \frac{1}{2}. \quad (72)$$

Hence, it is natural to extend the similar results to analytic functions with a fixed initial coefficient.

In this section, we try to obtain some inequalities in analytic functions with second fixed coefficients which improve

earlier results obtained in the literature. At first, we bring the following corollary.

Corollary 9. Let $p \in \mathcal{H}_\beta[1, n]$ and $z_0 \in \mathbb{U}$ such that $\Re p(z_0) > 0$ for $|z| < |z_0|$ and $\Re p(z_0) = 0$ with $p(z_0) \neq 0$. Then, we have

$$\frac{z_0 p'(z_0)}{p(z_0)} = im, |m| \geq \left(n + \frac{2-\beta}{2+\beta}\right), \quad (73)$$

$$z_0 p'(z_0) \leq -\frac{1}{2} \left(n + \frac{2-\beta}{2+\beta}\right) (1 + |p(z_0)|^2). \quad (74)$$

Proof. By letting $\alpha = \lambda = 1$ in Theorem 3, we obtain (73). For (74), we see that $p(z_0)$ is purely imaginary so $p(\bar{z}_0) = -p(z_0)$ and $|p(z_0)|^2 = -p(z_0)^2$. Let us put

$$q(z) = \frac{1+z}{1-z}. \quad (75)$$

Then, we have that $q \in Q$ with $q'(0) = 2$. From the assumption and Lemma 2, there exist $\zeta_0 \in \partial\mathbb{U}$ with $p(z_0) = q(\zeta_0)$ such that

$$\begin{aligned} z_0 p'(z_0) &= m \zeta_0 q'(\zeta_0) = m q^{-1}(p(z_0)) \frac{2}{(1 - q^{-1}(p(z_0))^2)} \\ &= -\frac{1}{2} m (1 + |p(z_0)|^2), \end{aligned} \quad (76)$$

which shows that

$$-z_0 p'(z_0) \geq \frac{1}{2} \left(n + \frac{2-\beta}{2+\beta}\right) (1 + |p(z_0)|^2), \quad (77)$$

and $z_0 p'(z_0)$ is a negative real number and (74) is obtained. $\square$

Remark 10. The special case when $\beta = 2$ reduces to the result due to Nunokawa [12].

Corollary 11. Let $p \in \mathcal{H}_\beta[1, n]$ and $p(z) \neq 0$ in $\mathbb{U}$. If there exist $z_0 \in \mathbb{U}$ such that $|\arg p(z)| < \pi\alpha/2$ for $|z| < |z_0|$ and $|\arg p(z_0)| = \pi\alpha/2$, where $\alpha > 0$. Then, we have

$$\frac{z_0 p'(z_0)}{p(z_0)} = -iam, m \leq -\frac{1}{2} \left(a + \frac{1}{a}\right) \left(n + \frac{2\alpha - \beta}{2\alpha + \beta}\right), \quad (78)$$

when $\arg p(z_0) = -\pi\alpha/2$ and

$$\frac{z_0 p'(z_0)}{p(z_0)} = iam, m \geq \frac{1}{2} \left(a + \frac{1}{a}\right) \left(n + \frac{2\alpha - \beta}{2\alpha + \beta}\right), \quad (79)$$

when $\arg p(z_0) = \pi\alpha/2$, where $p(z_0)^{1/\alpha} = \pm ia$.

Proof. Let $q(z) = p(z)^{1/\alpha}$. We have $\Re q(z) > 0$ for $|z| < |z_0|$ and $\Re q(z_0) = 0$. Let $\Re q(z_0) = \pm ia$ with $a > 0$. From Corollary 9, we have

$$\frac{z_0 q'(z_0)}{q(z_0)} = \frac{1}{\alpha} \frac{z_0 p'(z_0)}{p(z_0)} = im \implies \frac{z_0 p'(z_0)}{p(z_0)} = i\alpha m, \quad (80)$$

where

$$m \leq -\frac{1}{2} \left(a + \frac{1}{a}\right) \left(n + \frac{2\alpha - \beta}{2\alpha + \beta}\right), \text{ when } q(z_0) = -ia, \quad (81)$$

$$m \geq \frac{1}{2} \left(a + \frac{1}{a}\right) \left(n + \frac{2\alpha - \beta}{2\alpha + \beta}\right), \text{ when } q(z_0) = +ia.$$

So, the proof is done. $\square$

Remark 12. Note the special case of Corollary 11 when $\beta = 2\alpha$ reduces to the well-known lemma [9].

