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Research Article

Asymmetric Truncated Hankel Operators: Rank One, Matrix Representation

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Asymmetric truncated Hankel operators are the natural generalization of truncated Hankel operators. In this paper, we determine all rank one operators of this class. We explore these operators on finite-dimensional model spaces, in particular, their matrix representation. We also give their matrix representation and the one for asymmetric truncated Toeplitz operators in the case of model spaces associated to interpolating Blaschke products.

1. Introduction

Let H^2 be the standard Hardy space of the unit disc $\mathbb{D}$ identified with the subspace of the boundary functions of its functions in $L^2(\mathbb{T})$.

A function in $H^\infty(\mathbb{D})$ is inner if it is unimodular on the unit circle $\mathbb{T}$. To each inner function α we associate a model space, $K_\alpha = H^2 \ominus \alpha H^2$. The model space is a reproducing kernel Hilbert space with reproducing kernel, $k_\lambda^\alpha(z) = (1 - \overline{\alpha(\lambda)}\alpha(z))/(1-\bar{\lambda}z)$, for z, $\lambda \in \mathbb{D}$.

The inner function α has an angular derivative in the sense of Carathéodory (ADC) at a point $\eta \in \mathbb{T}$ if and only if every f in K_α has nontangential limit at η. In particular, $k_\eta^\alpha \in K_\alpha$.

If α is an inner function, then $\alpha^\# = \overline{\alpha(\bar{z})}$ is also an inner function and the associated model space is noted $K_{\alpha^\#}$.

To each model space, we associate a natural conjugation C_α such that $C_\alpha K_\alpha = K_\alpha$, given by $C_\alpha f(z) = \alpha \overline{z f(z)}$, for $z \in \mathbb{T}$. The image of the kernel function for C_α, called conjugate kernel function $\tilde{k}_\lambda^\alpha$, is given by $\tilde{k}_\lambda^\alpha(z) = (\alpha(z) - \alpha(\lambda))/(z-\lambda)$, z,$\lambda \in \mathbb{D}$. If α has an ADC at $\eta \in \mathbb{T}$,

$$\tilde{k}_\eta^\alpha(z) = \frac{\alpha(z)-\alpha(\eta)}{z-\eta} = \alpha(\eta)\bar{\eta}\frac{1-\overline{\alpha(\eta)}\alpha(z)}{1-\bar{\eta}z} = \alpha(\eta)\bar{\eta}k_\eta^\alpha(z). \quad (1)$$

The model spaces can also be defined as the only invariant subspaces of the backward shift operator S^* on H^2. Denote by S_α (S_α^*) the compression of the shift operator (resp., backward shift) to the model space K_α.

The only finite-dimensional model spaces are the one associated to finite Blaschke products, and its dimension is the same as the multiplicity of the associated Blaschke product. Since a finite Blaschke product is analytic on a neighborhood of the unit disc, it has an ADC everywhere on $\mathbb{T}$ and $k_\eta^\alpha \in K_\alpha$, for every $\eta \in \mathbb{T}$. In this case, let m denote the multiplicity of a finite Blaschke product α. If we arbitrarily choose a collection $\{\lambda_i, i=1,\cdots,m\}$ of distinct points in $\mathbb{D}$, $\{k_{\lambda_i}^\alpha, i=1,2,..,m\}$ forms a basis for K_α.

In infinite-dimensional case the kernel functions $(k_{\lambda_i}^\alpha)_{i\ge1}$ form a Riesz basis if and only if α is an interpolating Blaschke product or equivalently, $(\lambda_i)_{i\ge1}$ is a uniformly separated Blaschke sequence that satisfies $\inf_{k\ge1}\prod_{k\ge,k\ne n}|(\lambda_k-\lambda_n)/(1-\overline{\lambda_k}\lambda_n)|>0$.

Asymmetric truncated Toeplitz and Hankel operators were first introduced in [1, 2], respectively. For $\varphi \in L^2(\mathbb{T})$, let α and β be two inner functions, the asymmetric truncated Toeplitz operator $A_\varphi^{\alpha,\beta}$ and the asymmetric truncated Hankel operator $B_\varphi^{\alpha,\beta}$ are defined on $K_\alpha^\infty = K_\alpha \cap H^\infty$ by

$$A_{\varphi}^{\alpha,\beta}: K_{\alpha}^{\infty} \longrightarrow K_{\beta}, \tag{2}$$

$$f \mapsto A_{\varphi}^{\alpha,\beta}(f) = P_{\beta}(\varphi f), \tag{3}$$

$$B_{\varphi}^{\alpha,\beta}: K_{\alpha}^{\infty} \longrightarrow K_{\beta}, \tag{4}$$

$$f \mapsto B_{\varphi}^{\alpha,\beta}(f) = P_{\beta} J(I-P)(\varphi f), \tag{5}$$

where P and P_{β} denote the orthogonal projection on H^2 and K_{β}, respectively, and J is the flip operator from H^2 onto $\bar{H}_0^2 = \overline{zH^2}$ defined on $\mathbb{T}$ by $Jf(z) = \bar{z}f(\bar{z})$. We denote the set of bounded asymmetric truncated Toeplitz operators (ATTO) (bounded asymmetric truncated Hankel operators (ATHO)) by $\mathfrak{T}(\alpha,\beta)$ ($\mathfrak{H}(\alpha,\beta)$).

In [3], Sarason introduced truncated Toeplitz operators and showed that all rank one TTOs are of the form $ck_{\lambda}^{\alpha} \otimes \tilde{k}_{\lambda}^{\alpha}$ or $c\tilde{k}_{\lambda}^{\alpha} \otimes k_{\lambda}^{\alpha}$, where c in $\mathbb{C}$ and λ in $\mathbb{D}$ or in $\mathbb{T}$ such that α has an ADC at λ. Similar results were obtained in [4] for truncated Hankel operators ($\tilde{k}_{\bar{\lambda}}^{\alpha} \otimes \tilde{k}_{\lambda}^{\alpha}, k_{\bar{\lambda}}^{\alpha} \otimes k_{\lambda}^{\alpha}$). Surprisingly, this is not always true in the case of asymmetric truncated Toeplitz operators.

Theorem 1 (see [5, 6]). *Let α and β be two inner functions.*

(1) Let ω in $\mathbb{D}$ or ω in $\mathbb{T}$ such that α and β has an ADC at ω

$$k_{\omega}^{\beta} \otimes \tilde{k}_{\omega}^{\alpha} = A_{\frac{\alpha}{\overline{z\omega}}}^{\alpha,\beta} \in \mathfrak{T}(\alpha,\beta) \text{ and } \tilde{k}_{\omega}^{\beta} \otimes k_{\omega}^{\alpha} = A_{\frac{\beta}{z-\omega}}^{\alpha,\beta} \in \mathfrak{T}(\alpha,\beta). \tag{6}$$

(2) The only rank one asymmetric truncated Toeplitz operators in $\mathfrak{T}(\alpha,\beta)$ are nonzero scalar multiple of $k_{\omega}^{\beta} \otimes \tilde{k}_{\omega}^{\alpha}$ and $\tilde{k}_{\omega}^{\beta} \otimes k_{\omega}^{\alpha}$ where $\omega \in \mathbb{D}$ or $\omega \in \mathbb{T}$ such that α and β have an ADC at ω if and only if $\{mn \le 2\}$ or $\{m > 1 \text{ and } n > 1\}$, where $m, n \in \mathbb{N} \cup \{+\infty\}$ are the dimensions of K_{α} and K_{β}, respectively

In [7], Cima et al. raised the question of which linear transformations on finite-dimensional model spaces are truncated Toeplitz operators, and they proved that the matrix representation of TTOs with respect to kernel basis (conjugate kernel basis, Clark and modified Clark bases) is entirely determined by the entries of the main diagonal and first row. In [8, 9], Lanucha obtained similar results for TTOs acting on model spaces associated to interpolating Blaschke products and for THOs. In [6], Jurasik and Lanucha generalized these results to asymmetric truncated Toeplitz operators on finite-dimensional model spaces.

This paper determines all rank one asymmetric truncated Hankel operators and generalizes the results about matrix representation to ATHOs and ATTOs on special model spaces. In Section 2, we cite some results from [3, 10–12]. We precise all rank one ATHOs in Section 3. In Section 4, we explore the ATHOs on finite-dimensional spaces, we calculate the dimension of $\mathfrak{H}(\alpha,\beta)$ and exhibit a basis for it, and we give the matrix representation of ATHOs in finite dimensional model spaces associated to Blaschke products each with distinct zeros. Section 5 is dedicated to the matrix representation of ATHOs and ATTOS on special infinite-dimensional model spaces, in particular, we study the action of the unitary operator $V_{\xi,c} = \sqrt{\tau'_{\xi,c}} f \circ \tau_{\xi,c}$, where $\tau_{\xi,c}(z)$ is disc automorphism.

2. Preliminaries

Like for truncated Toeplitz and Hankel operators, the symbol of an ATTO and an ATHO is not unique, in fact, we have the following theorem.

Theorem 2 (see [10, 12]). *Let α and β be inner functions. For $\varphi \in L^2(\mathbb{T})$, we have $A_{\varphi}^{\alpha,\beta} = 0$ if and only if $\varphi \in \alpha\overline{H^2} + \beta H^2$ and $B_{\varphi}^{\alpha,\beta} = 0$ if and only if $\varphi \in H^2 + \overline{\alpha\beta^{\#} H^2}$. Where $\beta^{\#}(z) = \overline{\beta(\bar{z})}$.*

We will use the following technical lemma.

Lemma 3 (see [3]). *Let α be inner, for $\lambda \in \mathbb{D}$, we have*

$$S_{\alpha}^{*} k_{\lambda}^{\alpha} = \bar{\lambda} k_{\lambda}^{\alpha} - \overline{\alpha(\lambda)} \widetilde{k_0^{\alpha}}, \quad \text{and} \quad S_{\alpha} \tilde{k}_{\lambda}^{\alpha} = \lambda \tilde{k}_{\lambda}^{\alpha} - \alpha(\lambda) k_0^{\alpha}. \tag{7}$$

For $\lambda \in \mathbb{D} \setminus \{0\}$, we have

$$S_{\alpha}^{*} \widetilde{k_{\lambda}^{\alpha}} = \frac{1}{\bar{\lambda}} \widetilde{k_{\lambda}^{\alpha}} - \frac{1}{\lambda} \widetilde{k_0^{\alpha}}. \tag{8}$$

Clark, in [11], proved that the only unitary rank one perturbations of the compressed shift operator are

$$U_{\lambda}^{\alpha} = S_{\alpha} + \frac{\alpha(0) + \lambda}{1 - |\alpha(0)|^2} k_0^{\alpha} \otimes \widetilde{k_0^{\alpha}}, \quad \text{for} \quad \lambda \in \mathbb{T}, \tag{9}$$

and that the point spectrum of U_{λ}^{α} is the set of the solutions of the equation $\alpha(\eta) = \alpha(0) + \lambda / 1 + \overline{\alpha}(0)\lambda$ at which α has an ADC, denote them by η_m. The corresponding eigenvectors are the normalized boundary kernels $v_{\eta_m}^{\alpha} \coloneqq \|k_{\eta_m}^{\alpha}\|^{-1} k_{\eta_m}^{\alpha}$ corresponding to the points η_m. Whenever the point spectrum of U_{λ}^{α} is pure, the family of eigenvectors forms a basis for K_{α} called the Clark basis.

