Research Article

New Exact Soliton Solutions of the $(3 + 1)$ -Dimensional Conformable Wazwaz–Benjamin–Bona–Mahony Equation via Two Novel Techniques

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In this work, the $(3 + 1)$ -dimensional Wazwaz–Benjamin–Bona–Mahony equation is formulated in the sense of conformable derivative. Two novel methods of generalized Kudryashov and $\exp(-\varphi(\aleph))$ are investigated to obtain various exact soliton solutions. All algebraic computations are done with the help of the Maple software. Graphical representations are provided in 3D and 2D profiles to show the behavior and dynamics of all obtained solutions at various parameters' values and conformable orders using Wolfram Mathematica.

1. Introduction

Partial differential equations (PDEs) have attracted a particular interest from researchers in the fields of natural sciences and engineering due to the applicability of these equations in modeling various scientific phenomena in interdisciplinary sciences such as mathematical physics, mechanics, signal and image processing, and chemistry. Most physical systems are not linear; therefore, nonlinear partial differential equations (NLPDEs), particularly nonlinear evolution equations (NLEEqs) (see [1]), have inspired researchers to investigate the existence of exact solutions for such equations. Finding new exact solutions for NLPDEs can significantly provide a good interpretation for the physical meaning and dynamics of these equations. Therefore, several research studies have recently been done on developing new methods for solving NLPDEs exactly. Some of the most notable methods that have been applied to solve some interesting NLPDEs are

the methods of modified simple equation and extended simplest equation to solve $(4 + 1)$ -dimensional nonlinear Fokas equation [2], the methods of $\exp$ -function and $\exp(-\Phi(\aleph))$ -expansion to solve Vakhnenko–Parkes equation [3], and the method of generalized Kudryashov to solve nonlinear Jaulent–Miodek hierarchy and $(2 + 1)$ -dimensional Calogero–Bogoyavlenskii–Schiff equations [4]. To solve nonlinear integrable equations, a novel technique, known as Hirota bilinear method, was proposed by Ma and Ma et al. in [5, 6] in order to obtain new lump solutions for the investigated equations [7].

One of the most interesting NLPDEs is the Benjamin–Bona–Mahony equation (BBMEq), an extension of the Korteweg–de Vries equation (KdVEq), which is basically a model that represents the unidirectional propagation of long waves with small amplitude on the hydromagnetic and acoustic waves' surface in shallow water channel [8, 9]. Consider the following $(3 + 1)$ -dimensional modified form of BBMEq:

$$\Psi_t + \Psi_x + \Psi^2 \Psi_y - \Psi_{yzt} = 0. \quad (1)$$

Equation (1) was first proposed by Wazwaz [10] by formulating a new three-dimensional modified version of BBMEqs, known as the Wazwaz-Benjamin-Bona-Mahony equation (WBBMEq), via coupling or various generalized contexts or as combination of both of them. Higher dimensional problems with various applications can be described by the WBBMEq with its spatial and temporal variables [8, 9]. Therefore, there is a significant need to find exact solutions for WBBMEq to interpret their physical meaning and dynamics.

Fractional differential equations (FDiEqs) are generalized forms of integer-order differential equations. FDiEqs have attracted the interests of researchers due to the ability of these equations in modeling various scientific phenomena better than the integer-order ones (refer to [11–13]). The behavior of some physical systems can be interpreted better than the integer-order ones due to the nonlocality of fractional derivatives, and many systems have memory effects. There are many definitions of fractional derivatives, but the most common ones are Riemann-Liouville and Caputo fractional derivatives where the properties of linearity are commonly shared among these derivatives, but other properties such as product rule, constant, chain rule, and quotient rule are not satisfied. FDiEqs are considered as a powerful tool in modeling scenarios (see [14–16]), but this tool comes with various challenges when dealing with FDiEqs due to the difficulty of obtaining exact or analytical solutions where the solutions can be very complicated or impossible to obtain them for certain cases. As a result, to overcome some of challenges associated with nonlocal fractional derivatives, a new generalized fractional derivative of local-type, named conformable derivative, was proposed by Khalil et al. which is basically a stretch for the usual limit-based definition where all usual derivative's properties are satisfied [17]. Many research studies have discussed the mathematical analysis and applications of conformable derivative such as conformable Laplace's equation [18] and generalized conformable mean value theorems [19].