Theorem 13. Let $p \in \mathcal{H}_\beta[1, n]$, $0 \leq \beta < 2$, and suppose that

$$\left| \Im m \frac{z p'(z)}{p(z)} \right| < n + \frac{2-\beta}{2+\beta} (z \in \mathbb{U}). \quad (82)$$

Then, we have $\Re p(z) > 0$.

Proof. From the assumption (82), we have $p(z) \neq 0$. In fact, if $p(z)$ has a zero of order n at $z = \alpha$, then we have

$$p(z) = (z - \alpha)^n p_1(z), \quad (83)$$

where $p_1(z)$ is analytic in $\mathbb{U}$ and $p_1(\alpha) \neq 0$. Then, we have

$$\frac{z p'(z)}{p(z)} = \frac{nz}{z - \alpha} + \frac{z p_1'(z)}{p_1(z)}. \quad (84)$$

But n is fixed, so we have

$$\lim_{z \rightarrow \alpha} \left| \Im m \frac{nz}{z - \alpha} \right| = \infty. \quad (85)$$

Then, the imaginary part of the right-hand side of (84) can take any values when z approaches α . This contradicts (82). This shows that $p(z) \neq 0$ for $z \in \mathbb{U}$. Therefore, if there exists a point $z_0 \in \mathbb{U}$ such that $\Re p(z_0) = 0$, then we have $p(z_0) \neq 0$. From Corollary 9, we obtain

$$\frac{z_0 p'(z_0)}{p(z_0)} = im, |m| \geq \left(n + \frac{2-\beta}{2+\beta}\right). \quad (86)$$

This contradicts (82). This completes our proof. $\square$

Remark 14. The special case when $\beta = 2$ reduces to the result obtained in [9].

Corollary 15. Let $f \in \mathcal{A}_{n,b}$, $0 \leq b < 2$, and suppose that

$$\left| \Im \frac{zf'(z)}{f(z)} \right| < n + \frac{2-b}{2+b} (z \in \mathbb{U}). \quad (87)$$

Then, we have $\Re f(z)/z > 0$ for $z \in \mathbb{U}$.

Proof. Let $f \in \mathcal{A}_{n,b}$ and put $p(z) = f(z)/z$. Then, $p(z) \in \mathcal{H}_b[1, n]$. Applying Theorem 13, we get the result. $\square$

Corollary 16. Let $f \in \mathcal{A}_{n,b}$, $0 \leq b < 4/(n+1)$ and suppose that

$$\left| \Im \frac{zf''(z)}{f'(z)} \right| < 2n + 2 \frac{4-b(n+1)}{4+b(n+1)} (z \in \mathbb{U}). \quad (88)$$

Then, we have $\Re \sqrt{f'(z)} > 0$ for $z \in \mathbb{U}$.

Proof. Let $f \in \mathcal{A}_{n,b}$ and put $p(z) = \sqrt{f'(z)}$. Then, $p(z) \in \mathcal{H}_{1/2(n+1)b}[1, n]$. Applying Theorem 13, we get the result. $\square$

Corollary 17. Let $f \in \mathcal{A}_{n,b}$, $0 \leq b < 2/n$, and suppose that

$$\left| \Im \left\{ 1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} \right\} \right| < n + \frac{2-nb}{2+nb} (z \in \mathbb{U}). \quad (89)$$

Then, we have $\Re(zf'(z)/f(z)) > 0$ for $z \in \mathbb{U}$.

Proof. Let $f \in \mathcal{A}_{n,b}$ and put $p(z) = zf'(z)/f(z)$. Then, $p(z) \in \mathcal{H}_{nb}[1, n]$. Applying Theorem 13, we get the result. $\square$