The modified Clark basis $(e_{\eta_m}^{\alpha})_m$ satisfies $C_{\alpha} e_{\eta_m}^{\alpha} = e_{\eta_m}^{\alpha}$ and is given by $e_{\eta_m}^{\alpha} = \omega_m^{\alpha} v_{\eta_m}^{\alpha}$, where $\omega_m^{\alpha} = \exp(-i(\arg \eta_m - \arg \lambda))$.

One of the most important results about TTOs and THOs is their characterization in terms of compressed shift operator and operators of rank at most 2. Recently, the authors in [10] proved similar characterizations for both ATTO and ATHO.

Theorem 4 (see [10]). *Let A be a bounded linear operator from K_{α} to K_{β}. Then*

(1) $A \in \mathfrak{T}(\alpha, \beta)$ *if and only if*

$$A - S_\beta A S_\alpha^* = \psi \otimes k_0^\alpha + k_0^\beta \otimes \chi, \tag{10}$$

for some $\chi \in K_\alpha, \psi \in K_\beta$ *if and only if*

$$A - U_{\lambda_\beta}^\beta A \left(U_{\lambda_\alpha}^\beta\right)^* = \psi \otimes k_0^\alpha + k_0^\beta \otimes \chi, \tag{11}$$

for some $\chi \in K_\alpha, \psi \in K_\beta$ *and some* $\lambda_\alpha, \lambda_\beta$ *in* $\mathbb{T}$.

(2) $B \in \mathfrak{H}(\alpha, \beta)$ *if and only if*

$$B - S_\beta^* B S_\alpha^* = \psi \otimes k_0^\alpha + \tilde{k}_0^\beta \otimes \chi, \tag{12}$$

for some $\chi \in K_\alpha, \psi \in K_\beta$ *if and only if*

$$B - \left(U_{\lambda_\beta}^\beta\right)^* B \left(U_{\lambda_\alpha}^\alpha\right)^* = \psi \otimes k_0^\alpha + \tilde{k}_0^\beta \otimes \chi, \tag{13}$$

for some $\chi \in K_\alpha, \psi \in K_\beta$ *and some* $\lambda_\alpha, \lambda_\beta$ *in* $\mathbb{T}$.

Where $U_{\lambda_\alpha}^\alpha, U_{\lambda_\beta}^\beta$ *are Clark operators (9).*

In the same paper [10], we also have the following theorem.

Theorem 5 (see [10]). *Let* α *and* β *be two inner functions and* $\varphi \in L^2(\mathbb{T})$. *Then*

(1) $A \in \mathfrak{T}(\alpha, \beta)$ *if and only if* $C_\beta A C_\alpha \in \mathfrak{T}(\alpha, \beta)$. *In addition,* $C_\beta A_\varphi^{\alpha,\beta} C_\alpha = A^{\alpha,\beta}_{\overline{\alpha\beta\varphi}}$

(2) $B \in \mathfrak{H}(\alpha, \beta)$ *if and only if* $C_\beta B C_\alpha \in \mathfrak{H}(\alpha, \beta)$. *In addition,* $C_\beta B_\varphi^{\alpha,\beta} C_\alpha = B^{\alpha,\beta}_{\overline{\alpha\beta^\#}\varphi}$

(3) $C_\beta A_\varphi^{\alpha,\beta} J^\# = B^{\alpha^\#,\beta}_{\overline{\beta^\#}\varphi^\#}$, *where* $J^\#$ *is a conjugation on* L^2 *defined by* $J^\# f = f(\bar{z})$

3. Asymmetric Truncated Hankel Operators of Rank One

In this section, we describe all rank one asymmetric truncated Hankel operators through the results for asymmetric truncated Toeplitz operators. In what follows, let α and β denote two arbitrary inner functions.

The following proposition gives some rank one asymmetric truncated Hankel operators.

Proposition 6. *We have*

(1) *For every* $\lambda \in \mathbb{D}$, *the operators* $k_{\bar\lambda}^\beta \otimes k_\lambda^\alpha = B^{\alpha,\beta}_{1/z-\lambda}$ *and* $\tilde{k}_{\bar\lambda}^\beta \otimes \tilde{k}_\lambda^\alpha = B^{\alpha,\beta}_{\alpha\beta^\#/z\bar\lambda}$ *belong to* $\mathfrak{H}(\alpha, \beta)$

(2) *If* α *and* β *have an ADC at* η *and* $\bar\eta$, *respectively, the operators* $\tilde{k}_{\bar\eta}^\beta \otimes \tilde{k}_\eta^\alpha = B^{\alpha,\beta}_{\overline{\alpha\beta^\#}/z-\eta} = B^{\alpha,\beta}_{\overline{\alpha\beta^\#}/\tilde{k}_\eta}$ *and* $k_{\bar\eta}^\beta \otimes k_\eta^\alpha = B^{\alpha,\beta}_{\overline{\alpha\beta^\#}/\eta\tilde{k}_\eta}$ *belong to* $\mathfrak{H}(\alpha, \beta)$

Proof. From Theorem 5, we have

$$C_\beta A_\varphi^{\alpha,\beta} J^\# = B^{\alpha^\#,\beta}_{\overline{\beta^\#}\varphi^\#}, \tag{14}$$

or

$$C_\beta A_\varphi^{\alpha^\#,\beta} J^\# = B^{\alpha,\beta}_{\overline{\beta^\#}\varphi^\#}. \tag{15}$$

Choose an arbitrary $\lambda \in \mathbb{D}$ or $\lambda \in \mathbb{T}$ such that $\alpha^\#$ and β have an ADC at λ, by Theorem 1, we have

$$C_\beta \left(k_\lambda^\beta \otimes \tilde{k}_\lambda^{\alpha^\#}\right) J^\# = C_\beta A^{\alpha^\#,\beta}_{\frac{\overline{\alpha^\#}}{z\lambda}} J^\# = B^{\alpha,\beta}_{\frac{\overline{\alpha\beta^\#}}{\bar{z}-\lambda}} = \tilde{k}_\lambda^\beta \otimes \tilde{k}_{\bar\lambda}^\alpha, \tag{16}$$

$$C_\beta \left(\tilde{k}_\lambda^\beta \otimes k_\lambda^{\alpha^\#}\right) J^\# = C_\beta A^{\alpha^\#,\beta}_{\frac{\bar\beta}{z-\lambda}} J^\# = B^{\alpha,\beta}_{\frac{1}{\bar{z}-\lambda}} = k_\lambda^\beta \otimes k_{\bar\lambda}^\alpha. \tag{17}$$

For $\lambda \in \mathbb{D}$, the operators $\tilde{k}_\lambda^\beta \otimes \tilde{k}_{\bar\lambda}^\alpha$ and $k_\lambda^\beta \otimes k_{\bar\lambda}^\alpha$ belong to $\mathfrak{H}(\alpha, \beta)$. This is also true for $\lambda \in \mathbb{T}$, since $\alpha^\#$ has an ADC at λ, if and only if α has an ADC at $\bar\lambda$. ?

Now, we give the main theorem of this section.

Theorem 7. *All rank one operators in* $\mathfrak{H}(\alpha, \beta)$ *are the nonzero scalar multiples of* $k_{\bar\lambda}^\beta \otimes k_\lambda^\alpha$ *and* $\tilde{k}_{\bar\lambda}^\beta \otimes \tilde{k}_\lambda^\alpha$, *where* $\lambda \in \mathbb{D}$ *or* $\lambda \in \mathbb{T}$ *such that* α *and* β *have an ADC at* λ *and* $\bar\lambda$, *respectively, if and only if* $\{mn \le 2\}$ *or* $\{m > 1 \text{ and } n > 1\}$, *where* $m, n \in \mathbb{N} \cup \{+\infty\}$ *are the dimensions of* K_α *and* K_β, *respectively.*

Proof. Every rank one operator in $\mathfrak{H}(\alpha, \beta)$ is of the form $f \otimes g$ for $f \in K_\beta$ and $g \in K_\alpha$. Since $\mathfrak{H}(\alpha, \beta) = C_\beta \mathfrak{T}(\alpha^\#, \beta) J^\#$ ([10]), we can find $f' \in K_\beta$ and $g' \in K_{\alpha^\#}$ such that $f \otimes g = C_\beta f' \otimes g' J^\#$. For $f' \otimes g'$ is rank one in $\mathfrak{T}(\alpha^\#, \beta)$, by Theorem 1, there exist $c \in \mathbb{C}$ and $\lambda \in \mathbb{D}$ or $\lambda \in \mathbb{T}$ ($\alpha^\#, \beta$ have an ADC at λ) such that $f' \otimes g' = c k_\lambda^\beta \otimes \tilde{k}_\lambda^{\alpha^\#}$ or $f' \otimes g' = c\tilde{k}_\lambda^\beta \otimes k_\lambda^{\alpha^\#}$, if and only if $\{mn \le 2\}$ or $\{m > 1 \text{ and } n > 1\}$. Finally, $f \otimes g = cC_\beta(k_\lambda^\beta \otimes \tilde{k}_\lambda^{\alpha^\#})J^\# = c\tilde{k}_\lambda^\beta \otimes \tilde{k}_{\bar\lambda}^\alpha$ or $f \otimes g = cC_\beta(\tilde{k}_\lambda^\beta \otimes k_\lambda^{\alpha^\#})J^\# = ck_\lambda^\beta \otimes k_{\bar\lambda}^\alpha$ if and only if $\{mn \le 2\}$ or $\{m > 1 \text{ and } n > 1\}$. ☐

4. Asymmetric Truncated Hankel Operators in Finite-Dimensional Model Spaces

In this section, we suppose that both α and β are finite Blaschke products of respective multiplicities m and n.

4.1. Dimension and Basis of $\mathfrak{H}(\alpha, \beta)$

Theorem 8. *Let K_α and K_β have dimensions m and n, respectively, then the dimension of $\mathfrak{H}(\alpha, \beta)$ equals $m+n-1$.*

Proof. Using Theorem 2, we can write

$$\mathfrak{H}(\alpha, \beta) = \left\{ B^{\alpha,\beta}_{\bar{\chi}} : \chi \in K_{\alpha\beta^{\#}} \right\}, \tag{18}$$

where $\dim K_{\alpha\beta^{\#}} = m+n$. Since the constants are in $K_{\alpha\beta^{\#}}$ and $B^{\alpha,\beta}_c = 0$ for all $c \in \mathbb{C}$,

$$\dim \mathfrak{H}(\alpha, \beta) = \dim K_{\alpha\beta^{\#}} - 1 = m + n - 1. \tag{19}$$

□

Remark 9. From Theorem 4, $C_\beta \mathfrak{T}(\alpha^{\#}, \beta) J^{\#} = \mathfrak{H}(\alpha, \beta)$. Since C_β and $J^{\#}$ preserve the dimensions, $\dim \mathfrak{H}(\alpha, \beta) = \dim \mathfrak{T}(\alpha^{\#}, \beta) = m+n-1$.

