Hindawi
Journal of Function Spaces
Volume 2021, Article ID 4684290, 8 pages
https://doi.org/10.1155/2021/4684290

Research Article

Fixed Points of Generalized α-Meir-Keeler Contraction Mappings in S_b-Metric Spaces

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Received 21 May 2021; Revised 28 July 2021; Accepted 23 August 2021; Published 21 September 2021

Academic Editor: Nawab Hussain

In this note, we define Meir-Keeler contraction in S_b-metric spaces. Further, by adding the concept of α-admissible mappings, we define generalized α_s-Meir-Keeler contraction and used it for examining the existence and uniqueness of fixed points. Various results are also given as a consequence of our results.

1. Introduction and Preliminaries

The Banach contraction principle has been an important instrument for the study of a fixed point. It has been widely used in different areas like nonlinear analysis, applied mathematics, economics, and physics. Due to its importance, the result has been generalized in different ways. Meir and Keeler [1] introduce a generalization of the Banach contraction principle. According to them, self-mapping A in a metric space (X, d) is called Meir-Keeler contraction if for an $\varepsilon > 0$ there exists $\delta > 0$ such that $\varepsilon \le d(\theta, \phi) < \varepsilon + \delta(\varepsilon)$ implies $d(A\theta, A\phi) < \varepsilon$ for all $\theta, \phi \in X$. They also state and prove that if a self-mapping A in a complete metric space satisfies Meir-Keeler contraction, then there is a unique fixed point for the mapping A. There are a large number of works on Meir-Keeler contraction of which some of the recent works are mentioned here.

Pourhadi et al. [2] introduced the concept of Meir-Keeler expansive mappings and obtained Krasnosel'skii-type fixed point theorem in Banach spaces. A new fixed point theorem was obtained by Du and Rassias [3] for a Meir-Keeler type condition as a generalization of the Banach contraction principle, Kannan's fixed point theorem, Chatterjea's fixed point theorem, etc., simultaneously.

The idea of S_b-metric space [4–6] is defined by combining definitions of S-metric space [7] and b-metric space [8]. Samet et al. [9] introduced the concept of α-admissible mapping. This concept was further extended to G-metric space, S-metric space, S_b-metric space, etc. (for details, see [10–14]). There are various recent results on Meir-Keeler type and related topics which will be helpful to the readers for more information. Some of them can be seen in [15–21].

In this article, we give the concept of α-admissible and Meir-Keeler contraction in S_b-metric space. The new contraction will be known as generalized α_s-Meir-Keeler contraction. By using generalized α_s-Meir-Keeler contraction mappings, we study the existence and uniqueness of the fixed point in S_b-metric space.

The following definitions and properties will be needed.

Definition 1 (see [8]). In a set $X \neq \phi$, suppose $b \ge 1$ is a real number and $d : X \times X \longrightarrow [0,+\infty)$ is a function satisfying

(1) $d(\theta, \phi) = 0$ if and only if $\theta = \phi$ for all $\theta, \phi \in X$

(2) $d(\theta, \phi) = d(\phi, \theta)$ for all $\theta, \phi \in X$

(3) $d(\theta, \phi) \le b[d(\theta, \psi) + d(\psi, \phi)]$ for all $\theta, \phi, \psi \in X$

Then, d is called b-metric on X and the pair (X, d) is called a b-metric space with coefficient b.

Definition 2 (see [7]). In a set $X \ne \phi$, suppose $S : X \times X \times X \longrightarrow [0,+\infty)$ is a function satisfying

(1) $S(\theta, \phi, \psi) = 0$ if and only if $\theta = \phi = \psi$ for all $\theta, \phi, \psi \in X$

(2) $S(\theta, \phi, \psi) \le S(\theta, \theta, \omega) + S(\phi, \phi, \omega) + S(\psi, \psi, \omega)$, for all $\theta, \phi, \psi, \omega \in X$

Then, the pair (X, S) is said to be an S-metric space.

Definition 3 (see [5]). In a set $X \ne \phi$, suppose $b \ge 1$ is a real number and $S : X \times X \times X \longrightarrow [0,+\infty)$ is a function satisfying

(i) $S(\theta, \phi, \psi) = 0$ if and only if $\theta = \phi = \psi$

(ii) $S(\theta, \phi, \psi) \le b[S(\theta, \theta, \omega) + S(\phi, \phi, \omega) + S(\psi, \psi, \omega)]$, for all $\theta, \phi, \psi, \omega$ in X

Here, S is said to be a S_b-metric and (X, S) is said to be a S_b-metric space.

Definition 4 (see [4]). A S_b -metric S satisfying $S(\theta, \theta, \phi) = S(\phi, \phi, \theta)$ for all $\theta, \phi \in X$ is called a symmetric S_b-metric.