Seadawy et al. [20] and Bilal et al. [21] investigated various soliton solutions for conformable WBBMEq using the methods of simple ansatz and generalized exponential rational function, respectively.

Inspired by all above studies, this work is mainly aimed at obtaining new exact solitary solutions for a version of WBBMEq formulated in the sense of conformable derivative (ComD) with the help of two novel techniques: generalized Kudryashov method and $\exp(-\varphi(\aleph))$ method. The general fractional formulation of WBBMEq can be expressed as follows:

$$D_t^\zeta \Psi + D_x^\zeta \Psi + D_y^\zeta \Psi - D_{xzt}^{3\zeta} \Psi = 0, \quad (2)$$

where D^ζ is the fractional operator of order: $\zeta \in (0, 1]$. The exact solutions of Eq. (2) have been investigated in some research works using the (G'/G) -expansion method [9], modified simple equation method [8], and Riccati-Bernoulli

Sub-ODE method [8]. However, according to the best of our knowledge, none of previous research works has investigated Eq. (2) in the context of ComD via the methods of generalized Kudryashov and $\exp(-\varphi(\aleph))$.

Therefore, all results in our work are new and worthy.

This work is divided into the following sections: Essential notions about conformable derivative are presented, and the methodology of our two proposed approaches is discussed in Section 2. In Section 3, the main results of our work are presented. In Section 4, the graphical comparisons of our obtained exact solutions are represented in both 2D and 3D plots for various values of parameters and ζ . The conclusion of our work is provided in Section 5.

2. Fundamental Preliminaries and Methodology

Some important notions about conformable derivative are introduced in this section. In addition, the methodology of two different approaches, namely, generalized Kudryashov method and $\exp(-\varphi(\aleph))$ method, is also described, respectively. The conformable derivative can be defined as follows:

Definition 1. Given a function $\Psi : [0, \infty) \rightarrow \mathfrak{R} \ni \forall t > 0$, the conformable derivative of order $\zeta \in (0, 1]$ of Ψ can be expressed as follows:

$$D_t^\zeta (\Psi(t)) = \lim_{\Omega \rightarrow 0} \frac{\Psi(t + \Omega t^{1-\zeta}) - \Psi(t)}{\Omega}. \quad (3)$$

Let Ψ be ζ -differentiable in some $(0, q)$, where $q > 0$, and the limit of $D_t^\zeta (\Psi(t))$ exists as $t \rightarrow 0^+$; then, from this definition, we obtain the following:

$$D_t^\zeta (\Psi(0)) = \lim_{t \rightarrow 0^+} D_t^\zeta (\Psi(t)). \quad (4)$$

The theorem [17] below shows that D_t^ζ satisfies all usual limit-based derivative's properties as follows:

Theorem 2. For $\zeta \in (0, 1]$, let functions Ψ and Φ be ζ -differentiable at a point t ; then, we have the following:

- (a) $D_t^\zeta (\Psi(t)\Phi(t)) = \Psi(t)D_t^\zeta (\Phi(t)) + \Phi(t)D_t^\zeta (\Psi(t))$
- (b) $D_t^\zeta (m\Psi(t) + w\Phi(t)) = mD_t^\zeta \Psi(t) + wD_t^\zeta \Phi(t), \forall m, w \in \mathfrak{R}$
- (c) $D_t^\zeta (\Psi(t)/\Phi(t)) = \Phi(t)D_t^\zeta (\Psi(t)) - (\Psi(t))D_t^\zeta (\Phi(t))/\Phi^2(t)$
- (d) $D_t^\zeta (t^k) = kt^{k-\zeta}, \forall k \in \mathfrak{R}$
- (e) If $\Psi(t)$ is assumed to be a differentiable function, then $D_t^\zeta (\Psi(t)) = t^{1-\zeta} d\Psi/dt$
- (f) $\mathfrak{D}_t^\zeta (v) = 0, \forall \text{constant functions } \Psi(t) = v$