Theorem 10. *Let α, β be two finite Blaschke products of respective multiplicities m and n. Then for any $m+n-1$ distinct points from $\bar{\mathbb{D}}$, denoted by $\{\lambda_i\}_{i=1}^{m+n-1}$,*

(1) $\{k^\beta_{\bar{\lambda}_i} \otimes k^\alpha_{\lambda_i}, i = 1, .., m+n-1\}$ is a basis of $\mathfrak{H}(\alpha, \beta)$

(2) $\{\tilde{k}^\beta_{\bar{\lambda}_i} \otimes \tilde{k}^\alpha_{\lambda_i}, i = 1, .., m+n-1\}$ is also a basis of $\mathfrak{H}(\alpha, \beta)$

Proof. We only prove (1), the other case follows by application of the conjugations. From [6], a basis of $\mathfrak{T}(\alpha^{\#}, \beta)$ is $\{\tilde{k}^\beta_{\bar{\lambda}_i} \otimes k^{\alpha^{\#}}_{\bar{\lambda}_i}, i = 1, \cdots, m+n-1\}$. Using the proof of Proposition 6, $\{C_\beta \tilde{k}^\beta_{\bar{\lambda}_i} \otimes k^{\alpha^{\#}}_{\bar{\lambda}_i} J^{\#} = k^\beta_{\bar{\lambda}_i} \otimes k^\alpha_{\lambda_i}, i = 1, \cdots, m+n-1\}$ is a basis of $\mathfrak{H}(\alpha, \beta)$. □

Remark 11. We can prove the result directly as in [6] for ATTOs.

4.2. Matrix Representation of ATHO on Finite-Dimensional Spaces. In this subsection, we give the matrix representation of an ATHO acting on two finite dimensional model spaces, with respect to kernel and conjugate kernel bases, Clark and modified Clark bases. In all this section, suppose that the inner function α has distinct zeros $(a_i)_{i=1}^m$ and β has distinct zeros $(b_j)_{j=1}^n$.

4.2.1. Matrix Representation with respect to the Kernel Bases and Conjugate Kernel Bases. Choose $m+n-1$ points $\{\lambda_i\}_{i=1}^{m+n-1}$ in $\mathbb{D}$ distinct from $(\bar{b}_j)_{j=1}^n$, then $\{k^\beta_{\bar{\lambda}_i} \otimes k^\alpha_{\lambda_i}, i = 1, .., m+n-1\}$ is a basis for $\mathfrak{H}(\alpha, \beta)$ (Theorem 10). Since the zeros of the considered inner functions are distinct, $\{k^\alpha_{a_j}, j = 1, \cdots, m\}$ and $\{k^\beta_{b_i}, i = 1, \cdots, n\}$ are bases of K_α and K_β (same for the conjugate kernel functions). We denote the matrix representation of a bounded operator B with respect to the abovementioned kernel bases (conjugate kernel bases) by $(s_{i,j})((p_{i,j}))$, where $1 \le i \le n$ and $1 \le j \le m$. For the asymmetric case, we also have (see [9]),

$$s_{i,j} = \frac{1}{\beta'^{(b_i)}} \left\langle B k^\alpha_{a_j}, \tilde{k}^\beta_{b_i} \right\rangle. \tag{20}$$

Theorem 12. *Let B be a bounded linear transformation from K_α to K_β.*

$B \in \mathfrak{H}(\alpha, \beta)$ if and only if for $1 \le i \le n$ and $1 \le j \le m$,

$$s_{i,j} = \frac{\bar{\beta}'(b_1)(1 - a_j \bar{b}_1) s_{1,j} - \bar{\beta}'(b_1)(1 - a_1 \bar{b}_1) s_{1,1} + \bar{\beta}'(b_i)(1 - a_1 \bar{b}_i) s_{i,1}}{\bar{\beta}'(b_i)(1 - a_j \bar{b}_i)}. \tag{21}$$

Proof. Suppose that $B = B^{\alpha,\beta}_\varphi \in \mathfrak{H}(\alpha, \beta)$ for some $\varphi \in L^2$. We show that its matrix representation with respect to the kernel basis is of the form (21). From Theorem 10, $B^{\alpha,\beta}_\varphi = \sum_{q=1}^{m+n-1} d_q k^\beta_{\bar{\lambda}_q} \otimes k^\alpha_{\lambda_q}$. We have

$$B^{\alpha,\beta}_\varphi k^\alpha_{a_j} = \left(\sum_{q=1}^{m+n-1} d_q k^\beta_{\bar{\lambda}_q} \otimes k^\alpha_{\lambda_q} \right) k^\alpha_{a_j} = \sum_{q=1}^{m+n-1} d_q \frac{1}{1 - \bar{a}_j \lambda_q} k^\beta_{\bar{\lambda}_q}. \tag{22}$$

Now replace in (20), we have

$$s_{i,j} = \frac{1}{\beta'^{(b_i)}} \sum_{q=1}^{m+n-1} d_q \frac{1}{1 - \bar{a}_j \lambda_q} \frac{\beta^{\#}(\lambda_q)}{\lambda_q - \bar{b}_i}. \tag{23}$$

We can write

$$\frac{1}{1 - \bar{a}_j \lambda_q} \frac{1}{\lambda_q - \bar{b}_i} = \frac{1}{(1 - \bar{a}_j \bar{b}_i)(1 - \lambda_q)} \left(\frac{1 - \bar{b}_i}{\lambda_q - \bar{b}_i} - \frac{1 - \bar{a}_j}{1 - \lambda_q \bar{a}_j} \right). \tag{24}$$

By adding and subtracting $1 - \bar{b}_1 / \lambda_q - \bar{b}_1$ and $1 - \bar{a}_1 / 1 - \lambda_q \bar{a}_1$, we find

$$s_{i,j} = \frac{\bar{\beta}'(b_1)(1 - a_j \bar{b}_1) s_{1,j} - \bar{\beta}'(b_1)(1 - a_1 \bar{b}_1) s_{1,1} + \bar{\beta}'(b_i)(1 - a_1 \bar{b}_i) s_{i,1}}{\bar{\beta}'(b_i)(1 - a_j \bar{b}_i)}. \tag{25}$$

Conversely, suppose that the matrix representation of the bounded transformation B satisfies (21). From the proof of the first implication, we know that the subspace of the matrices satisfying (21) is a subspace of $\mathfrak{H}(\alpha, \beta)$, and its dimension is obviously $m + n - 1$. But $\dim \mathfrak{H}(\alpha, \beta) = m + n - 1$, we have the equality. □

The following theorem establishes the matrix representation with respect to conjugate kernel bases.

Theorem 13. *Let B be a bounded linear transformation from K_α to K_β. $B \in \mathfrak{H}(\alpha, \beta)$ if and only if for $1 \le i \le n$ and $1 \le j \le m$,*

$$p_{i,j} = \frac{\beta'(b_1)(1 - a_j b_1) p_{1,j} - \beta'(b_1)(1 - a_1 b_1) p_{1,1} + \beta'(b_i)(1 - a_1 b_i) p_{i,1}}{\beta'(b_i)(1 - a_j b_i)}. \tag{26}$$

Proof. We have

$$\begin{aligned} p_{i,j} &= \frac{1}{\beta'(b_i)} \left\langle B\tilde{k}^{\alpha}_{a_j}, k^{\beta}_{b_i} \right\rangle = \frac{1}{\beta'(b_i)} \left\langle BC_\alpha k^{\alpha}_{a_j}, C_\beta \tilde{k}^{\beta}_{b_i} \right\rangle \\ &= \frac{1}{\beta'\overline{(b_i)}} \left\langle C_\beta B C_\alpha k^{\alpha}_{a_j}, \tilde{k}^{\beta}_{b_i} \right\rangle = \frac{1}{\beta'\overline{(b_i)}} \left\langle B^{\alpha,\beta}_{\alpha\bar{\beta}^{\#}\varphi} k^{\alpha}_{a_j}, \tilde{k}^{\beta}_{b_i} \right\rangle. \end{aligned} \tag{27}$$

Therefore

$$p_{i,j} = \overline{s_{i,j}}, \tag{28}$$

where $s_{i,j}$ is the entry in the ith row and jth column of the matrix representing $B^{\alpha,\beta}_{\alpha\bar{\beta}^{\#}\varphi}$ with respect to the kernel bases, by Theorem 12, it satisfies (21) and we have

$$\begin{aligned} p_{i,j} &= \overline{\frac{\bar{\beta}'(b_1)(1 - a_j \bar{b}_1) s_{1,j} - \bar{\beta}'(b_1)(1 - a_1 \bar{b}_1) s_{1,1} + \bar{\beta}'(b_i)(1 - a_1 \bar{b}_i) s_{i,1}}{\bar{\beta}'(b_i)(1 - a_j \bar{b}_i)}} \\ &= \frac{\beta'(b_1)(1 - a_j b_1) \overline{s_{1,j}} - \beta'(b_1)(1 - a_1 b_1) \overline{s_{1,1}} + \beta'(b_i)(1 - a_1 b_i) \overline{s_{i,1}}}{\beta'(b_i)(1 - a_j b_i)} \\ &= \frac{\beta'(b_1)(1 - a_j b_1) p_{1,j} - \beta'(b_1)(1 - a_1 b_1) p_{1,1} + \beta'(b_i)(1 - a_1 b_i) p_{i,1}}{\beta'(b_i)(1 - a_j b_i)}. \end{aligned} \tag{29}$$

□

4.2.2. Matrix Representation with respect to Clark and Modified Clark Bases. Let α be m-finite Blaschke product and β be n-finite Blaschke product. Choose arbitrary $\lambda_1, \lambda_2 \in \mathbb{T}$, then the following equations

$$\alpha(\eta) = \frac{\alpha(0) + \lambda_1}{1 + \bar{\alpha}(0)\lambda_1} := \alpha_{\lambda_1} \text{ and } \beta(\eta) = \frac{\beta(0) + \lambda_2}{1 + \bar{\beta}(0)\lambda_2} := \beta_{\lambda_2}, \tag{30}$$

have exactly m and n distinct solutions, respectively, denoted by $(\eta_j)_{j=1}^{m}$ and $(\zeta_i)_{i=1}^{n}$. The families of normalized eigenvectors $v^{\alpha}_{\eta_j} := \|k^{\alpha}_{\eta_j}\|^{-1} k^{\alpha}_{\eta_j}$ and $v^{\beta}_{\zeta_i} := \|k^{\beta}_{\zeta_i}\|^{-1} k^{\beta}_{\zeta_i}$ form orthonormal bases for K_α and K_β, respectively. We also have

$$\left\| k^{\alpha}_{\eta_j} \right\| = \sqrt{\left|\alpha'\left(\eta_j\right)\right|} \text{ and } \left\| k^{\beta}_{\zeta_i} \right\| = \sqrt{|\beta'(\zeta_i)|}. \tag{31}$$

Reorder the sequences $(\eta_j)_{j=1}^{m}$ and $(\zeta_i)_{i=1}^{n}$ so that l be the maximal integer such that $\bar{\eta}_i = \zeta_i$, for any $i \le l$. $l = 0$ when $\bar{\eta}_i = \zeta_i$ for every i. Denote by $(t_{i,j})$, for $1 \le i \le n$ and $1 \le j \le m$, the matrix representation of any bounded transformation from K_α into K_β with respect to Clark bases, and by $(u_{i,j})$, for $1 \le i \le n$ and $1 \le j \le m$, the matrix representation with respect to the modified Clark bases, we have

$$u_{i,j} = \left\langle Be^{\alpha}_{\eta_j}, e^{\beta}_{\zeta_i} \right\rangle = \omega^{\alpha}_{j} \bar{\omega}^{\beta}_{i} \left\langle Bv^{\alpha}_{\eta_j}, v^{\beta}_{\zeta_i} \right\rangle = \omega^{\alpha}_{j} \bar{\omega}^{\beta}_{i} t_{i,j}. \tag{32}$$