Definition 5 (see [5]). In a S_b-metric space (X, S), a sequence $\{\theta_n\}$ is called

(i) convergent if and only if $S(\theta_n, \theta_n, \theta) \longrightarrow 0$ as $n \longrightarrow \infty$, where $\theta \in X$ and is expressed as $\lim_{n\longrightarrow\infty} \theta_n = \theta$

(ii) Cauchy if and only if $S(\theta_n, \theta_m, \theta) \longrightarrow 0$ as $n, m \longrightarrow \infty$, where $\theta \in X$

(iii) complete S_b-metric space if every Cauchy sequence $\{\theta_n\}$ is convergent and converging to θ in X

We recall some types of α-admissible mappings in a metric space (X, d).

Definition 6 (see [9]). Let $A : X \longrightarrow X$ and $\alpha : X \times X \longrightarrow [0,+\infty)$ be functions. Here, A is said to be α-admissible if $\alpha(\theta, \phi) \ge 1$ implies $\alpha(A\theta, A\phi) \ge 1$ for all $\theta, \phi \in X$.

Definition 7 (see [15]). Let $A, B : X \longrightarrow X$ and $\alpha : X \times X \longrightarrow 0,+\infty)$ are functions. Here, the pair of mappings (A, B) is said to be *an* α-admissible if $\alpha(\theta, \phi) \ge 1$ implies $\alpha(A\theta, B\phi) \ge 1$ and $\alpha(B\theta, A\phi) \ge 1$ for all $\theta, \phi \in X$.

Definition 8 (see [16]). Let $A : X \longrightarrow X$ and $\alpha : X \times X \longrightarrow [0,+\infty)$ be functions. Here, A is known as triangular α-admissible, if

(i) $\alpha(\theta, \phi) \ge 1$, which implies $\alpha(A\theta, A\phi) \ge 1, \theta, \phi \in X$

(ii) $\alpha(\theta, \phi) \ge 1$, $\alpha(\phi, \psi) \ge 1$, which implies $\alpha(\theta, \psi) \ge 1$, for all $\theta, \phi, \psi \in X$

Definition 9 (see [15]). Let $A, B : X \longrightarrow X$ and $\alpha : X \times X \longrightarrow [0,+\infty)$ be functions. Here, the pair (A, B) is said to be a triangular α-admissible, if

(i) $\alpha(\theta, \phi) \ge 1$, which implies $\alpha(A\theta, B\phi) \ge 1$ and $\alpha(B\theta, A\phi) \ge 1, \theta, \phi \in X$

(ii) $\alpha(\theta, \phi) \ge 1$, $\alpha(\phi, \psi) \ge 1$, which implies $\alpha(\theta, \psi) \ge 1$, for all $\theta, \phi, \psi \in X$

We extend the concept of α-admissible mapping to be suitable for S-metric and S_b-metric spaces. Here, we consider X as S-metric space or S_b-metric space.

Definition 10. Let $A : X \longrightarrow X$ and $\alpha_s : X \times X \times X \longrightarrow [0,+\infty)$ are functions, then A is called α_s-admissible, if $\theta, \phi, \psi \in X$, $\alpha_s(\theta, \phi, \psi) \ge 1$ implies $\alpha_s(A\theta, A\phi, A\psi) \ge 1$.

Example 11. Consider $X = [0,+\infty)$ and define $A : X \longrightarrow X$ and $\alpha_s : X \times X \times X \longrightarrow [0,+\infty)$ by $A\theta = 4\theta$, for all $\theta, \phi, \psi \in X$, and

$$\alpha_s(\theta, \phi, \psi) = \begin{cases} e^{\psi/\theta\phi}, & \text{if } \theta \ge \phi \ge \psi, \theta, \phi \ne 0, \\ 0, & \text{if } \theta < \phi < \psi. \end{cases} \tag{1}$$

Then, A is an α_s-admissible mapping.

Definition 12. Let $A, B : X \longrightarrow X$ and $\alpha_s : X \times X \times X \longrightarrow [0,+\infty)$ be three functions. The pair (A, B) is called α_s-admissible if $\theta, \phi, \psi \in X$ such that $\alpha_s(\theta, \phi, \psi) \ge 1$, then we have $\alpha_s(A\theta, A\phi, B\psi)) \ge 1$ and $\alpha_s(B\theta, B\phi, A\psi) \ge 1$.

2. Main Result

Here, we give various types of Meir-Keeler contractive mappings in order to extend various results of Gülyaz et al. [17] in S_b-metric space. Throughout this paper, assume (X, S) is a S_b-metric space, $b \ge 1$ is a real number, and $A : X \longrightarrow X$ is a mapping.