Theorem 18. Let $p \in \mathcal{H}_\beta[1, n]$ and $p(z) \neq 0$ in $\mathbb{U}$. Also, suppose that $0 \leq \gamma < \delta < \gamma + 2$ and $0 \leq \beta \leq 2(1-\alpha)$. If there exist $z_0 \in \mathbb{U}$ such that $\Re p(z) > \alpha$ for $|z| < |z_0|$ and $\Re p(z_0) = \alpha$ but $p(z_0) \neq \alpha$ where $0 < \alpha < 1$. Then, we have

$$\Re \frac{z_0 p'(z_0)}{\delta p(z_0) + \gamma} \leq \Gamma(\alpha) = \begin{cases} \frac{-(\delta\alpha + \gamma)m}{2(1-\alpha)\delta^2}, & \alpha \in \left(0, \frac{\delta-\gamma}{2}\right], \\ \frac{(\alpha-1)m}{2(\delta\alpha + \gamma)}, & \alpha \in \left(\frac{\delta-\gamma}{2}, 1\right), \end{cases} \quad (90)$$

where

$$m \geq n + \frac{2-2\alpha-\beta}{2-2\alpha+\beta}. \quad (91)$$

Proof. Let

$$q(z) = \frac{1-(2\alpha-1)z}{1-z}, \quad (92)$$

where $0 < \alpha < 1$. We know that q is analytic and univalent in $\mathbb{U}$ with $q'(0) = 2-2\alpha$ and $\Re q(\mathbb{U}) > \alpha$. So, $q \in \mathcal{Q}$ with $E(q)$

$= 1$. From Lemma 2, there exist ζ_0 on $\partial\mathbb{U}$ with $p(z_0) = q(\zeta_0)$ such that

$$z_0 p'(z_0) = m \zeta_0 q'(\zeta_0), \text{ where } m \geq n + \frac{|q'(0)| - \beta |z_0|^n}{|q'(0)| + \beta |z_0|^n}. \quad (93)$$

We have

$$\begin{aligned} z_0 p'(z_0) &= m \zeta_0 q'(\zeta_0) = m \times \frac{p(z_0) - 1}{p(z_0) - (2\alpha - 1)} \\ &\quad \times \frac{2 - 2\alpha}{(1 - (p(z_0) - 1)/(p(z_0) - (2\alpha - 1)))^2} \\ &= \frac{m}{2 - 2\alpha} (p(z_0) - 1)(p(z_0) - (2\alpha - 1)), \end{aligned} \quad (94)$$

where

$$m \geq n + \frac{2 - 2\alpha - \beta}{2 - 2\alpha + \beta}. \quad (95)$$

By taking $p(z_0) = \alpha + it$ for a fixed real t , we have

$$\begin{aligned} \Re \frac{z_0 p'(z_0)}{\delta p(z_0) + \gamma} &= \frac{-(\delta\alpha + \gamma)m}{2 - 2\alpha} \left(\frac{t^2 + (\alpha - 1)^2}{\delta^2 t^2 + (\delta\alpha + \gamma)^2} \right), m \\ &\geq n + \frac{2 - 2\alpha - \beta}{2 - 2\alpha + \beta}. \end{aligned} \quad (96)$$

If $A = (t^2 + (\alpha - 1)^2)/\delta^2 t^2 + (\delta\alpha + \gamma)^2$, then for $\alpha \in (0, \delta - \gamma/2]$ and for all real number t , we have $A \geq 1/\delta^2$ which leads to

$$\begin{aligned} \Re \frac{z_0 p'(z_0)}{\delta p(z_0) + \gamma} &\leq \frac{-(\delta\alpha + \gamma)m}{2(1-\alpha)\delta^2}, 0 < \alpha \leq \frac{\delta-\gamma}{2}, m \\ &\geq n + \frac{2 - 2\alpha - \beta}{2 - 2\alpha + \beta}. \end{aligned} \quad (97)$$

Also, we have $(\alpha - 1/(\delta\alpha + \gamma))^2 \leq A < 1$ when $\alpha \in (\delta - \gamma/2, 1)$, and t is an arbitrary real number. This leads to

$$\begin{aligned} \Re \frac{z_0 p'(z_0)}{\delta p(z_0) + \gamma} &\leq \frac{(\alpha - 1)m}{2(\delta\alpha + \gamma)}, \frac{\delta - \gamma}{2} < \alpha < 1, m \\ &\geq n + \frac{2 - 2\alpha - \beta}{2 - 2\alpha + \beta}. \end{aligned} \quad (98)$$