Theorem 14. *A bounded linear transformation B from K_α to K_β belongs to $\mathfrak{H}(\alpha, \beta)$ if and only if*

(1) For $l \ge 1$ with $i \le l$ and $i=j$

$$\begin{aligned} t_{i,j} = &\frac{\sqrt{|\beta'(\zeta_1)|}\left(\bar{\eta}_j - \zeta_1\right)}{\sqrt{|\beta'(\zeta_i)|}\left(\bar{\eta}_j - \zeta_i\right)} t_{1,j} \\ &+ \frac{\sqrt{|\alpha'(\eta_i)|}\sqrt{|\beta'(\zeta_1)|}\left(\bar{\eta}_j - \zeta_i\right)}{\sqrt{\left|\alpha'\left(\eta_j\right)\right|}\sqrt{|\beta'(\zeta_i)|}\left(\bar{\eta}_j - \zeta_i\right)} t_{1,i}. \end{aligned} \tag{33}$$

For $l \ge 1$ with $i > l$,

$$t_{i,j} = \frac{\sqrt{|\beta'(\zeta_1)|}\left(\bar{\eta}_j - \zeta_1\right)}{\sqrt{|\beta'(\zeta_i)|}\left(\bar{\eta}_j - \zeta_i\right)} t_{1,j} + \frac{\sqrt{|\alpha'(\eta_1)|}\left(\bar{\eta}_j - \zeta_i\right)}{\sqrt{\left|\alpha'\left(\eta_j\right)\right|}\left(\bar{\eta}_j - \zeta_i\right)} t_{i,1}. \tag{34}$$

(2) For $l = 0$

$$\begin{aligned} t_{i,j} = &\frac{\sqrt{|\beta'(\zeta_1)|}\left(\bar{\eta}_j - \zeta_1\right)}{\sqrt{|\beta'(\zeta_i)|}\left(\bar{\eta}_j - \zeta_i\right)} t_{1,j} - \frac{\sqrt{|\alpha'(\eta_1)|}\sqrt{|\beta'(\zeta_1)|}(\bar{\eta}_1 - \zeta_1)}{\sqrt{\left|\alpha'\left(\eta_j\right)\right|}\sqrt{|\beta'(\zeta_i)|}\left(\bar{\eta}_j - \zeta_i\right)} t_{1,1} \\ &+ \frac{\sqrt{|\alpha'(\eta_1)|}(\bar{\eta}_1 - \zeta_i)}{\sqrt{\left|\alpha'\left(\eta_j\right)\right|}\left(\bar{\eta}_j - \zeta_i\right)} t_{i,1}. \end{aligned} \tag{35}$$

Proof. We proceed as in the proof of Theorem 12. □

We deduce the following theorem from Theorem 14 and formula (32).

Theorem 15. *A bounded linear transformation B from K_α to K_β belongs to $\mathfrak{H}(\alpha,\beta)$ if and only if for $1 \le i \le n$ and $1 \le j \le m$, except that $i = j \le l$.*

(1) For $l \ge 1$ with $i \le l$ and i=j

$$u_{i,j} = \frac{\omega_1^\beta}{\omega_i^\beta} \frac{\sqrt{|\beta'(\zeta_1)|}\left(\bar{\eta}_j - \zeta_1\right)}{\sqrt{|\beta'(\zeta_i)|}\left(\bar{\eta}_j - \zeta_i\right)} u_{1,j} + \frac{\omega_j^\alpha}{\omega_1^\alpha} \frac{\sqrt{|\alpha'(\eta_i)|}\sqrt{|\beta'(\zeta_1)|}\left(\bar{\eta}_j - \zeta_i\right)}{\sqrt{\left|\alpha'\left(\eta_j\right)\right|}\sqrt{|\beta'(\zeta_i)|}\left(\bar{\eta}_j - \zeta_i\right)} u_{1,i}. \tag{36}$$

For $l \ge 1$ with $i > l$,

$$u_{i,j} = \frac{\omega_1^\beta}{\omega_i^\beta} \frac{\sqrt{|\beta'(\zeta_1)|}\left(\bar{\eta}_j - \zeta_i\right)}{\sqrt{|\beta'(\zeta_i)|}\left(\bar{\eta}_j - \zeta_i\right)} u_{1,j} + \frac{\omega_j^\alpha}{\omega_1^\alpha} \frac{\sqrt{|\alpha'(\eta_1)|}\left(\bar{\eta}_j - \zeta_i\right)}{\sqrt{\left|\alpha'\left(\eta_j\right)\right|}\left(\bar{\eta}_j - \zeta_i\right)} u_{i,1}. \tag{37}$$

(2) For $l = 0$

$$u_{i,j} = \frac{\omega_1^\beta}{\omega_i^\beta} \frac{\sqrt{|\beta'(\zeta_1)|}\left(\bar{\eta}_j - \zeta_i\right)}{\sqrt{|\beta'(\zeta_i)|}\left(\bar{\eta}_j - \zeta_i\right)} u_{1,j} - \frac{\omega_j^\alpha}{\omega_i^\beta} \frac{\omega_1^\beta}{\omega_1^\alpha} \frac{\sqrt{|\alpha'(\eta_1)|}\sqrt{|\beta'(\zeta_1)|}\left(\bar{\eta}_j - \zeta_i\right)}{\sqrt{\left|\alpha'\left(\eta_j\right)\right|}\sqrt{|\beta'(\zeta_i)|}\left(\bar{\eta}_j - \zeta_i\right)} u_{1,1} + \frac{\omega_j^\alpha}{\omega_1^\alpha} \frac{\sqrt{|\alpha'(\eta_1)|}\left(\bar{\eta}_j - \zeta_i\right)}{\sqrt{|\beta'(\zeta_i)|}\left(\bar{\eta}_j - \zeta_i\right)} u_{i,1}. \tag{38}$$

5. Matrix Representation in Infinite-Dimensional Case

In all this section, unless mentioned, we will suppose that α and β are two interpolating Blaschke sequences with respective zeros $(a_m)_{m\ge1}$ and $(b_n)_{n\ge1}$, then the kernel functions $(k^\alpha_{a_i})_{i\ge1}$ and conjugate kernel functions $(\tilde{k}^\alpha_{a_i})_{i\ge1}$ form two Riesz bases for K_α (same for K_β) (see [8, 9]). For all sequence of complex numbers $(f_m)_{m\ge1}$ such that $\sum_{m=1}^{\infty}|f_m|^2(1-|a_m|^2) < \infty$, the unique solution f in K_α of the interpolation problem $f(a_m) = f_m$ is given by

$$f = \sum_{i=1}^{+\infty} \frac{f(a_i)}{\alpha'(a_i)} \tilde{k}^\alpha_{a_i} = \sum_{i=1}^{+\infty} \frac{\tilde{f}(\bar{a}_i)}{\overline{\alpha'(a_i)}} k^\alpha_{a_i}. \tag{39}$$

We will also keep the previous notations of the matrix representations of an operator with respect to the different bases ($s_{i,j}, p_{i,j}, t_{i,j}$, and $u_{i,j}$).

5.1. Matrix Representation of ATHOs. Denote by $(a_{l_k})_{k\ge1} = (\bar{b}_{l_k})_{k\ge1}$ the subsequence of common elements between $(a_m)_m$ and $(\bar{b}_n)_n$ ordered such that $a_{l_k} = \bar{b}_{l_k}$ and that $1 = l_1 \in (l_k)_{k\ge1}$. We will prove that the above matrix representations are also true in the case of model spaces associated to interpolating Blaschke products.

Theorem 16. *Let B be a bounded linear transformation from K_α to K_β. Then, $B \in \mathfrak{H}(\alpha,\beta)$ if and only if for any i, j*

$$s_{i,j} = \frac{\bar{\beta}'(b_1)(1-a_j\bar{b}_1)s_{1,j} - \bar{\beta}'(b_1)(1-a_1\bar{b}_1)s_{1,1} + \bar{\beta}'(b_i)(1-a_1\bar{b}_i)s_{i,1}}{\bar{\beta}'(b_i)(1-a_j\bar{b}_i)}. \tag{40}$$

Proof. Suppose that $B = B^{\alpha,\beta}_\varphi$ and we will prove that it satisfies (40). As in the proof of Theorem 8, for any φ in $L^2(\mathbb{T})$, there is a $\phi \in K_{\alpha\beta^\#}$ such that $B^{\alpha,\beta}_\varphi = B^{\alpha,\beta}_{\bar{\phi}}$, where $K_{\alpha\beta^\#} = K_\alpha \oplus \alpha K_{\beta^\#}$. Then, there are $\chi \in K_\alpha$ and $\psi \in K_{\beta^\#}$ such that $B^{\alpha,\beta}_\varphi = B^{\alpha,\beta}_{\bar{\chi}} + B^{\alpha,\beta}_{\overline{\alpha\psi}}$. Let $\beta^\#$ be an interpolating Blaschke product with zeros $(\bar{b}_m)_{m\ge1}$. For the bases $(k^\alpha_{a_m})_{m\ge1}$ and $(\tilde{k}^{\beta^\#}_{\bar{b}_n})_{n\ge1}$, we have

$$\chi = \sum_{m=1}^{\infty} \bar{c}_m k^\alpha_{a_m} \text{ and } \psi = \sum_{n=1}^{\infty} \bar{d}_n \tilde{k}^{\beta^\#}_{\bar{b}_n}. \tag{41}$$

Replacing in $B^{\alpha,\beta}_\varphi$,

$$B^{\alpha,\beta}_\varphi = \sum_{m=1}^{\infty} c_m B^{\alpha,\beta}_{\overline{k^{\alpha}_{a_m}}} + \sum_{n=1}^{\infty} d_n B^{\alpha,\beta}_{\overline{\alpha\tilde{k}^{\beta^\#}_{\bar{b}_n}}}. \tag{42}$$

By (6), we have

$$B^{\alpha,\beta}_{\overline{k^{\alpha}_{a_m}}} = B^{\alpha,\beta}_{\frac{1}{\overline{1\bar{a}_m z}}} = B^{\alpha,\beta}_{\frac{a_m}{z-a_m}} = a_m k^\beta_{\bar{a}_m} \otimes k^\alpha_{a_m}, B^{\alpha,\beta}_{\overline{\alpha\tilde{k}^{\beta^\#}_{\bar{b}_n}}} = B^{\alpha,\beta}_{\frac{\overline{\alpha\beta^\#}}{z-b_n}} = \tilde{k}^\beta_{b_n} \otimes \tilde{k}^\alpha_{\bar{b}_n}. \tag{43}$$

So

$$B^{\alpha,\beta}_\varphi = \sum_{m=1}^{\infty} c_m a_m k^\beta_{\bar{a}_m} \otimes k^\alpha_{a_m} + \sum_{n=1}^{\infty} \tilde{k}^\beta_{b_n} \otimes \tilde{k}^\alpha_{\bar{b}_n}. \tag{44}$$