Definition 13. An α_s-admissible mapping A in (X, S) is known as α_s-Meir-Keeler contraction mapping of type I, if there esists $\delta > 0$ for all $\varepsilon > 0$ such that

$$\varepsilon \le S(\theta, \phi, \psi) < \varepsilon + \delta \tag{2}$$

implies

$$\alpha_s(\theta, \phi, \psi)S(A\theta, A\phi, A\psi) < \frac{\varepsilon}{b} \quad (3)$$

for all $\theta, \phi, \psi \in X$.

Definition 14. An α_s-admissible mapping A in (X, S) is known as α_s-Meir-Keeler contraction mapping of type II, if there exists $\delta > 0$ for all $\varepsilon > 0$ such that

$$\varepsilon \leq S(\theta, \theta, \phi) < \varepsilon + \delta \quad (4)$$

implies

$$\alpha_s(\theta, \theta, \phi)S(A\theta, A\theta, A\phi) < \frac{\varepsilon}{b} \quad (5)$$

for all $\theta, \phi \in X$.

Remark 15.

(i) If A is an α_s-Meir-Keeler contraction of type I, then

$$\alpha_s(\theta, \phi, \psi)S(A\theta, A\phi, A\psi) \leq \frac{S(\theta, \phi, \psi)}{b}, \quad (6)$$

for all $\theta, \phi, \psi \in X$ and equality is true, when $\theta = \phi = \psi$

(ii) If A is an α_s-Meir-Keeler contraction of type II, then

$$\alpha_s(\theta, \theta, \phi)S(A\theta, A\theta, A\phi) \leq \frac{S(\theta, \theta, \phi)}{b}, \quad (7)$$

for all $\theta, \phi \in X$ and equality is true, when $\theta = \phi$

Now, we introduce the following generalization of Meir-Keeler mappings.

Definition 16. An α_s-admissible mapping A in (X, S) is known as generalized α_s-Meir-Keeler contraction mapping of type AI, if there exists $\delta > 0$ for all $\varepsilon > 0$ such that

$$\varepsilon \leq \Lambda(\theta, \phi, \psi) < \varepsilon + \delta \quad (8)$$

implies

$$\alpha_s(\theta, \phi, \psi)S(A\theta, A\phi, A\psi) < \frac{\varepsilon}{b}, \quad (9)$$

where

$$\Lambda(\theta, \phi, \psi) = \max \{S(\theta, \phi, \psi), S(\theta, \theta, A\theta), S(\phi, \phi, A\phi), S(\psi, \psi, A\psi)\} \quad (10)$$

for all $\theta, \phi, \psi \in X$.

Definition 17. An α_s-admissible mapping A in (X, S) is known as generalized α_s-Meir-Keeler contraction mapping of type AII, if there exists $\delta > 0$ for all $\varepsilon > 0$ such that

$$\varepsilon \leq \Lambda(\theta, \theta, \phi) < \varepsilon + \delta \quad (11)$$

implies

$$\alpha_s(\theta, \theta, \phi)S(A\theta, A\theta, A\phi) < \frac{\varepsilon}{b}, \quad (12)$$

where

$$\Lambda(\theta, \theta, \phi) = \max \{S(\theta, \theta, \phi), S(\theta, \theta, A\theta), S(\phi, \phi, A\phi)\} \quad (13)$$

for all $\theta, \phi \in X$.

Definition 18. An α_s-admissible mapping A in (X, S) is known as generalized α_s-Meir-Keeler contraction mapping of type BI, if there exists $\delta > 0$ for all $\varepsilon > 0$ such that

$$\varepsilon \leq \Lambda(\theta, \phi, \psi) < \varepsilon + \delta \quad (14)$$

implies

$$\alpha_s(\theta, \phi, \psi)S(A\theta, A\phi, A\psi) < \frac{\varepsilon}{b}, \quad (15)$$

where

$$\Lambda(\theta, \phi, \psi) = \max \{S(\theta, \phi, \psi), S(\theta, \theta, A\theta), S(\phi, \phi, A\phi), S(\psi, \psi, A\psi), \frac{1}{4}(S(\theta, \theta, A\phi) + S(\phi, \phi, A\psi) + S(\psi, \psi, A\theta))\} \quad (16)$$

for all $\theta, \phi, \psi \in X$.