Therefore, the proof is complete. $\square$

Corollary 19. Let $p \in \mathcal{H}_\beta[1, n]$ and $p(z) \neq 0$ in $\mathbb{U}$. Suppose also that $0 < \lambda, 0 \leq \gamma < \delta < \gamma + 2$, $0 < \alpha < 1$, $0 \leq \beta \leq 2(1 - \alpha)$ and

$$\Re \left\{ p(z) + \lambda \frac{zp'(z)}{\delta p(z) + \gamma} \right\} > \begin{cases} \alpha + \lambda \frac{-(\delta\alpha + \gamma)m_1}{2\delta^2(1 - \alpha)}, & \alpha \in \left(0, \frac{\delta - \gamma}{2}\right], \\ \alpha + \lambda \frac{(\alpha - 1)m_1}{2(\delta\alpha + \gamma)}, & \alpha \in \left(\frac{\delta - \gamma}{2}, 1\right), \end{cases} \quad (99)$$

where

$$m_1 = n + (2 - 2\alpha - \beta)/(2 - 2\alpha + \beta). \quad (100)$$

Then, we have $\Re\{p(z)\} > \alpha$ for $|z| < 1$.

Proof. If there exists a point, z_0 , $|z_0| < 1$, such that

$$\begin{aligned} \Re\{p(z)\} &> \alpha(|z| < |z_0|), \\ \Re\{p(z_0)\} &= \alpha. \end{aligned} \quad (101)$$

Then, from Theorem 18, we have

$$\Re \left\{ \frac{z_0 p'(z_0)}{\delta p(z_0) + \gamma} \right\} \leq \begin{cases} \frac{-(\delta\alpha + \gamma)m_1}{2(1 - \alpha)\delta^2}, & \alpha \in \left(0, \frac{\delta - \gamma}{2}\right], \\ \frac{(\alpha - 1)m_1}{2(\delta\alpha + \gamma)}, & \alpha \in \left(\frac{\delta - \gamma}{2}, 1\right), \end{cases} \quad (102)$$

and so,

$$\begin{aligned} \Re \left\{ p(z_0) + \lambda \frac{z_0 p'(z_0)}{\delta p(z_0) + \gamma} \right\} \\ \leq \begin{cases} \alpha + \lambda \frac{-(\delta\alpha + \gamma)m_1}{2\delta^2(1 - \alpha)}, & \alpha \in \left(0, \frac{\delta - \gamma}{2}\right], \\ \alpha + \lambda \frac{(\alpha - 1)m_1}{2(\delta\alpha + \gamma)}, & \alpha \in \left(\frac{\delta - \gamma}{2}, 1\right). \end{cases} \end{aligned} \quad (103)$$

This contradicts (99), and the proof is complete. $\square$

By taking $\delta = 1$, $\gamma = 0$, and $p(z) = zf'(z)/f(z)$ in Corollary 19, we obtain the following theorem.

Theorem 20. Let $f \in \mathcal{A}_{n,b}$, $0 < \alpha \leq 1/2$ and $0 \leq b < 2(1 - \alpha)/n$. If

$$\Re \left\{ (1 - \lambda) \frac{zf'(z)}{f(z)} + \lambda \left(1 + \frac{zf''(z)}{f'(z)} \right) \right\} > \alpha + \lambda \frac{-\alpha m_1}{2(1 - \alpha)}, \quad (104)$$

where

$$m_1 = n + \frac{2 - 2\alpha - nb}{2 - 2\alpha + nb}, \quad 0 < \lambda. \quad (105)$$

Then,

$$\Re \frac{zf'(z)}{f(z)} > \alpha. \quad (106)$$

Remark 21. We remark that the special case of Theorem 20 with $\alpha = 1/2$, $\lambda = 1$, and $n = 1$ was proved in [8]. Also, the special case when $\alpha = 1/2$, $\lambda = n = 1$, and $b = 0$ reduces to the well-known Mark-strohhäcker result [13].

Also, by taking $\delta = 1$, $\gamma = 0$, and $p(z) = zf'(z)/f(z)$ in Corollary 19, we obtain the following theorem.

Theorem 22. Let $f \in \mathcal{A}_{n,b}$, $1/2 < \alpha < 1$, and $0 \leq b < 2(1 - \alpha)/n$. If

$$\Re \left\{ (1 - \lambda) \frac{zf'(z)}{f(z)} + \lambda \left(1 + \frac{zf''(z)}{f'(z)} \right) \right\} > \alpha + \lambda \frac{(\alpha - 1)m_1}{2\alpha}, \quad (107)$$

where

$$m_1 = n + \frac{2 - 2\alpha - nb}{2 - 2\alpha + nb}, \quad 0 < \lambda. \quad (108)$$