Replacing in (20), we get

$$s_{i,j} = \frac{1}{\beta'^{\overline{(b_i)}}} \left(\sum_{m=1}^{\infty} \frac{c_m a_m}{1 - \bar{a}_j a_m} \overline{\tilde{k}^{\beta}_{b_i}(\bar{a_m})} + \sum_{n=1}^{\infty} \frac{d_n}{1 - \bar{b}_i b_n} \overline{\tilde{k}^{\alpha}_{\bar{b_n}}(a_j)} \right). \tag{45}$$

When $(l_k)_k$ is empty, we have

$$s_{i,j} = \frac{1}{\beta'^{\overline{(b_i)}}} \left(\sum_{m=1}^{\infty} \frac{c_m a_m}{1 - \bar{a}_j a_m} \frac{\beta^{\#}(a_m)}{a_m - \bar{b}_i} + \sum_{n=1}^{\infty} \frac{d_n}{1 - \bar{b}_i b_n} \frac{\alpha^{\#}(b_n)}{b_n - \bar{a}_j} \right). \tag{46}$$

When $(l_k)_k$ is not empty, we have

$$\overline{\tilde{k}^{\beta}_{b_i}(\bar{a_m})} = \begin{cases} i \in (l_k)_k : \begin{cases} 0 & : m \in (l_k)_k, m \neq i, \\ \beta'^{\overline{(b_i)}} & : m = i, \\ \dfrac{\beta^{\#}(a_m)}{a_m - \bar{b}_i} & : m \in (l_k)_k, \end{cases} \\ i \in (l_k)_k : \begin{cases} 0 & : m \in (l_k)_k, \\ \dfrac{\beta^{\#}(a_m)}{a_m - \bar{b}_i} & : m \in (l_k)_k, \end{cases} \end{cases} \tag{47}$$

$$\overline{\tilde{k}^{\alpha}_{\bar{b_n}}(a_j)} = \begin{cases} j \in (l_k)_k : \begin{cases} 0 & : n \in (l_k)_k, n \neq j, \\ \overline{\alpha'^{(a_j)}} & : n = j, \\ \dfrac{\alpha^{\#}(b_n)}{b_n - \overline{a_j}} & : n \in (l_k)_k, \end{cases} \\ j \in (l_k)_k : \begin{cases} 0 & : n \in (l_k)_k, \\ \dfrac{\alpha^{\#}(b_n)}{b_n - \overline{a_j}} & : n \in (l_k)_k. \end{cases} \end{cases} \tag{48}$$

Therefore, we have 4 cases, $\{j \in (l_k)_k \text{ and } i \in (l_k)_k\}$, $\{j \in (l_k)_k \text{ and } i \in (l_k)_k\}$, $\{j \in (l_k)_k \text{ and } i \in (l_k)_k\}$, and $\{j \in (l_k)_k \text{and} i \in (l_k)_k\}$. In all these cases, we decompose, add, and subtract as in the finite-dimensional case. Using the fact that $a_1 = \overline{b_1}$, we obtain

$$s_{i,j} = \frac{\bar{\beta}'(b_1)\left(1 - \overline{a_j b_1}\right)s_{1,j} - \bar{\beta}'(b_1)\left(1 - \overline{a_1 b_1}\right)s_{1,1} + \bar{\beta}'(b_i)\left(1 - \overline{a_1 b_i}\right)s_{i,1}}{\beta'^{\overline{(b_i)}}\left(1 - \overline{a_j b_i}\right)}. \tag{49}$$

Conversely, we proceed as in [9] for truncated Hankel operators but using the generalized characterization in Theorem 4. □

As for Theorem 10, we also generalize the matrix representation of the ATHOs with respect to conjugate kernel bases.

Theorem 17. *Let B be a bounded linear transformation from K_α to K_β. We have $B \in \mathfrak{H}(\alpha, \beta)$ if and only if for any $i, j \geq 1$, we have*

$$p_{i,j} = \frac{\beta'(b_1)(1 - a_j b_1)p_{1,j} - \beta'(b_1)(1 - a_1 b_1)p_{1,1} + \beta'(b_i)(1 - a_1 b_i)p_{i,1}}{\beta'(b_i)(1 - a_j b_i)}. \tag{50}$$

Suppose that α and β are inner functions such that K_α and K_β have Clark bases $(v^{\alpha}_{\eta_j})_{j\geq 1}$ and $(v^{\beta}_{\zeta_i})_{i\geq 1}$, where $(\eta_j)_j$ and $(\zeta_i)_i$ are sequences of eigenvalues for some $U^{\alpha}_{\lambda_1}$ and some $U^{\beta}_{\lambda_2}$ satisfying the equations (30). We have the relations $\sqrt{|\alpha'(\eta_j)|} = \|k^{\alpha}_{\eta_j}\|$ and $\sqrt{|\beta'(\zeta_i)|} = \|k^{\beta}_{\zeta_i}\|$. Reorder these sequences such that $\eta_{l_k} = \bar{\zeta}_{l_k}$, for any $k \geq 1$ and that $1 = l_1 \in (l_k)_k$. We have the following theorem.

Theorem 18. *Let B be a bounded linear transformation from K_α to K_β. $B \in \mathfrak{H}(\alpha, \beta)$ if and only if*

(1) When $(l_k)_{k\geq 1}$ is not empty, for $\{i \in (l_k)_k \text{ and } j \in (l_k)_k\}$ or $\{i \in (l_k)_k \text{ and } j \in (l_k)_k \text{ and } j \neq i\}$

$$t_{i,j} = \frac{\sqrt{|\beta'(\zeta_1)|}\left(\bar{\eta}_j - \zeta_1\right)}{\sqrt{|\beta'(\zeta_i)|}\left(\bar{\eta}_j - \zeta_i\right)} t_{1,j} + \frac{\sqrt{|\alpha'(\eta_i)|}\sqrt{|\beta'(\zeta_1)|}(\bar{\eta}_1 - \zeta_j)}{\sqrt{\left|\alpha'\left(\eta_j\right)\right|}\sqrt{|\beta'(\zeta_i)|}\left(\bar{\eta}_j - \zeta_i\right)} t_{1,i}. \tag{51}$$

For $\{i \in (l_k)_k \text{ and } j \in (l_k)_k\}$ or $\{i \in (l_k)_k \text{ and } j \in (l_k)_k\}$,

$$t_{i,j} = \frac{\sqrt{|\beta'(\zeta_1)|}\left(\bar{\eta}_j - \zeta_1\right)}{\sqrt{|\beta'(\zeta_i)|}\left(\bar{\eta}_j - \zeta_i\right)} t_{1,j} + \frac{\sqrt{|\alpha'(\eta_1)|}(\bar{\eta}_1 - \zeta_j)}{\sqrt{\left|\alpha'\left(\eta_j\right)\right|}\left(\bar{\eta}_j - \zeta_i\right)} t_{i,1}. \tag{52}$$

(2) When $(l_k)_{k\geq 1}$ is empty, for any $i, j \geq 1$

$$t_{i,j} = \frac{\sqrt{|\beta'(\zeta_1)|}\left(\bar{\eta}_j - \zeta_1\right)}{\sqrt{|\beta'(\zeta_i)|}\left(\bar{\eta}_j - \zeta_i\right)} t_{1,j} - \frac{\sqrt{|\alpha'(\eta_1)|}\sqrt{|\beta'(\zeta_1)|}(\bar{\eta}_1 - \zeta_1)}{\sqrt{\left|\alpha'\left(\eta_j\right)\right|}\sqrt{|\beta'(\zeta_i)|}\left(\bar{\eta}_j - \zeta_i\right)} t_{1,1} + \frac{\sqrt{|\alpha'(\eta_1)|}(\bar{\eta}_1 - \zeta_j)}{\sqrt{\left|\alpha'\left(\eta_j\right)\right|}\left(\bar{\eta}_j - \zeta_i\right)} t_{i,1}. \tag{53}$$

Proof. The proof is similar to the one of Theorem 14, but we use formula (44). For the converse implication, we proceed as in Theorem 21, except using the following characterization of ATHOs in Theorem 4, there exist $\chi \in K_\alpha$ and $\psi \in K_\beta$,

$$B - \left(U^{\beta}_{\lambda_2}\right)^* B \left(U^{\alpha}_{\lambda_1}\right)^* = \psi \otimes k_0^\alpha + \tilde{k}_0^\beta \otimes \chi. \tag{54}$$

□

As for Theorem 15, we deduce the matrix representation with respect to modified Clark bases.

Theorem 19. *A bounded linear transformation B from K_α to K_β belongs to $\mathfrak{H}(\alpha,\beta)$ if and only if*

(1) *When $(l_k)_{k\geq 1}$ is not empty, for $\{i \in (l_k)_k$ and $j \in (l_k)_k\}$ or $\{i \in (l_k)_k$ and $j \in (l_k)_k$ and $j \neq i\}$*

$$u_{i,j} = \frac{\omega_1^\beta}{\omega_i^\beta} \frac{\sqrt{|\beta'(\zeta_1)|}\left(\overline{\eta_j} - \zeta_1\right)}{\sqrt{|\beta'(\zeta_i)|}\left(\overline{\eta_j} - \zeta_i\right)} u_{1,j} + \frac{\omega_j^\alpha}{\omega_1^\alpha} \frac{\sqrt{|\alpha'(\eta_i)|}\sqrt{|\beta'(\zeta_1)|}\left(\overline{\eta_1} - \zeta_j\right)}{\sqrt{\left|\alpha'\left(\eta_j\right)\right|}\sqrt{|\beta'(\zeta_i)|}\left(\overline{\eta_j} - \zeta_i\right)} u_{1,i}. \tag{55}$$

For $\{i \in (l_k)_k$ and $j \in (l_k)_k\}$ or $\{i \in (l_k)_k$ and $j \in (l_k)_k\}$,

$$u_{i,j} = \frac{\omega_1^\beta}{\omega_i^\beta} \frac{\sqrt{|\beta'(\zeta_1)|}\left(\overline{\eta_j} - \zeta_1\right)}{\sqrt{|\beta'(\zeta_i)|}\left(\overline{\eta_j} - \zeta_i\right)} u_{1,j} + \frac{\omega_j^\alpha}{\omega_1^\alpha} \frac{\sqrt{|\alpha'(\eta_1)|}\left(\overline{\eta_1} - \zeta_j\right)}{\sqrt{\left|\alpha'\left(\eta_j\right)\right|}\left(\overline{\eta_j} - \zeta_i\right)} u_{i,1}. \tag{56}$$

(2) *When $(l_k)_{k\geq 1}$ is empty*

$$u_{i,j} = \frac{\omega_1^\beta}{\omega_i^\beta} \frac{\sqrt{|\beta'(\zeta_1)|}\left(\overline{\eta_j} - \zeta_1\right)}{\sqrt{|\beta'(\zeta_i)|}\left(\overline{\eta_j} - \zeta_i\right)} u_{1,j} - \frac{\omega_j^\alpha \omega_1^\beta}{\omega_1^\alpha \omega_i^\beta} \frac{\sqrt{|\alpha'(\eta_1)|}\sqrt{|\beta'(\zeta_1)|}\left(\overline{\eta_1} - \zeta_1\right)}{\sqrt{\left|\alpha'\left(\eta_j\right)\right|}\sqrt{|\beta'(\zeta_i)|}\left(\overline{\eta_j} - \zeta_i\right)} u_{1,1} + \frac{\omega_j^\alpha}{\omega_1^\alpha} \frac{\sqrt{|\alpha'(\eta_1)|}\left(\overline{\eta_1} - \zeta_j\right)}{\sqrt{\left|\alpha'\left(\eta_j\right)\right|}\left(\overline{\eta_j} - \zeta_i\right)} u_{i,1}. \tag{57}$$

5.2. Matrix Representation of ATTOs. To get the matrix representation of ATTOs on interpolating model spaces, we will work as in the proof of Theorem 2.2 in [8]. We first need to explore the action of $V_{\xi,c}$ on $\mathfrak{T}(\alpha,\beta)$, where $V_{\xi,c}$ acts from K_α into $K_{\alpha\circ\tau_{\xi,c}}$ and is defined by $V_{\xi,c} f = \sqrt{\tau_{\xi,c}{}'}(f \circ \tau_{\xi,c})$, $\tau_{\xi,c}(z) = \xi(c-z)/(1-\bar{c}z)$. ([13]).