Definition 19. An α_s-admissible mapping A in (X, S) is known as generalized α_s-Meir-Keeler contraction mapping of type BII, if there exists $\delta > 0$ for all $\varepsilon > 0$ such that

$$\varepsilon \leq \Lambda(\theta, \theta, \phi) < \varepsilon + \delta \quad (17)$$

implies

$$\alpha_s(\theta, \theta, \phi)S(A\theta, A\theta, A\phi) < \frac{\varepsilon}{b}, \quad (18)$$

where

$$\Lambda(\theta, \theta, \phi) = \max \left\{S(\theta, \theta, \phi), S(\theta, \theta, A\theta), S(\phi, \phi, A\phi), \frac{1}{4}(S(\theta, \theta, A\theta) + S(\theta, \theta, A\phi) + S(\phi, \phi, A\theta))\right\} \quad (19)$$

for all $\theta, \phi \in X$.

Remark 20.

(i) Let $A : X \longrightarrow X$ be a generalized α_s-Meir-Keeler contraction of type AI or BI. Then

$$\alpha_s(\theta, \phi, \psi) S(A\theta, A\phi, A\psi) \leq \frac{\Lambda(\theta, \phi, \psi)}{b} \tag{20}$$

for all $\theta, \phi, \psi \in X$, where the equality holds only when $\theta = \phi = \psi$

(ii) Let $A : X \longrightarrow X$ be a generalized α_s-Meir-Keeler contraction of type AII or BII. Then

$$\alpha_s(\theta, \theta, \phi) S(A\theta, A\theta, A\phi) \leq \frac{\Lambda(\theta, \theta, \phi)}{b}, \tag{21}$$

for all $\theta, \phi \in X$, where the equality holds only when $\theta = \phi$

Lemma 21. *Let (X, S) be a S_b-metric space and $\{\theta_n\}$ be a sequence satisfying*

(i) $\theta_m \neq \theta_n$ for all $m \neq n$, $m, n \in \mathbb{N}$

(ii) $S(\theta_n, \theta_n, \theta_{n+1}) \leq 1/b S(\theta_{n-1}, \theta_{n-1}, \theta_n)$, for all $n \in \mathbb{N}$

Then, $\{\theta_n\}$ is a Cauchy sequence in (X, S).

Proof. In order to show that sequence $\{\theta_n\}$ is Cauchy, we must prove that $\lim_{n\longrightarrow\infty} S(\theta_n, \theta_n, \theta_{n+k}) = 0$ for any $k \in \mathbb{N}$.

From (ii), we have

$$S(\theta_n, \theta_n, \theta_{n+1}) \leq \frac{1}{b^n} S(\theta_0, \theta_0, \theta_1), \text{for all } n \in \mathbb{N}. \tag{22}$$

Applying limit as $n \longrightarrow \infty$, we get

$$\begin{aligned} 0 &\leq \lim_{n\longrightarrow\infty} S(\theta_n, \theta_n, \theta_{n+1}) \\ &\leq \frac{1}{b^n} S(\theta_0, \theta_0, \theta_1) . \therefore \lim_{n\longrightarrow\infty} S(\theta_n, \theta_n, \theta_{n+1}) = 0. \end{aligned} \tag{23}$$

Now,

$$\begin{aligned} S(\theta_n, \theta_n, \theta_{n+k}) &\leq 2bS(\theta_n, \theta_n, \theta_{n+1}) + b^2 S(\theta_{n+1}, \theta_{n+1}, \theta_{n+k}) \\ &\leq 2bS(\theta_n, \theta_n, \theta_{n+1}) + 2b^3 S(\theta_{n+1}, \theta_{n+1}, \theta_{n+2}) \\ &\quad + b^4 S(\theta_{n+2}, \theta_{n+2}, \theta_{n+k}) \\ &\leq 2\left\{ bS(\theta_n, \theta_n, \theta_{n+1}) + b^3 S(\theta_{n+1}, \theta_{n+1}, \theta_{n+2}) \right. \\ &\quad \left. + \cdots + b^{2(k-1)+1} S(\theta_{n+k-1}, \theta_{n+k-1}, \theta_{n+k}) \right\} \\ &\leq 2\left\{ b\frac{S(\theta_0, \theta_0, \theta_1)}{b^n} + b^3 \frac{S(\theta_0, \theta_0, \theta_1)}{b^{n+1}} \right. \\ &\quad \left. + \cdots + b^{2(k-1)+1} \frac{S(\theta_0, \theta_0, \theta_1)}{b^{n+k-1}} \right\} \\ &= \frac{2}{b^{n-1}} \left\{ 1 + b + \cdots + b^k \right\} S(\theta_0, \theta_0, \theta_1) \\ &= \frac{2\left(b^k - 1\right)}{b^{n-1}(b-1)} S(\theta_0, \theta_0, \theta_1) . \therefore \lim_{n\longrightarrow\infty} S(\theta_n, \theta_n, \theta_{n+k}) \\ &\leq \lim_{n\longrightarrow\infty} \frac{2\left(b^k - 1\right)}{b^{n-1}(b-1)} S(\theta_0, \theta_0, \theta_1) = 0. \end{aligned} \tag{24}$$