The methodology of the generalized Kudryashov method (GKuM) and $\exp(-\varphi(\aleph))$ method (ExpM) can be presented as follows:

Consider the following form of NLEEq, with 4 independent variables: x, y, z , and t , formulated generally in the sense of fractional derivative:

$$T\left(\Psi, D_t^\zeta \Psi, D_x^\zeta \Psi, D_y^\zeta \Psi, D_z^\zeta \Psi, D_t^{2\zeta} \Psi, D_x^{2\zeta} \Psi, D_y^{2\zeta} \Psi, D_z^{2\zeta} \Psi, \dots\right) = 0; 0 < \zeta \leq 1, \quad (5)$$

where $\Psi = \Psi(x, y, z, t)$ is an unknown function and T is a polynomial of Ψ and its partial derivatives in which all of the nonlinear terms and highest-order derivatives are included in Eq. (5). First of all, to solve Eq. (5), we use the following traveling wave transformations for ComD:

For ComD,

$$\Psi(x, y, z, t) = \Psi(\aleph); \aleph = p \frac{x^\zeta}{\zeta} + q \frac{y^\zeta}{\zeta} + \gamma \frac{z^\zeta}{\zeta} - \delta \frac{t^\zeta}{\zeta}, \quad (6)$$

where p, q, γ , and δ are all constants with the condition: $p, q, \gamma, \delta \neq 0$, and δ is the wave speed.

According to the above transformations, Eq. (5) is reduced to the following ODE.

$$L\left(\Psi, \Psi', \Psi'', \Psi''', \dots\right) = 0. \quad (7)$$

The derivative with respect to $\aleph$ is represented by a prime. Equation (7) should be integrated term by term one or more times.

2.1. The Generalized Kudryashov Method. From GKUM, the obtained solution for the reduced equation is constructed by a polynomial in $\hbar(\aleph)$ as [4, 22]:

$$\Psi(\aleph) = \frac{\sum_{k=0}^J p_k \hbar^k(\aleph)}{\sum_{l=0}^W q_l \hbar^l(\aleph)}, \quad (8)$$

where $p_k (k=0, 1, \dots, J)$, $b_l (l=0, 1, \dots, W)$ are constants which are needed to be determined $\exists p_j \neq 0, q_w \neq 0$, and $L = L(\aleph)$ is the solution of the following equation:

$$\frac{d\hbar}{d\aleph} = \hbar^2(\aleph) - \hbar(\aleph). \quad (9)$$

The solution of Eq. (9) can be expressed as follows:

$$\hbar(\aleph) = \frac{1}{1 + I_1 e^\aleph}, \quad (10)$$

where I_1 is an integration constant.

According to the homogeneous balance principle (HBPrp), the positive integers: J and W in Eq. (8) can be obtained with the help of Eq. (7). In addition, a polynomial, $\hbar$, can be determined by the substitution of Eq. (8) into Eq. (7) along with Eq. (9). Now, by equating all the coefficients of polynomial $\hbar$ to 0 in order to construct a system of algebraic equations, this system is solved with the aid of the com-

puter software such as Maple and Wolfram Mathematica in order to find the values of $p_k (k=0, 1, \dots, J)$, $q_l (l=0, 1, \dots, W)$. At the end, all soliton-type solutions of the reduced Eq. (7) can be found by the substitution of these obtained values and Eq. (9) into Eq. (8).