Proposition 20. *Let α and β be two inner functions, $\varphi \in L^2$ and $A_\varphi^{\alpha,\beta} \in \mathfrak{T}(\alpha,\beta)$. Then for some $c \in \mathbb{D}$ and $\xi \in \mathbb{T}$,*

$$V_{\xi,c} A_\varphi^{\alpha,\beta} V_{\xi,c}^{-1} = A_{\varphi\circ\tau_{\xi,c}}^{\alpha\circ\tau_{\xi,c},\beta\circ\tau_{\xi,c}}. \tag{58}$$

Proof. For $f \in K_\alpha$, $\varphi \in L^2$, $\xi \in \mathbb{T}$ and $a \in \mathbb{D}$, we have

$$\begin{aligned} V_{\xi,c}\left[P_\beta(\varphi f)\right](z) &= \sqrt{\tau'_{\xi,c}(z)}\left[P_\beta(\varphi f)\right] \circ \tau_{\xi,c}(z) \\ &= \sqrt{\tau'_{\xi,c}(z)}\left\langle \varphi f, k^\beta_{\tau_{\xi,c}(z)} \right\rangle \\ &= \sqrt{\tau'_{\xi,c}(z)} \int_{\mathbb{T}} \varphi(\chi) f(\chi) \\ &\quad \cdot \frac{1 - \beta\left(\tau_{\xi,c}(z)\right)\overline{\beta(\chi)}}{1 - \tau_{\xi,c}(z)\bar{\chi}} d\chi. \end{aligned} \tag{59}$$

Let $\chi = \tau_{\xi,c}(\omega)$. Since $\tau'_{\xi,c}(z) = \xi(|c|^2 - 1)/((1-\bar{c}z)^2)$, we have

$$\begin{aligned} V_{\xi,c}\left[P_\beta(\varphi f)\right](z) &= \sqrt{\tau'_{\xi,c}(z)} \int_{\mathbb{T}} \varphi\left(\tau_{\xi,c}(\omega)\right) f\left(\tau_{\xi,c}(\omega)\right) \\ &\quad \cdot \frac{1 - \beta\left(\tau_{\xi,c}(z)\right)\overline{\beta\left(\tau_{\xi,c}(\omega)\right)}}{1 - \tau_{\xi,c}(z)\overline{\tau_{\xi,c}(\omega)}} \left|\tau'_{\xi,c}(\omega)\right| d\omega \\ &= \sqrt{\tau'_{\xi,c}(z)} \int_{\mathbb{T}} \varphi\left(\tau_{\xi,c}(\omega)\right) f\left(\tau_{\xi,c}(\omega)\right) \\ &\quad \cdot \frac{1 - z\bar{\omega}}{1 - \tau_{\xi,c}(z)\overline{\tau_{\xi,c}(\omega)}} \overline{k_z^{\beta\circ\tau_{\xi,c}}(\omega)} \left|\tau'_{\xi,c}(\omega)\right| d\omega. \end{aligned} \tag{60}$$

Since

$$\sqrt{\tau'_{\xi,c}(z)} \left|\tau'_{\xi,c}(\omega)\right| \frac{1 - z\bar{\omega}}{1 - \tau'_{\xi,c}(z)\overline{\tau'_{\xi,c}(\omega)}} = \sqrt{\tau'_{\xi,c}(\omega)}, \tag{61}$$

we have

$$\begin{aligned} V_{\xi,c}\left[P_\beta(\varphi f)\right](z) &= \int_{\mathbb{T}} \varphi\left(\tau_{\xi,c}(\omega)\right) f\left(\tau_{\xi,c}(\omega)\right) \\ &\quad \cdot \sqrt{\tau'_{\xi,c}(\omega)} \overline{k_z^{\beta\circ\tau_{\xi,c}}(\omega)} d\omega \\ &= \int_{\mathbb{T}} \varphi \circ \tau_{\xi,c}(\omega) V_{\xi,c} f(\omega) \overline{k_z^{\beta\circ\tau_{\xi,c}}(\omega)}(\omega) d\omega \\ &= \left\langle \varphi \circ \tau_{\xi,c} V_{\xi,c} f, k_z^{\beta\circ\tau_{\xi,c}}(\omega) \right\rangle \\ &= P_{\beta\circ\tau_{\xi,c}}\left(\varphi \circ \tau_{\xi,c} V_{\xi,c} f\right)(z). \end{aligned} \tag{62}$$

In conclusion,

$$V_{\xi,c}A_{\varphi}^{\alpha,\beta} = A_{\varphi\circ\tau_{\xi,c}}^{\alpha\circ\tau_{\xi,c},\beta\circ\tau_{\xi,c}}V_{\xi,c}. \tag{63}$$

Applying $V_{\xi,c}^{-1} = V_{\xi,c}^{*}$, we get the result. □

As before, the formula (20) is also true for ATTOs. We will prove that the matrix characterizations of ATTOs on finite-dimensional model space obtained in [6] are also true in the infinite case. Denote by $(a_{l_k})_{k\geq 1}$ the subsequence of common zeros between α and β ordered such that $a_{l_k} = b_{l_k}$, for any $k \geq 1$ and that $1 = l_1 \in (l_k)_k$ when $(l_k)_k$ is not empty.

Theorem 21. *A bounded linear transformation A from K_α to K_β belongs to $\mathfrak{T}(\alpha,\beta)$ if and only if its matrix representation with respect to the kernel bases satisfies*

(1) When $(l_k)_{k\geq 1}$ is not empty, for $\{i \in (l_k)_k$ and $j\notin(l_k)_k\}$ or $\{i \notin (l_k)_k$ and $j \notin (l_k)_k$ and $j \neq i\}$

$$s_{i,j} = \frac{\bar{\beta}'(b_1)\left(\overline{a_j} - \overline{b_1}\right)s_{1,j} + \bar{\beta}'(b_1)\left(\overline{a_1} - \overline{b_i}\right)s_{1,i}}{\bar{\beta}'(b_i)\left(\overline{a_j} - \overline{b_i}\right)}. \tag{64}$$

For $\{i \notin (l_k)_k$ and $j \in (l_k)_k\}$ or $\{i \notin (l_k)_k$ and $j \notin (l_k)_k\}$,

$$s_{i,j} = \frac{\bar{\beta}'(b_1)\left(\overline{a_j} - \overline{b_1}\right)s_{1,j} + \bar{\beta}'(b_i)\left(\overline{a_1} - \overline{b_i}\right)s_{i,1}}{\bar{\beta}'(b_i)\left(\overline{a_j} - \overline{b_i}\right)}. \tag{65}$$

(2) When $(l_k)_{k\geq 1}$ is empty

$$s_{i,j} = \frac{\bar{\beta}'(b_1)\left(\overline{a_j} - \overline{b_1}\right)s_{1,j} - \bar{\beta}'(b_1)\left(\overline{a_1} - \overline{b_1}\right)s_{1,1} + \bar{\beta}'(b_i)\left(\overline{a_1} - \overline{b_i}\right)s_{i,1}}{\bar{\beta}'(b_i)\left(\overline{a_j} - \overline{b_i}\right)}. \tag{66}$$

Proof. The proof of necessity is the same as in [6] for ATTOs acting on finite-dimensional model spaces. Conversely, consider any bounded linear transformation from K_α into K_β whose matrix representation satisfies Theorem 21. To show that $A \in \mathfrak{T}(\alpha,\beta)$, or equivalently to show that A satisfies the characterization in Theorem 4, we will find a $\chi \in K_\alpha$ and a $\psi \in K_\beta$ such that

$$A - S_\beta A S_\alpha^* = \psi \otimes k_0^\alpha + k_0^\beta \otimes \chi. \tag{67}$$

This is equivalent to

$$\begin{aligned}&\left\langle Ak_{a_j}^\alpha, \tilde{k}_{b_i}^\beta\right\rangle - \left\langle S_\beta A S_\alpha^* k_{a_j}^\alpha, \tilde{k}_{b_i}^\beta\right\rangle \\ &\quad = \left\langle \psi \otimes k_0^\alpha\left(k_{a_j}^\alpha\right), \tilde{k}_{b_i}^\beta\right\rangle + \left\langle k_0^\beta \otimes \chi\left(k_{a_j}^\alpha\right), \tilde{k}_{b_i}^\beta\right\rangle,\end{aligned} \tag{68}$$

for any $i, j \geq 1$. Using the relations in Lemma 3, when $b_i \neq 0$, for any $i \geq 1$, we have for every $i, j \geq 1$

$$\begin{aligned}&\left(1 - \frac{\overline{a_j}}{\overline{b_i}}\right)\left\langle Ak_{a_j}^\alpha, \tilde{k}_{b_i}^\beta\right\rangle + \frac{\overline{a_j}}{\overline{b_i}}\left\langle k_{a_j}^\alpha, A^*\tilde{k}_0^\beta\right\rangle \\ &\quad = k_{a_j}^\alpha(0)\left\langle \psi, \tilde{k}_{b_i}^\beta\right\rangle + \overline{\tilde{k}_{b_i}^\beta(0)}\left\langle k_{a_j}^\alpha, \chi\right\rangle,\end{aligned} \tag{69}$$

or

$$\left(1 - \frac{\overline{a_j}}{\overline{b_i}}\right)\overline{\beta'(b_i)}s_{i,j} + \frac{\overline{a_j}}{\overline{b_i}}\left\langle k_{a_j}^\alpha, A^*\tilde{k}_0^\beta\right\rangle = \overline{\tilde{\psi}(b_i)} - \frac{\overline{\beta(0)}}{\overline{b_i}}\overline{\chi(a_j)}, \tag{70}$$

for any $i, j \geq 1$.