Thus, $\{\theta_n\}$ is a Cauchy sequence in S_b-metric space (X, S).□ □

Theorem 22. *Let (X, S) be a complete S_b-metric space and $\alpha_s : X \times X \times X \longrightarrow [0, +\infty)$ be a mapping. Let $A : X \longrightarrow X$ satisfy the following:*

(i) A is a generalized α_s-Meir-Keeler contraction mapping of type AI

(ii) A is α_s-admissible

(iii) There is $\theta_0 \in X$ so that $\alpha_s(\theta_0, \theta_0, A\theta_0) \geq 1$

(iv) A is continuous

Then, there exists a fixed point of A in X.

Proof. Suppose $\theta_0 \in X$ and $\alpha_s(\theta_0, \theta_0, A\theta_0) \geq 1$. Define the sequence $\{\theta_n\}$ in X as

$$\theta_{n+1} = A\theta_n, \text{for all } n \in \mathbb{N}. \tag{25}$$

Suppose $\theta_{n_0} = \theta_{n_0+1}$ for some $n_0 \in \mathbb{N}$ that is $S(\theta_{n_0}, \theta_{n_0}, \theta_{n_0+1}) = 0$ implies that θ_{n_0} is a fixed point of A. Thus, assume that $\theta_n \neq \theta_{n+1}$ for all $n \geq 0$. From (ii), we have

$$\alpha_s(\theta_0, \theta_0, A\theta_0) = \alpha_s(\theta_0, \theta_0, \theta_1) \geq 1 \tag{26}$$

implies that

$$\alpha_s(A\theta_0, A\theta_0, A\theta_1) = \alpha_s(\theta_1, \theta_1, \theta_2) \geq 1 ; \tag{27}$$

continuing on the same lines, we have

$$\alpha_s(\theta_n, \theta_n, \theta_{n+1}) \geq 1, \quad \forall n \in \mathbb{N}. \tag{28}$$

Here, we need to show that sequence $\{\theta_n\}$ satisfies the conditions of Lemma 21. If we put $\theta = \phi = \theta_n$ and $\psi = \theta_{n+1}$ in (9), for all $\varepsilon > 0$, there is $\delta > 0$ satisfying

$$\varepsilon \leq \Lambda(\theta_n, \theta_n, \theta_{n+1}) < \varepsilon + \delta \tag{29}$$

implies

$$\alpha_s(\theta_n, \theta_n, \theta_{n+1}) S(A\theta_n, A\theta_n, A\theta_{n+1}) < \frac{\varepsilon}{b}, \tag{30}$$

where

$$\Lambda(\theta_n,\theta_n,\theta_{n+1}) = \max\{S(\theta_n,\theta_n,\theta_{n+1}), S(\theta_n,\theta_n,A\theta_n), S(\theta_{n+1},\theta_{n+1},A\theta_{n+1})\}. \tag{31}$$

From Remark 20(ii), we have

$$\begin{aligned} S(\theta_{n+1},\theta_{n+1},\theta_{n+2}) &= S(A\theta_n,A\theta_n,A\theta_{n+1}) \\ &\le \alpha_s(\theta_n,\theta_n,\theta_{n+1})S(A\theta_n,A\theta_n,A\theta_{n+1}) \\ &\le \frac{\Lambda(\theta_n,\theta_n,\theta_{n+1})}{b}; \end{aligned} \tag{32}$$

due to the fact that $\theta_n \neq \theta_{n+1}$, we see that equality does not hold, hence,

$$S(\theta_{n+1},\theta_{n+1},\theta_{n+2}) < \frac{\Lambda(\theta_n,\theta_n,\theta_{n+1})}{b}. \tag{33}$$

If $\Lambda(\theta_n,\theta_n,\theta_{n+1}) = S(\theta_{n+1},\theta_{n+1},\theta_{n+2})$ for some $n \in \mathbb{N}$, then (11) implies

$$S(\theta_{n+1},\theta_{n+1},\theta_{n+2}) < \frac{S(\theta_{n+1},\theta_{n+1},\theta_{n+2})}{b} \tag{34}$$

which is not possible. Then, $\Lambda(\theta_n,\theta_n,\theta_{n+1}) = S(\theta_n,\theta_n,\theta_{n+1})$ for all $n \in \mathbb{N}$, so that (11) yields

$$S(\theta_{n+1},\theta_{n+1},\theta_{n+2}) < \frac{S(\theta_n,\theta_n,\theta_{n+1})}{b}, \tag{35}$$

which shows that Lemma 21(ii) is true.□ □

Next, we consider the case for $\theta_n \neq \theta_m$ for all $n \neq m$.