2.2. The $\exp(-\varphi(\aleph))$ Method. From ExpM [3], the obtained solution for the reduced equation is constructed by a polynomial in $\exp(-\Phi(\aleph))$ as follows:

$$\Psi(\aleph) = \sum_{j=0}^w p_j (\exp(-\phi(\aleph)))^j, \quad (11)$$

where $p_j (p_w \neq 0)$ are constants which are needed to be found, and $\varphi(\aleph)$ satisfies the following auxiliary ODE:

$$\varphi'(\aleph) = \exp(-\varphi(\aleph)) + \vartheta \exp(\varphi(\aleph)) + \chi. \quad (12)$$

Note that Eq. (12) has distinct solutions which are expressed as follows:

Case 1. When $\chi^2 - 4\vartheta > 0$ and $\vartheta \neq 0$, the hyperbolic function solutions are expressed as follows:

$$\phi_1(\aleph) = \ln \left(\frac{-\sqrt{\chi^2 - 4\vartheta} \tanh \left(\frac{\sqrt{\chi^2 - 4\vartheta/2}(\aleph + I)}{2\vartheta} \right) - \chi}{2\vartheta} \right). \quad (13)$$

Case 2. When $\chi^2 - 4\vartheta < 0$ and $\vartheta \neq 0$, the trigonometric function solutions are expressed as follows:

$$\phi_2(\aleph) = \ln \left(\frac{\sqrt{4\vartheta - \chi^2} \tan \left(\frac{\sqrt{4\vartheta - \chi^2/2}(\aleph + C)}{2\vartheta} \right) - \chi}{2\vartheta} \right). \quad (14)$$

Case 3. When $\chi^2 - 4\vartheta > 0, \vartheta = 0$ and $\chi \neq 0$, the hyperbolic function solutions are expressed as follows:

$$\phi_3(\aleph) = -\ln \left(\frac{\chi}{\cosh(\chi(\aleph + I)) + \sinh(\chi(\aleph + I)) - 1} \right). \quad (15)$$

Case 4. When $\chi^2 - 4\vartheta = 0, \vartheta \neq 0$ and $\chi \neq 0$, the rational function solutions are expressed as follows:

$$\phi_4(\aleph) = \ln \left(-\frac{2(\chi(\aleph + I) + 2)}{\chi^2(\aleph + I)} \right). \quad (16)$$

Case 5. When $\chi^2 - 4\vartheta = 0, \vartheta = 0$ and $\chi = 0$, we have the following:

$$\phi_5(\aleph) = \ln(\aleph + I). \quad (17)$$

From the above cases, the integration constant is represented by I . By the substitution of Eq. (11) into the reduced Eq. (7) and collecting all terms together that are in the same order of $\exp(-\varphi(\aleph))^j (j=0, 1, 2, \dots)$, the polynomial in terms of $\exp(-\varphi(\aleph))$ is verified. Then, by equating all coefficients to 0, a set of algebraic equations is constructed for $p_j (j=0, 1, \dots)$.

m), χ , δ and ϑ . With the help of Maple, Wolfram Mathematica, or MATLAB, the system can be solved to obtain diverse exact solutions for Eq. (6).

3. Solutions of the (3 + 1)-Dimensional Conformable WBBMEq

Exact soliton solutions of the proposed Eq. (2) are obtained in this section via GKUM and ExpM.

$$p\gamma\delta\Psi''' + q(\Psi^3)' + (-\gamma + p)\Psi' = 0. \quad (18)$$

By integrating Eq. (18) once with respect to $\aleph$, we obtain the following:

$$p\gamma\delta\Psi'' + q\Psi^3(-\delta + p)\Psi = 0. \quad (19)$$

3.1. Exact Solutions via the Generalized Kudryashov Method. According to the HBPrp, it is obvious to have $J = W + 1$. Let us set $W = 1$, and we obtain $J = 2$. Thus, the solution can be written as follows:

$$\Psi(\aleph) = \frac{p_0 + p_1\hbar + p_2\hbar^2}{q_0 + q_1\hbar}, \quad (20)$$

where $\hbar = \hbar(\aleph)$ is the solution of Eq. (9). As a result, by substituting Eq. (20) into Eq. (19) and using Eq. (9), we obtain system of algebraic equations by equating all coefficients of the functions $\hbar^0, \hbar^1, \hbar^2, \hbar^3, \hbar^4, \hbar^5, \hbar^6$ to 0. Now, p_0, p_1, p_2, q_0 , and q_1 are all parameters.