When $(l_k)_k$ is not empty, suppose that $\beta(0) \neq 0$. Since the matrix representation of A satisfies the formulas in Theorem 21, the above system is equivalent to

$$\begin{cases}\left(1 - \dfrac{\overline{a_j}}{\overline{b_1}}\right)\bar{\beta}'(b_1)s_{1,j} + \dfrac{\overline{a_j}}{\overline{b_1}}\left\langle k_{a_j}^\alpha, A^*\tilde{k}_0^\beta\right\rangle = \overline{\tilde{\psi}(b_1)} - \dfrac{\overline{\beta(0)}}{\overline{b_1}}\overline{\chi(a_j)}, & j \geq 1; \\ \left\langle k_{a_{l_k}}^\alpha, A^*\tilde{k}_0^\beta\right\rangle = \overline{\tilde{\psi}(b_{l_k})} - \dfrac{\overline{\beta(0)}}{\overline{b_{l_k}}}\chi(\bar{a}_{l_k}), & k \geq 1; \\ \left(1 - \dfrac{\overline{a_1}}{\overline{b_i}}\right)\overline{\beta'(b_i)}s_{i,1} + \dfrac{\overline{a_1}}{\overline{b_i}}\left\langle k_{a_1}^\alpha, A^*\tilde{k}_0^\beta\right\rangle = \overline{\tilde{\psi}(b_i)} - \dfrac{\overline{\beta(0)}}{\overline{b_i}}\overline{\chi(a_1)}, & i \geq 1, i\notin(l_k)_k.\end{cases} \tag{71}$$

Set an arbitrary $\tilde{\psi}(b_1)$, then the solution of the system is

$$\begin{cases} \overline{\chi(a_j)} = \dfrac{\overline{b_1}}{\overline{\beta(0)}}\left(\overline{\tilde{\psi}(b_1)} - \left(1 - \dfrac{\overline{a_j}}{\overline{b_1}}\right)\overline{\beta'^{(b_1)}} s_{1,j} - \dfrac{\overline{a_j}}{\overline{b_1}}\left\langle k^{\alpha}_{a_j}, A^*\tilde{k}^{\beta}_0\right\rangle\right), & j \geq 1; \\ \overline{\tilde{\psi}(b_i)} = -\left\langle k^{\alpha}_{a_i}, A^*\tilde{k}^{\beta}_0\right\rangle - \dfrac{\overline{\beta(0)}}{\overline{b_i}}\overline{\chi(a_i)}; & i \in (l_k)_k. \\ \overline{\tilde{\psi}(b_i)} = \left(1 - \dfrac{\overline{a_1}}{\overline{b_i}}\right)\overline{\beta'^{(b_i)}} s_{i,1} + \dfrac{\overline{a_1}}{\overline{b_i}}\left\langle k^{\alpha}_{a_1}, A^*\tilde{k}^{\beta}_0\right\rangle + \dfrac{\overline{\beta(0)}}{\overline{b_i}}\overline{\chi(a_1)}. & i \geq 1, i \notin (l_k)_k. \end{cases} \tag{72}$$

To show that the solutions of the system χ and ψ are in K_α and K_β, respectively, it suffices to prove that χ and ψ are the unique solutions in K_α and K_β of the interpolation problems corresponding to $(a_m)_{m\geq 1}$ and $(b_n)_{n\geq 1}$ (39). In fact, since $A^*\tilde{k}^{\beta}_0, A^*\tilde{k}^{\beta}_{b_1}(a_j), \widetilde{Ak^{\alpha}_{a_1}} \in K_\beta$, and $(a_j)_{j\geq 1}, (b_i)_{i\geq 1}$ are Blaschke sequences, we have

$$\begin{aligned} &\sum_{j=1}^{+\infty} |\chi(a_j)|^2\left(1 - |a_j|^2\right) \\ &\quad = \sum_{j=1}^{+\infty}\left|\frac{\overline{b_1}}{\overline{\beta(0)}}\right|\left|\overline{\tilde{\psi}(b_1)} - \left(1 - \frac{\overline{a_j}}{\overline{b_1}}\right)\overline{\beta}'(b_1)s_{1,j} - \frac{\overline{a_j}}{\overline{b_1}}\left\langle k^{\alpha}_{a_j}, A^*\tilde{k}^{\beta}_0\right\rangle\right|^2 \\ &\quad\quad \cdot \left(1 - |a_j|^2\right) \leq C\sum_{j=1}^{+\infty}\left|A^*\tilde{k}^{\beta}_{b_1}(a_j)\right|^2\left(1 - |a_j|^2\right) + C\left|\widetilde{\psi(b_1)}\right|^2 \\ &\quad\quad \cdot \sum_{j=1}^{+\infty}\left(1 - |a_j|^2\right) + C\sum_{j=1}^{+\infty}\left|A^*\tilde{k}^{\beta}_0(a_j)\right|^2\left(1 - |a_j|^2\right) < \infty. \end{aligned} \tag{73}$$

$$\begin{aligned} &\sum_{i=1}^{+\infty} |\tilde{\psi}(b_i)|^2\left(1 - |b_i|^2\right) \\ &\quad = \sum_{i=1, i\in(l_k)_k}^{+\infty}\left| <k^{\alpha}_{a_i}, A^*\tilde{k}^{\beta}_0> + \frac{\overline{\beta(0)}}{\overline{b_i}}\overline{\chi(a_i)}\right|^2\left(1 - |b_i|^2\right) \\ &\quad\quad + \sum_{i=1, i\notin(l_k)_k}^{+\infty}\left|\left(1 - \frac{\overline{a_1}}{\overline{b_i}}\right)\overline{\beta'^{(b_i)}}s_{i,1} + \frac{\overline{a_1}}{\overline{b_i}}\right. \\ &\quad \left. <k^{\alpha}_{a_1}, A^*\tilde{k}^{\beta}_0> + \frac{\overline{\beta(0)}}{\overline{b_i}}\overline{\chi(a_1)}\right|^2\left(1 - |b_i|^2\right) \\ &\quad \leq C\sum_{i=1}^{+\infty}\left|A^*\tilde{k}^{\beta}_0(a_i)\right|^2\left(1 - |a_i|^2\right) + C\sum_{i=1}^{+\infty}|\chi(a_i)|^2 \\ &\quad\quad \cdot \left(1 - |a_i|^2\right) + C\sum_{i=1}^{+\infty}\left|\widetilde{Ak^{\alpha}_{a_1}}(b_i)\right|\left(1 - |b_i|^2\right) \\ &\quad\quad + C\left(|\chi(a_1)|^2 + \left|A^*\widetilde{k^{\beta}_0}(a_1)\right|^2\right)\sum_{i=1}^{+\infty}\left(1 - |b_i|^2\right) < \infty. \end{aligned} \tag{74}$$

If $\beta(0) = 0$, $\beta(c) \neq 0$ for some $c \in \mathbb{D}$. Let $\xi = 1$, by Proposition 20, $A \in \mathfrak{T}(\alpha, \beta)$ if and only if $V_c A V_c^* \in \mathfrak{T}(\alpha \circ \tau_c, \beta \circ \tau_c)$. Since $\beta \circ \tau_c(0) = \beta(c) \neq 0$, we need to show that the matrix representation of $V_c A V_c^*$, $(w_{i,j})$ with respect to the kernel bases $\left(k^{\alpha\circ\tau_c}_{\tau_c(a_m)}\right)_m$ and $\left(k^{\beta\circ\tau_c}_{\tau_c(b_n)}\right)_n$ satisfies the relations in Theorem 21. In fact, as in [8], we have

$$\begin{aligned} V_c^* k^{\alpha\circ\tau_c}_{\tau_c(a_j)}(z) &= i\frac{1 - c\overline{a_j}}{\sqrt{1 - |c|^2}}k^{\alpha}_{a_j}(z), V_c^*\tilde{k}^{\beta\circ\tau_c}_{\tau_c(b_i)}(z) \\ &= -i\frac{1 - \overline{c}b_i}{\sqrt{1 - |c|^2}}\tilde{k}^{\beta}_{b_i}(z), \end{aligned} \tag{75}$$

$$\begin{aligned} w_{i,j} &= \frac{1}{\overline{(\beta \circ \tau_c)'^{(b_i)}}}\left\langle V_c A V_c^* k^{\alpha\circ\tau_c}_{\tau_c(a_j)}, \tilde{k}^{\beta\circ\tau_c}_{\tau_c(b_i)}\right\rangle \\ &= \frac{\overline{(\tau_c)'^{(b_i)}}}{\overline{(\beta)'^{(b_i)}}} i\frac{1 - c\overline{a_j}}{\sqrt{1 - |c|^2}}\overline{i}\frac{\overline{(1\overline{c}b_i)}}{\sqrt{1 - |c|^2}}\left\langle Ak^{\alpha}_{a_j}, \tilde{k}^{\beta}_{b_i}\right\rangle \\ &= \frac{1 - c\overline{a_j}}{1 - c\overline{b_i}}s_{i,j}, \end{aligned} \tag{76}$$

for any $i, j \geq 1$. We also have

$$\begin{aligned} (\beta \circ \tau_c)'(\tau_c(b_i)) &= \frac{\beta'(b_i)}{\tau_c'(b_i)}, \tau_c(z) = \frac{c - z}{1 - \overline{c}z} \text{ and } \tau'_c(z) \\ &= \frac{|c|^2 - 1}{(1 - \overline{c}z)^2}, \end{aligned} \tag{77}$$

using these formulas we can prove that every $w_{i,j}$ satisfies first relation in Theorem 21,

$$\begin{aligned} &\frac{\overline{(\beta \circ \tau_c)'^{(\tau_c(b_1))}}\left(\overline{\tau_c(a_j)} - \overline{\tau_c(b_1)}\right)w_{1,j} + \overline{(\beta \circ \tau_c)'^{(\tau_c(b_i))}}\left(\overline{\tau_c(a_1)} - \overline{\tau_c(b_i)}\right)w_{i,1}}{\overline{(\beta \circ \tau_c)'^{(\tau_c(b_i))}}\left(\overline{\tau_c(a_j)} - \overline{\tau_c(b_i)}\right)} \\ &\quad = \frac{1 - c\overline{a_j}}{1 - c\overline{b_i}}s_{i,j} = w_{i,j}. \end{aligned} \tag{78}$$

If $(l_k)_k$ is empty, then the equation (70) becomes with the assumption $\beta(0) \neq 0$,

$$\begin{cases} \left(1-\frac{\overline{a_j}}{\overline{b_1}}\right)\overline{\beta'(b_1)}s_{1,j} + \frac{\overline{a_j}}{\overline{b_1}}\left\langle k^\alpha_{a_j}, A^*\tilde{k}^\beta_0\right\rangle = \overline{\tilde{\psi}(b_1)} - \frac{\overline{\beta(0)}}{\overline{b_1}}\overline{\chi(a_j)}, & j \geq 1; \\ \left(1-\frac{\overline{a_1}}{\overline{b_i}}\right)\overline{\beta'(b_i)}s_{i,1} + \frac{\overline{a_1}}{\overline{b_i}}\left\langle k^\alpha_{a_1}, A^*\tilde{k}^\beta_0\right\rangle = \overline{\tilde{\psi}(b_i)} - \frac{\overline{\beta(0)}}{\overline{b_i}}\overline{\chi(a_1)}, & i \geq 2. \end{cases} \tag{79}$$

If we set an arbitrary $\tilde{\psi}(b_1)$, then, the solutions are

$$\begin{cases} \overline{\chi(a_j)} = \frac{\overline{b_1}}{\overline{\beta(0)}}\left(\overline{\tilde{\psi}(b_1)} - \left(1-\frac{\overline{a_j}}{\overline{b_1}}\right)\overline{\beta'(b_1)}s_{1,j} - \frac{\overline{a_j}}{\overline{b_1}}\left\langle k^\alpha_{a_j}, A^*\tilde{k}^\beta_0\right\rangle\right), & j \geq 1; \\ \overline{\tilde{\psi}(b_i)} = \left(1-\frac{\overline{a_1}}{\overline{b_i}}\right)\bar{\beta}'(b_i)s_{i,1} + \frac{\overline{a_1}}{\overline{b_i}}\left\langle k^\alpha_{a_1}, A^*\tilde{k}^\beta_0\right\rangle + \frac{\overline{\beta(0)}}{\overline{b_i}}\overline{\chi(a_1)}, & i \geq 2. \end{cases} \tag{80}$$

It remains to check that $\chi \in K_\alpha$ and $\psi \in K_\beta$ by showing that they are solutions of the corresponding interpolation problems. The case $\beta(0) = 0$ can be treated in the same way as in the previous case. □

To get the matrix representation of ATTOs with respect to conjugate kernel bases $(\tilde{k}^\alpha_{a_m})_m$ and $(\tilde{k}^\beta_{b_n})_n$, we proceed as in Theorem 13 and use the fact that $C_\beta A C_\alpha = A^{\alpha,\beta}_{\overline{\alpha\beta\varphi}}$ from the Theorem 5.