If possible, let $\theta_n = \theta_m$ for some $m, n \in \mathbb{N}$. We have $S(\theta_n,\theta_n,\theta_{n+1}) \ge 0$ for some $n \in \mathbb{N}$. In general, let $m > n+1$.

We have $S(\theta_m,\theta_m,\theta_{m+1}) = S(\theta_n,\theta_n,\theta_{n+1})$; by inequality (12), we have

$$\begin{aligned} S(\theta_n,\theta_n,\theta_{n+1}) = S(\theta_m,\theta_m,\theta_{m+1}) &< \frac{S(\theta_{m-1},\theta_{m-1},\theta_m)}{b} \\ &< \frac{S(\theta_{m-2},\theta_{m-2},\theta_{m-1})}{b^2} \cdots < \frac{S(\theta_n,\theta_n,\theta_{n+1})}{b^{m-n}} \end{aligned} \tag{36}$$

becomes impossible. Thus, for some $m \neq n, \lambda_n = \lambda_m$ is not true, and hence, it must be $\theta_n \neq \theta_m$ for all $n \neq m$. So, due to Lemma 21, $\{\theta_n\}$ is a Cauchy sequence in (X, S). Thus, $\{\theta_n\}$ converges to $u \in X$, i.e.,

$$\lim_{n\longrightarrow\infty} S(\theta_n,\theta_n,u) = 0. \tag{37}$$

By the continuity of A, we have

$$\lim_{n\longrightarrow\infty} S(A\theta_n,A\theta_n,Au) = \lim_{n\longrightarrow\infty} S(\theta_{n+1},\theta_{n+1},Au) = 0, \tag{38}$$

so $\{\theta_n\}$ converges to Au. Since the limit is unique, $Au = u$.

Theorem 23. *Let (X, S) be a complete S_b-metric space and $\alpha_s : X\times X\times X \longrightarrow [0,+\infty)$ be a mapping. Let $A : X \longrightarrow X$ be a mapping such that*

(v) for a pair of fixed points (θ, ϕ) of A, $\alpha_s(\theta,\theta,\phi) \ge 1$

together with the four conditions of Theorem 22, then A has a unique fixed point in X.

Proof. The existence of a fixed point is proved in Theorem 22. Now, for uniqueness, consider θ and ϕ as two different fixed points of A in X.

By (9), we have

$$\varepsilon \le \Lambda(\theta,\theta,\phi) < \varepsilon + \delta \tag{39}$$

implies

$$\alpha_s(\theta,\theta,\phi)S(A\theta,A\theta,A\phi) < \frac{\varepsilon}{b}, \tag{40}$$

where

$$\begin{aligned} \Lambda(\theta,\theta,\phi) &= \max\{S(\theta,\theta,\phi), S(\theta,\theta,A\theta), S(\phi,\phi,A\phi)\} \\ &= \max\{S(\theta,\theta,\phi), 0, 0\} = S(\theta,\theta,\phi). \end{aligned} \tag{41}$$

By (v), $\alpha_s(\theta,\theta,\phi) \ge 1$, since $S(\theta,\theta,\phi) > 0$, Remark 20(ii) becomes

$$\begin{aligned} S(\theta,\theta,\phi) = S(A\theta,A\theta,A\phi) &\le \alpha_s(\theta,\theta,\phi)S(A\theta,A\theta,A\phi) \\ &< \frac{\Lambda(\theta,\theta,\phi)}{b} = \frac{S(\theta,\theta,\phi)}{b}, \end{aligned} \tag{42}$$

which is a contradiction, hence, $S(\theta,\theta,\phi) = 0$, i.e., $\theta = \phi$. Thus, the fixed point of A is unique.□ □

Definition 24. In S_b-metric space (X, S), $\alpha_s : X\times X\times X \longrightarrow [0,+\infty)$ is a mapping. Then, S_b-metric space (X, S) is known as an α-regular if for any sequence $\{\theta_n\}$, $\lim_{n\longrightarrow\infty} S(\theta_n,\theta_n,\theta) = 0$ and $\alpha_s(\theta_n,\theta_n,\theta_{n+1}) \ge 1$ for all $n \in \mathbb{N}$; we have $\alpha_s(\theta_n,\theta_n,\theta) \ge 1$ for all $n \in \mathbb{N}$.