$$\begin{aligned} \hbar^6 : & qp_2^3 + 2p\gamma\delta p_2 q_1^2 = 0, \\ \hbar^5 : & 3qp_1 p_2^2 - 3p\gamma\delta p_2 q_1^2 + 6p\gamma\delta p_2 q_0 q_1 = 0, \\ \hbar^4 : & -9p\gamma\delta p_2 q_0 q_1 - \delta p_2 q_1^2 + p p_2 q_1^2 + 6p\gamma\delta p_2 q_0^2 \\ & + p\gamma\delta p_2 q_1^2 + 3qp_0 p_2^2 + 3qp_1^2 p_2 = 0, \\ \hbar^3 : & -\delta p_1 q_1^2 + 2p\gamma\delta p_1 q_0^2 + qp_1^3 - 10p\gamma\delta p_2 q_0^2 \\ & + p\gamma\delta \hbar_1 q_0 q_1 - p\gamma\delta q_1^2 p_0 + p p_1 q_1^2 - 2\delta p_2 q_0 q_1 \\ & + 3p\gamma\delta p_2 q_0 q_1 + 6qp_0 p_1 p_2 + 2p p_2 q_0 q_1 \\ & - 2p\gamma\delta q_1 p_0 q_0 = 0, \\ \hbar^2 : & -2\delta p_1 q_0 q_1 - p\gamma\delta p_1 q_0 q_1 + p p_0 q_1^2 + 3qp_0^2 p_2 \\ & - \delta p_0 q_1^2 + 3qp_0 p_1^2 - 3p\gamma\delta p_1 q_0^2 + 2p p_1 q_0 q_1 \\ & + p p_2 q_0^2 - \delta p_2 q_0^2 + 4p\gamma\delta p_2 q_0^2 + p\gamma\delta q_1^2 p_0 \\ & + 3p\gamma\delta q_1 p_0 q_0 = 0, \\ \hbar^1 : & -p\gamma\delta q_1 p_0 q_0 + 2p p_0 q_0 q_1 + p\gamma\delta p_1 q_0^2 - \delta p_1 q_0^2 \\ & + 3qp_0^2 p_1 - 2\delta p_0 q_0 q_1 + p p_1 q_0^2 = 0, \\ \hbar^0 : & p p_0 q_0^2 + qp_0^3 = 0. \end{aligned} \quad (21)$$

From the above set of algebraic equations, various cases are presented as follows:

Case 1.

$$\begin{aligned} p_0 = 0, p_1 &= \pm p q_1 \sqrt{-\frac{\gamma}{2q + p\gamma}}, \\ p_2 &= \pm \frac{2p\gamma q_1}{(p\gamma + 2)q\sqrt{-\gamma/2q + p\gamma}}, \\ q_0 = 0, q_1 &= q_1, \delta = \frac{2p}{p\gamma + 2}. \end{aligned} \quad (22)$$

Then, by the substitution of the obtained values into Eq. (20) with Eq. (10), the soliton-type solutions of the following WBBMEq in the sense of ComD are as follows:

$$\Psi_1(x, y, z, t) = \pm \frac{(1 - I_1(\cosh(\aleph) + \sinh(\aleph)))p\gamma}{\sqrt{-\gamma/q(p\gamma + 2)}q(p\gamma + 2)(1 + I_1(\cosh(\aleph) + I_1 \sinh(\aleph)))}, \quad (23)$$

where $\aleph = p(x^\zeta/\zeta) + q(y^\zeta/\zeta) + \gamma(z^\zeta/\zeta) - (2p/p\gamma + 2)t^\zeta/\zeta$ for ComD. I_1 is an arbitrary constant.