Then,

$$\Re \frac{zf'(z)}{f(z)} > \alpha. \quad (109)$$

For $\gamma \geq 0$, $\delta > 0$, let $I_{\delta,\gamma}$ be the integral operator defined as

$$F(z) = I_{\delta,\gamma}[f(z)] = \left[\frac{\delta + \gamma}{z^\gamma} \int_0^z f^\delta(t) t^{\gamma-1} dt \right]^{1/\delta}. \quad (110)$$

We note that the integral operator $I_{\delta,\gamma}$ maps $\mathcal{A}_{n,b}$ to $\mathcal{A}_{n,(\delta+\gamma)b/n+\delta+\gamma}$. Now, we state the next following theorems.

Theorem 23. Let $f \in \mathcal{A}_{n,b}$, $0 < \alpha \leq 1/2$, and $0 \leq b < 2(n + \delta + \gamma)(1 - \alpha)/n(\delta + \gamma)$. If $0 \leq \gamma < \delta < \gamma + 2$ and

$$\Re \frac{zf'(z)}{f(z)} > \alpha + \frac{-(\delta\alpha + \gamma)m_1}{2\delta^2(1 - \alpha)}, \quad (111)$$

where

$$m_1 = n + \frac{2 - 2\alpha - nb(\delta + \gamma)/(n + \delta + \gamma)}{2 - 2\alpha + nb(\delta + \gamma)/(n + \delta + \gamma)}. \quad (112)$$

Then,

$$\Re \frac{zF'(z)}{F(z)} > \alpha, \quad (113)$$

where $F(z) = I_{\delta, \gamma}[f(z)]$ is defined in (110).

Proof. Let us define $p(z) = zF'(z)/F(z)$. It is easy to show that

$$p(z) + \frac{zp'(z)}{\delta p(z) + \gamma} = \frac{zf'(z)}{f(z)}. \quad (114)$$

Now considering Corollary 19, we get our result. $\square$

Putting $\delta = b = n = 1$, $\alpha = 1/2$, and $\gamma = 0$ in the above theorem, we have the following result.

Corollary 24. Let $f \in \mathcal{A}_{1,1}$. If

$$\Re \frac{zf'(z)}{f(z)} > -\frac{1}{6}. \quad (115)$$

Then,

$$\Re \frac{zF'(z)}{F(z)} > \frac{1}{2}. \quad (116)$$

By considering Corollary 19, we have the following result.

Theorem 25. Let $f \in \mathcal{A}_{n,b}$, $1/2 < \alpha \leq 1$, and $0 \leq b < 2(n + \delta + \gamma)(1 - \alpha)/n(\delta + \gamma)$. If $0 \leq \gamma < \delta < \gamma + 2$ and

$$\Re \frac{zf'(z)}{f(z)} > \alpha + \lambda \frac{(\alpha - 1)m_1}{2(\delta\alpha + \gamma)}, \quad (117)$$

where

$$m_1 = n + \frac{2 - 2\alpha - nb(\delta + \gamma)/(n + \delta + \gamma)}{2 - 2\alpha + nb(\delta + \gamma)/(n + \delta + \gamma)}. \quad (118)$$

Then,

$$\Re \frac{zF'(z)}{F(z)} > \alpha, \quad (119)$$

where $F(z) = I_{\delta, \gamma}[f(z)]$ is defined in (110).

Finally, we state the following theorem which is very interesting and in the especial case improves some earlier results in the literature.

Theorem 26. Suppose that $f \in \mathcal{A}_{n,b}$ and $0 \leq b < 1/1 + n$. Also, let

$$\Re \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} < \frac{m_1 + 2}{2}, \quad (120)$$

where

$$m_1 = n + \frac{1 - b(1 + n)}{1 + b(1 + n)}. \quad (121)$$

If $zf'(z)/f(z)$ is analytic in $\mathbb{U}$ and omits $2(1 + m_1)/2 + m_1$, then f is starlike in the unit disc $\mathbb{U}$ and

$$\Re \frac{zf'(z)}{f(z)} < \frac{2(1 + m_1)}{2 + m_1}. \quad (122)$$

Proof. Let us define the function p as

$$\frac{zf'(z)}{f(z)} = \frac{2(1 + m_1)p(z)}{m_1 + (2 + m_1)p(z)}. \quad (123)$$