Theorem 22. *A bounded linear transformation from K_α to K_β belongs to $\mathfrak{T}(\alpha,\beta)$ if and only if*

(1) When $(l_k)_{k\geq 1}$ is not empty, for $\{i \in (l_k)_k$ and $j \in (l_k)_k\}$ or $\{i \in (l_k)_k$ and $j \in (l_k)_k$ and $j \neq i\}$

$$p_{i,j} = \frac{\beta'(b_1)(a_j - b_1)p_{1,j} + \beta'(b_1)(a_1 - b_i)p_{1,i}}{\beta'(b_i)(a_j - b_i)}. \tag{81}$$

For $\{i \in (l_k)_k$ and $j \in (l_k)_k\}$ or $\{i \in (l_k)_k$ and $j \in (l_k)_k\}$,

$$p_{i,j} = \frac{\beta'(b_1)(a_j - b_1)p_{1,j} + \beta'(b_i)(a_1 - b_i)p_{i,1}}{\beta'(b_i)(a_j - b_i)}. \tag{82}$$

(2) When $(l_k)_{k\geq 1}$ is empty

$$p_{i,j} = \frac{\beta'(b_1)(a_j - b_1)p_{1,j} - \beta'(b_1)(a_1 - b_1)p_{1,1} + \beta'(b_i)(a_1 - b_i)p_{i,1}}{\beta'(b_i)(a_j - b_i)}. \tag{83}$$

Suppose that α and β are inner functions such that the spaces K_α and K_β have Clark bases, $(v^\alpha_{\eta_j})_j$ and $(v^\beta_{\zeta_i})_i$. Denote the subsequence of common elements of $(\eta_j)_j$ and $(\zeta_i)_i$ by $(\eta_{l_k})_k = (\zeta_{l_k})_k$ ordered such that $\eta_{l_k} = \zeta_{l_k}$, for any $k \geq 1$ and that $1 = l_1 \in (l_k)_k$.

Theorem 23. *A bounded linear transformation belongs to $\mathfrak{T}(\alpha,\beta)$ if and only if*

(1) When $(l_k)_{k\geq 1}$ is not empty, for $\{i \in (l_k)_k$ and $j \in (l_k)_k\}$ or $\{i \in (l_k)_k$ and $j \in (l_k)_k$ and $j \neq i\}$

$$t_{i,j} = \frac{\sqrt{|\beta'(\zeta_1)|}\left(1-\zeta_1\overline{\eta_j}\right)}{\sqrt{|\beta'(\zeta_i)|}\left(1-\zeta_i\overline{\eta_j}\right)}t_{1,j} + \frac{\sqrt{|\alpha'(\eta_i)|}\sqrt{|\beta'(\zeta_1)|}\zeta_1(1-\zeta_i\overline{\eta_1})}{\sqrt{\left|\alpha'\left(\eta_j\right)\right|}\sqrt{|\beta'(\zeta_i)|}\eta_i\left(1-\zeta_i\overline{\eta_j}\right)}t_{1,i}. \tag{84}$$

For $\{i \in (l_k)_k$ and $j \in (l_k)_k\}$ or $\{i \in (l_k)_k$ and $j \in (l_k)_k\}$,

$$t_{i,j} = \frac{\sqrt{|\beta'(\zeta_1)|}\left(1-\zeta_1\overline{\eta_j}\right)}{\sqrt{|\beta'(\zeta_i)|}\left(1-\zeta_i\overline{\eta_j}\right)}t_{1,j} + \frac{\sqrt{|\alpha'(\eta_1)|}(1-\zeta_i\overline{\eta_1})}{\sqrt{\left|\alpha'\left(\eta_j\right)\right|}\left(1-\zeta_i\overline{\eta_j}\right)}t_{i,1}. \tag{85}$$

(2) When $(l_k)_{k\geq 1}$ is empty

$$t_{i,j} = \frac{\sqrt{|\beta'(\zeta_1)|}\left(1-\zeta_1\overline{\eta_j}\right)}{\sqrt{|\beta'(\zeta_i)|}\left(1-\zeta_i\overline{\eta_j}\right)}t_{1,j} - \frac{\sqrt{|\alpha'(\eta_1)|}\sqrt{|\beta'(\zeta_1)|}(1-\zeta_1\overline{\eta_1})}{\sqrt{\left|\alpha'\left(\eta_j\right)\right|}\sqrt{|\beta'(\zeta_i)|}\left(1-\zeta_i\overline{\eta_j}\right)}t_{1,1} + \frac{\sqrt{|\alpha'(\eta_1)|}(1-\zeta_i\overline{\eta_1})}{\sqrt{\left|\alpha'\left(\eta_j\right)\right|}\left(1-\zeta_i\overline{\eta_j}\right)}t_{i,1}. \tag{86}$$

Proof. The proof is the same as in Theorem 21, except using the equivalent characterization from Theorem 4 instead. ?

We also deduce the matrix representation with respect to the modified Clark bases.

Theorem 24. *A bounded linear transformation belongs to $\mathfrak{T}(\alpha,\beta)$ if and only if*

(1) When $(l_k)_{k\geq 1}$ *is not empty, for* $\{i \in (l_k)_k$ *and* $j \in (l_k)_k\}$ *or* $\{i \in (l_k)_k$ *and* $j \in (l_k)_k$ *and* $j \neq i\}$

$$\begin{aligned} u_{i,j} = {} & \frac{\omega_1^\beta}{\omega_i^\beta} \frac{\sqrt{|\beta'(\zeta_1)|}\left(1-\zeta_1\overline{\eta_j}\right)}{\sqrt{|\beta'(\zeta_i)|}\left(1-\zeta_i\overline{\eta_j}\right)} u_{1,j} \\ & + \frac{\omega_j^\alpha}{\omega_1^\alpha} \frac{\sqrt{|\alpha'(\eta_i)|}\sqrt{|\beta'(\zeta_1)|}\zeta_1(1-\zeta_i\overline{\eta_1})}{\sqrt{\left|\alpha'\left(\eta_j\right)\right|}\sqrt{|\beta'(\zeta_i)|}\eta_i\left(1-\zeta_i\overline{\eta_j}\right)} u_{1,i}. \end{aligned} \tag{87}$$

For $\{i \in (l_k)_k$ *and* $j \in (l_k)_k\}$ *or* $\{i \in (l_k)_k$ *and* $j \in (l_k)_k\}$,

$$\begin{aligned} u_{i,j} = {} & \frac{\omega_1^\beta}{\omega_i^\beta} \frac{\sqrt{|\beta'(\zeta_1)|}\left(1-\zeta_1\overline{\eta_j}\right)}{\sqrt{|\beta'(\zeta_i)|}\left(1-\zeta_i\overline{\eta_j}\right)} u_{1,j} \\ & + \frac{\omega_j^\alpha}{\omega_1^\alpha} \frac{\sqrt{|\alpha'(\eta_1)|}(1-\zeta_i\overline{\eta_1})}{\sqrt{\left|\alpha'\left(\eta_j\right)\right|}\left(1-\zeta_i\overline{\eta_j}\right)} u_{i,1}. \end{aligned} \tag{88}$$

(2) When $(l_k)_{k\geq 1}$ *is empty*

$$\begin{aligned} u_{i,j} = {} & \frac{\omega_1^\beta}{\omega_i^\beta} \frac{\sqrt{|\beta'(\zeta_1)|}\left(1-\zeta_1\overline{\eta_j}\right)}{\sqrt{|\beta'(\zeta_i)|}\left(1-\zeta_i\overline{\eta_j}\right)} u_{1,j} \\ & - \frac{\omega_j^\alpha}{\omega_i^\beta}\frac{\omega_1^\beta}{\omega_1^\alpha} \frac{\sqrt{|\alpha'(\eta_1)|}\sqrt{|\beta'(\zeta_1)|}(1-\zeta_1\overline{\eta_1})}{\sqrt{\left|\alpha'\left(\eta_j\right)\right|}\sqrt{|\beta'(\zeta_i)|}\left(1-\zeta_i\overline{\eta_j}\right)} u_{1,1} \\ & + \frac{\omega_j^\alpha}{\omega_1^\alpha} \frac{\sqrt{|\alpha'(\eta_1)|}(1-\zeta_i\overline{\eta_1})}{\sqrt{\left|\alpha'\left(\eta_j\right)\right|}\left(1-\zeta_i\overline{\eta_j}\right)} u_{i,1}. \end{aligned} \tag{89}$$

Remark 25. Note that since $J^\# k^\alpha_{a_j} = k^{\alpha^\#}_{\overline{a_j}}$ and $J^\# \tilde{k}^\alpha_{a_j} = \tilde{k}^{\alpha^\#}_{\overline{a_j}}$, we have

$$\begin{aligned} s_{i,j} &= \frac{1}{\overline{\beta'(b_i)}} \left\langle Bk^\alpha_{a_j}, \tilde{k}^\beta_{b_i} \right\rangle = \frac{1}{\overline{\beta'(b_i)}} \overline{\left\langle C_\beta B J^\# k^{\alpha^\#}_{\overline{a_j}}, k^\beta_{b_i} \right\rangle} \\ &= \frac{1}{\overline{\beta'(b_i)}} \left\langle J^\# B C_\alpha \tilde{k}^\alpha_{a_j}, \tilde{k}^{\beta^\#}_{\overline{b_i}} \right\rangle, \end{aligned} \tag{90}$$

and since $C_\beta B J^\#$ and $J^\# B C_\alpha$ are ATTOs by Theorem 4 ([10]), then we can deduce the matrix representation of an ATTO with respect to kernel and conjugate kernel bases, and to conjugate kernel and kernel bases in both finite and infinite dimensional cases, which matches with the work done in [14].

Similarly, we also can obtain the matrix representation of an ATHO with respect to kernel and conjugate kernel bases, and to conjugate kernel and kernel bases via the passage formulas.

Data Availability

There is no underlying data in this paper, and all of its research is the derivation of basic theory.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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