Case 2.

$$\begin{aligned} p_0 = 0, p_1 &= \pm 2q_0 p \sqrt{\frac{2\gamma}{-q + p\gamma}}, \\ p_2 &= \mp 2q_0 p \sqrt{\frac{2\gamma}{-q + p\gamma}}, \\ q_0 = q_0, q_1 &= -2q_0, \delta = \frac{p}{1 - p\gamma}. \end{aligned} \quad (24)$$

Then, by the substitution of the obtained values into Eq. (20) with Eq. (10), the soliton-type solutions of the following WBBMEq in the sense of ComD are as follows:

$$\begin{aligned} \Psi_2(x, y, z, t) &= \pm \frac{2pI_1 \sqrt{2\gamma/q(-1 + p\gamma)}(\cosh(\aleph) + \sinh(\aleph))}{-1 + I_1^2(2 \cosh(\aleph) \sinh(\aleph) + 2 \cosh^2(\aleph) - 1)}, \end{aligned} \quad (25)$$

where $\aleph = p(x^\zeta/\zeta) + q(y^\zeta/\zeta) + \gamma(z^\zeta/\zeta) - (p/1 - p\gamma)t^\zeta/\zeta$ for ComD. I_1 is a constant.

3.2. Exact Solutions via the $\exp(-\varphi(\aleph))$ Method. The ExpM is a very helpful technique to construct the solution of Eq. (7) in the following form:

$$\Psi(\xi) = p_0 + p_1 \exp(-\varphi(\aleph)). \quad (26)$$

Let us substitute Eq. (19) into Eq. (7) and collect the coefficient of each power of $\exp(-\Phi(\aleph))^j$. Now, a set of algebraic equations of $p, q, \gamma, \delta, p_0, p_1, \chi$ and ϑ is obtained by equating all coefficients to 0.

$$\begin{aligned} \exp(3\aleph): q p_0^3 \delta p_0 + p \gamma \delta p_1 \vartheta \chi + p p_0 &= 0, \\ \exp(2\aleph): 3 q p_0^2 p_1 - \delta p_1 + 2 p \gamma \delta p_1 \vartheta + p \gamma \delta p_1 \chi^2 + p p_1 &= 0, \\ \exp(\aleph): 3 p \gamma \delta p_1 \chi + 3 q p_0 p_1^2 &= 0, \\ \exp(0\aleph): 2 p \gamma \delta p_1 + q p_1^3 &= 0. \end{aligned} \quad (27)$$

With the help of software programs such as Maple or Mathematica, the solution is obtained as follows:

$$\begin{aligned} p_0 &= \pm \chi p \sqrt{\frac{\gamma}{-p q \gamma \chi^2 - 2q + 4p q \gamma \vartheta}}, \\ p_1 &= \pm 2p \sqrt{\frac{\gamma}{-p q \gamma \chi^2 - 2q + 4p q \gamma \vartheta}}, \\ \delta &= \frac{2p}{p \gamma \chi^2 + 2 - 4p \gamma \vartheta}. \end{aligned} \quad (28)$$

From all the above obtained values, the technique's algorithm, and its auxiliary equations, different cases for the conformable WBBMEq are given as follows:

Case 1. When $\chi^2 - 4\vartheta > 0$ and $\vartheta \neq 0$, the hyperbolic function solutions are expressed as follows:

$$\Psi_1(x, y, z, t) = \pm \frac{p \sqrt{\gamma/q(-p \gamma \chi^2 - 2 + 4p \gamma \vartheta)} \left(\chi^2 + \chi \tanh \left(\sqrt{\chi^2 - 4\vartheta/2}(\aleph + I) \right) \sqrt{\chi^2 - 4\vartheta - 4\vartheta} \right)}{\chi + \tanh \left(\sqrt{\chi^2 - 4\vartheta/2}(\aleph + I) \right) \sqrt{\chi^2 - 4\vartheta}}, \quad (29)$$

where $\aleph = p(x^\varsigma/\varsigma) + q(y^\varsigma/\varsigma) + \gamma(z^\varsigma/\varsigma) - (2p/p \gamma \chi^2 + 2 - 4p \gamma \vartheta)t^\varsigma/\varsigma$ for ComD. I is a constant.