Then, it is easy to see that $p \in \mathcal{H}_{2b(1+n)}[1, n]$, and

$$1 + \frac{zf''(z)}{f'(z)} = \frac{zf'(z)}{f(z)} + \frac{zp'(z)}{p(z)} - \frac{(2 + m_1)zp'(z)}{m_1 + (2 + m_1)p(z)}. \quad (124)$$

If there exists a point z_0 such that $\Re p(z) > 0$ for $|z| < |z_0|$ and $\Re p(z_0) = 0$. Then, using Corollary 9, we have

$$\frac{z_0 p'(z_0)}{p(z_0)} = ik, \quad (125)$$

where k is a real number with $|k| \geq m_1$. For the case $\arg p(z_0) = \pi/2$, and $p(z_0) = ia$ with $a > 0$, we obtain

$$\begin{aligned} 1 + \frac{z_0 f''(z_0)}{f'(z_0)} &= \frac{z_0 f'(z_0)}{f(z_0)} + \frac{z_0 p'(z_0)}{p(z_0)} - \frac{(2 + m_1)z_0 p'(z_0)}{m_1 + (2 + m_1)p(z_0)} \\ &= \frac{2(1 + m_1)ia}{m_1 + (2 + m_1)ia} + \frac{z_0 p'(z_0)}{p(z_0)} \\ &\quad - \frac{z_0 p'(z_0)}{p(z_0)} \frac{(2 + m_1)p(z_0)}{m_1 + (2 + m_1)p(z_0)} \\ &= \frac{2(1 + m_1)ia(m_1 - (2 + m_1)ia)}{m_1^2 + (2 + m_1)^2 a^2} \\ &\quad + ik + \frac{k(2 + m_1)a(m_1 - (2 + m_1)ia)}{m_1^2 + (2 + m_1)^2 a^2}. \end{aligned} \quad (126)$$

Therefore, we obtain

$$\Re \left\{ 1 + \frac{z_0 f''(z_0)}{f'(z_0)} \right\} \geq \frac{2(1+m_1)(2+m_1)a^2}{m_1^2 + (2+m_1)^2 a^2} + \frac{m_1^2(2+m_1)(1+a^2)}{2(m_1^2 + (2+m_1)^2 a^2)} = \frac{2+m_1}{2}, \quad (127)$$

and that contradicts the hypothesis of the theorem. In the case $\arg p(z_0) = -\pi/2$ and $p(z_0) = -ia$ with $a > 0$, by applying the same method as above, we also obtain

$$\Re \left\{ 1 + \frac{z_0 f''(z_0)}{f'(z_0)} \right\} \geq \frac{2+m_1}{2}, \quad (128)$$

and this also contradicts with the assumption of the theorem. Hence, we obtain

$$\Re p(z) > 0, z \in \mathbb{U}. \quad (129)$$

But the inequality

$$\left| \frac{zf'(z)}{f(z)} - \frac{1+m_1}{2+m_1} \right| < \frac{1+m_1}{2+m_1} \quad (130)$$

holds true if only if

$$\left| \frac{2(1+m_1)p(z)}{m_1 + (2+m_1)p(z)} - \frac{1+m_1}{2+m_1} \right| < \frac{1+m_1}{2+m_1}, \quad (131)$$

or equivalently

$$\left| \frac{(2+m_1)p(z) - m_1}{m_1 + (2+m_1)p(z)} \right| < 1. \quad (132)$$

Finally, we note that the last inequality holds true if and only if $\Re p(z) > 0$, and this is proved in the above. Therefore, from (130), we have

$$0 < \frac{zf'(z)}{f(z)} < \frac{2(1+m_1)}{2+m_1}, \quad (133)$$

and the proof is complete. $\square$

Putting $n = 1$, $b = 0$ in Theorem 26, we obtain the following result.

Corollary 27. Suppose that $f \in \mathcal{A}_{1,0}$ and

$$\Re \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} < 2. \quad (134)$$

If $zf'(z)/f(z)$ is analytic in $\mathbb{U}$ and omits $4/3$, then f is starlike in the unit disc $\mathbb{U}$ and

$$\Re \frac{zf'(z)}{f(z)} < \frac{4}{3}. \quad (135)$$

Remark 28. Note that Corollary 27 improves the result of Sumit and Ravichandran [8].

Data Availability

In this paper, we do not use any data from elsewhere, since this paper does not need data.

Conflicts of Interest

The authors declare that there is no conflict of interests regarding the publication of this paper.

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