Theorem 25. *In a complete S_b-metric space (X, S), $b \ge 1$ is a parameter and $\alpha_s : X\times X\times X \longrightarrow [0,+\infty)$ is an α_s-admissible mapping. Let $A : X \longrightarrow X$ be a generalized α_s-Meir-Keeler contraction of type AI satisfying the following:*

(i) There is $\theta_0 \in X$ so that $\alpha_s(\theta_0,\theta_0,A\theta_0) \ge 1$

(ii) The S_b-metric space (X, S) is an α-regular, then there exists a fixed point of A in X

(iii) For all pairs of fixed points, $\theta, \phi \in X, \alpha_s(\theta,\theta,\phi) \ge 1$

Then, A has unique fixed point.

Proof. Suppose $\theta_0 \in X$ such that $\alpha_s(\theta_0,\theta_0,A\theta_0) \ge 1$. Define a sequence $\{\theta_n\} \in X$ such that $\theta_{n+1} = A\theta_n$ for all $n \in \mathbb{N}$ and converges to $u \in X$ uniquely.

As (X, S) is α_s-regular, $\alpha_s(\theta_n,\theta_n,u) \ge 1$.

By (9), we have

$$\varepsilon \leq \Lambda(\theta_n, \theta_n, u) < \varepsilon + \delta \tag{43}$$

implies

$$\alpha_s(\theta_n, \theta_n, u)S(A\theta_n, A\theta_n, Au) < \frac{\varepsilon}{b}, \tag{44}$$

where

$$\Lambda(\theta_n, \theta_n, u) = \max\{S(\theta_n, \theta_n, u), S(\theta_n, \theta_n, A\theta_n), S(u, u, Au)\}. \tag{45}$$

On the other hand, from Remark 20(ii), we have

$$\begin{aligned} S(\theta_{n+1}, \theta_{n+1}, Au) &= S(A\theta_n, A\theta_n, Au) \\ &\leq \alpha_s(\theta_n, \theta_n, u)S(A\theta_n, A\theta_n, Au) \\ &< \frac{\Lambda(\theta_n, \theta_n, u)}{b}. \end{aligned} \tag{46}$$

We have

$$\lim_{n\longrightarrow\infty} S(\theta_{n+1}, \theta_{n+1}, Au) = S(u, u, Au). \tag{47}$$

Also,

$$\begin{aligned} \lim_{n\longrightarrow\infty} \Lambda(\theta_n, \theta_n, u) &= \lim_{n\longrightarrow\infty} \max\{S(\theta_n, \theta_n, u), S(\theta_n, \theta_n, A\theta_n), S(u, u, Au)\} \\ &= S(u, u, Au). \end{aligned} \tag{48}$$

Taking the limit as $n \longrightarrow \infty$ in (46), we have

$$S(u, u, Au) \leq \frac{S(u, u, Au)}{b}, \tag{49}$$

which conclude that $S(u, u, Au) = 0$.□ □

The uniqueness part is identical to Theorem 23.

Note: Theorems 22, 23, and 25 will be true for generalized α_s-Meir-Keeler contraction mapping of type BI and BII.

Example 26. Let $X = [0,\infty)$ be endowed with S_b-metric

$$S(x, y, z) = |y + z - 2x|, \text{where } b = 2. \tag{50}$$

Define $A : X \longrightarrow X$ by

$$Ax = \begin{cases} \dfrac{x^2}{8}, & x \in [0, 1], \\ \dfrac{1}{8} + \log x, & x \in (1,\infty), \end{cases} \tag{51}$$

$$\alpha_s(x, y, z) = \begin{cases} 1, & x, y, z \in [0, 1], \\ 0, & \text{otherwise.} \end{cases}$$

Clearly, mapping A is α_s-admissible and continuous mapping. Let $x, y \in [0, 1]$, without loss of generality, assume that $x \leq y$, then

$$S(Ax, Ax, Ay) = S\left(\frac{x^2}{8}, \frac{x^2}{8}, \frac{y^2}{8}\right) = \left|\frac{y^2}{8} - \frac{x^2}{8}\right|. \tag{52}$$

Now, to calculate

$$\Lambda(x, y, z) = \max\{S(x, y, z), S(x, x, Ax), S(y, y, Ay), S(z, z, Az)\}; \tag{53}$$

in our case, if we take $x = y$, then after a simple calculation, we have

$$\begin{aligned} \Lambda(x, x, y) &= \max\{S(x, x, y), S(x, x, Ax), S(y, y, Ay)\} \\ &= \max\left\{|y - x|, \left|x - \frac{x^2}{8}\right|, \left|y - \frac{y^2}{8}\right|\right\}. \end{aligned} \tag{54}$$