Case 2. When $\chi^2 - 4\vartheta < 0$ and $\vartheta \neq 0$, the trigonometric function solutions are expressed as follows:

$$\Psi_2(x, y, z, t) = \pm \frac{p \sqrt{\gamma/q(-p \gamma \chi^2 - 2 + 4p \gamma \vartheta)} \left(-\chi^2 + \chi \tan \left(\sqrt{4\vartheta - \chi^2/2}(\aleph + I) \right) \sqrt{4\vartheta - \chi^2 + 4\vartheta} \right)}{-\chi + \tan \left(\sqrt{4\vartheta - \chi^2/2}(\aleph + I) \right) \sqrt{4\vartheta - \chi^2}}, \quad (30)$$

where $\aleph = p(x^\varsigma/\varsigma) + q(y^\varsigma/\varsigma) + \gamma(z^\varsigma/\varsigma) - (2p/p \gamma \chi^2 + 2 - 4p \gamma \vartheta)t^\varsigma/\varsigma$ for ComD. I is a constant.

Case 3. When $\chi^2 - 4\vartheta > 0, \vartheta = 0$ and $\chi \neq 0$, the hyperbolic function solutions are expressed as follows:

$$\Psi_3(x, y, z, t) = \pm \frac{\chi p \sqrt{\gamma/q(-p \gamma \chi^2 - 2 + 4p \gamma \vartheta)} (\cosh(\chi(\aleph + I)) + \sinh(\chi(\aleph + I)) + 1)}{\cosh(\chi(\aleph + I)) + \sinh(\chi(\aleph + I)) - 1}, \quad (31)$$

where $\aleph = p(x^\varsigma/\varsigma) + q(y^\varsigma/\varsigma) + \gamma(z^\varsigma/\varsigma) - (2p/p \gamma \chi^2 + 2 - 4p \gamma \vartheta)t^\varsigma/\varsigma$ for ComD. I is a constant.

where $\aleph = p(x^\varsigma/\varsigma) + q(y^\varsigma/\varsigma) + \gamma(z^\varsigma/\varsigma) - (2p/p \gamma \chi^2 + 2 - 4p \gamma \vartheta)t^\varsigma/\varsigma$ for ComD. I is a constant.

Case 4. When $\chi^2 - 4\vartheta = 0, \vartheta \neq 0$ and $\chi \neq 0$, the rational function solutions are expressed as follows:

$$\Psi_4(x, y, z, t) = \pm \frac{2 \chi p \sqrt{\gamma/q(-p \gamma \chi^2 - 2 + 4p \gamma \vartheta)}}{\chi(\aleph + I) + 2}, \quad (32)$$

Case 5. Finally, when $\chi^2 - 4\vartheta = 0, \vartheta = 0$ and $\chi = 0$, we have the following:

$$\Psi_5(x, y, z, t) = \pm \frac{\sqrt{\gamma/q(-p \gamma \chi^2 - 2 + 4p \gamma \vartheta)} p (\chi(\aleph + I) + 2)}{\aleph + I}, \quad (33)$$

FIGURE 1: The plots of Eq. (23) are represented in 3D in (a), (c), and (e) and in 2D in (b), (d), and (f) for $\gamma = -1$; $q = p = I_1 = 1$; $t = 15$; $y = z = 0$; $\zeta = 0.50$; $\zeta = 0.80$; $\zeta = 0.90$ for ComD, respectively.

FIGURE 2: The plots of Eq. (25) are represented in 3D in (a), (c), and (e) and in 2D in (b), (d), and (f) for $\gamma = -1$; $q = p = I_1 = 1$; $t = 15$; $y = z = 0$; $\zeta = 0.50$; $\zeta = 0.80$; $\zeta = 0.90$ for ComD, respectively.

where $\aleph = p(x^\zeta/\zeta) + q(y^\zeta/\zeta) + \gamma(z^\zeta/\zeta) - (2p/p\gamma\chi^2 + 2 - 4p\gamma\vartheta)t^\zeta/\zeta$ for ComD. I is a constant.

4. The Graphical Comparisons of Solutions

To show the dynamics and behavior of our obtained solutions, various soliton solutions in Eqs. (23), (25), (29), (30), (31), (32), and (33) are graphically represented and compared in both 3D and 2D plots in Figures 1–7 for various parameters' values and ζ . In this work, two techniques are employed in the sense of ComD. Therefore, all our results are new and worthy because studying WBBMEq is very important to investigate various nonlinear scientific phe-

nomena. ComD is a local-type fractional derivative which is a generalized formulation of usual limit-based derivative. Since the measurements in physics are local, this can make ComD suitable for modeling many physical phenomena, and it is also efficient to work with ComD to obtain exact solutions for NLPDEs although ComD does not have some of the essential properties to be categorized as a fractional derivative. From the authors' opinion, formulating and solving NLPDEs in the sense of ComD are always recommended since all local and nonlocal fractional derivatives have both advantages and disadvantages. Therefore, exploring new properties and definitions for all local and nonlocal fractional derivatives are helpful while working on modeling

FIGURE 3: The plots of Eq. (29) are represented in 3D in (a), (c), and (e) and in 2D in (b), (d), and (f) for $\gamma = 1$; $\vartheta = -1$; $q = p = \chi = I = 1$; $t = 15$; $y = z = 0$; $\zeta = 0.50$; $\zeta = 0.75$; $\zeta = 0.90$ for ComD, respectively.

nonlinear scientific phenomena. In addition, studying the ComD's definition is also interesting because any new mathematical definition deserves to be explored and investigated to show its validity and potentiality for more application.

5. Conclusion

Two novel techniques of generalized Kudryashov and $\exp(-\varphi(\mathcal{N}))$ have been applied in this work to investigate

FIGURE 4: The plots of Eq. (30) are represented in 3D in (a), (c), and (e) and in 2D in (b), (d), and (f) for $\gamma = 1$; $\vartheta = 1$; $q = p = \chi = I = 1$; $t = 15$; $y = z = 0$; $\zeta = 0.50$; $\zeta = 0.75$; $\zeta = 0.90$ for ComD, respectively.

FIGURE 5: The plots of Eq. (31) are represented in 3D in (a), (c), and (e) and in 2D in (b), (d), and (f) for $\gamma = -1$; $\vartheta = 0$; $q = p = \chi = I = 1$; $t = 15$; $y = z = 0$; $\zeta = 0.50$; $\zeta = 0.75$; $\zeta = 0.90$ for ComD, respectively.

FIGURE 6: The plots of Eq. (32) are represented in 3D in (a), (c), and (e) and in 2D in (b), (d), and (f) for $\gamma = 1$; $\vartheta = 0$; $q = p = \chi = I = 1$; $t = 15$; $y = z = 0$; $\zeta = 0.50$; $\zeta = 0.75$; $\zeta = 0.90$ for ComD, respectively.

FIGURE 7: The plots of Eq. (33) are represented in 3D in (a), (c), and (e) and in 2D in (b), (d), and (f) for $\gamma = -1$; $\vartheta = 0$; $q = p = I = 1$; $\chi = 0$; $t = 15$; $y = z = 0$; $\zeta = 0.50$; $\zeta = 0.75$; $\zeta = 0.90$ for ComD, respectively.

exact soliton solutions of the $(3 + 1)$ -dimensional conformable WBBMEq in the sense of ComD. The obtained solutions are new which imply that the studied techniques provide efficient results. All algebraic computations in this work have been done using the Maple software. Graphical representations have been provided for all obtained solutions at various parameters' values and conformable orders with the help of Wolfram Mathematica. All in all, the studied methods can be potentially applied to solve various NLPDEs that are apparent in many important nonlinear scientific phenomena in physics and engineering. Our results can be further extended in future research works into solving various classes of higher dimensional nonlinear partial differential equations which will provide a major contribution to soliton theory and mathematical physics.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

The authors declare that the study was realized in collaboration with equal responsibility. All authors read and approved the final manuscript.

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