Now, suppose that

$$\varepsilon < \Lambda(x, x, y) = \max\left\{|y - x|, \left|x - \frac{x^2}{8}\right|, \left|y - \frac{y^2}{8}\right|\right\} < \varepsilon + \delta \tag{55}$$

for $\delta = 3\varepsilon$. Now, observe that $\max_{x,y\in[0,1]}\{|y - x|\} = 1$ and $\max_{x,y\in[0,1]}\{|y + x|\} = 2$, and assume that $\varepsilon \in (1/2, 1)$, then we have

$$\frac{|y - x||y + x|}{8} < \frac{2}{8} = \frac{1}{4} < \frac{\varepsilon}{2}, \tag{56}$$

which implies that

$$S(Ax, Ax, Ay) = \left|\frac{y^2}{8} - \frac{x^2}{8}\right| < \frac{\varepsilon}{2}. \tag{57}$$

Since $\alpha_s(x, y, z) = 1$ for all $x, y, z \in [0, 1]$; otherwise, $\alpha_s(x, y, z) = 0$, and we have

$$0 = \alpha_s(x, y, z)S(Ax, Ay, Az) < \frac{\varepsilon}{b} = \frac{\varepsilon}{2}. \tag{58}$$

Hence, A satisfies the conditions of generalized α_s-Meir-Keeler contraction mapping of type AI. Also, all the conditions of Theorem 22 are satisfied, and hence, $x = 0$ is the unique fixed point of mapping A.

3. Consequences

Here, we consider some consequences of Theorems 22, 23, and 25.

Corollary 27. *Let (X, S) be complete S_b-metric space and $A : X \longrightarrow X$ be an α_s-admissible mapping satisfying the following:*

(i) For all $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\varepsilon \leq N(\theta, \phi, \psi) < \varepsilon + \delta \tag{59}$$

implies

$$\alpha_s(\theta, \phi, \psi) S(A\theta, A\phi, A\psi) < \frac{\varepsilon}{b}, \tag{60}$$

where

$$N(\theta, \phi, \psi) = \max\left\{S(\theta, \phi, \psi), \frac{1}{3}[S(\theta, \theta, A\theta) + S(\phi, \phi, A\phi) + S(\psi, \psi, A\psi)]\right\} \tag{61}$$

for all $\theta, \phi, \psi \in X$

(ii) There exists $\theta_0 \in X$ such that $\alpha_s(\theta_0, \theta_0, A\theta_0) \geq 1$

(iii) A is continuous or S_b-metric space (X, S) is α_s-regular

Then, A has a fixed point in X.
Also,

(iv) for every pair of fixed points (θ, ϕ) of A, if $\alpha_s(\theta, \theta, \phi) \geq 1$

Then, the fixed point of A is unique in X.

Proof. As $N(\theta, \phi, \psi) \leq \Lambda(\theta, \phi, \psi)$ for all $\theta, \phi, \psi \in X$, the proof is obvious from Theorems 22, 23, and 25.□ □

Corollary 28. *Let (X, S) be complete S_b-metric space and $A : X \longrightarrow X$ be an α_s-Meir-Keeler contraction of type I; that is, there exists $\delta > 0$ for every $\varepsilon > 0$ such that*

$$\varepsilon \leq S(\theta, \phi, \psi) < \varepsilon + \delta \tag{62}$$

implies

$$\alpha_s(\theta, \phi, \psi) S(A\theta, A\phi, A\psi) < \frac{\varepsilon}{b} \tag{63}$$

for all $\theta, \phi, \psi \in X$.

If A is continuous or S_b-metric space (X, S) is α-regular, then A has a fixed point. Further, with condition (v) in Theorem 23, the fixed point of A is unique.

Proof. The proof follows easily from the relation $S(\theta, \phi, \psi) \leq \Lambda(\theta, \phi, \psi)$ for all $\theta, \phi, \psi \in X$.□ □

Taking $\alpha(\theta, \phi, \psi) = 1$ in Theorem 25, we get the following.

Corollary 29. *Let (X, S) be a complete S_b-metric space and $A : X \longrightarrow X$ be a continuous mapping. If there exists $\delta > 0$ for every $\varepsilon > 0$ such that*

$$\varepsilon \leq \Lambda(\theta, \phi, \psi) < \varepsilon + \delta \tag{64}$$

implies

$$S(A\theta, A\phi, A\psi) < \frac{\varepsilon}{b}, \tag{65}$$

where

$$\Lambda(\theta, \phi, \psi) = \max\{S(\theta, \phi, \psi), S(\theta, \theta, A\theta), S(\phi, \phi, A\phi), S(\psi, \psi, A\psi)\} \tag{66}$$

for all $\theta, \phi, \psi \in X$. Then, the fixed point of A is unique.

Corollary 30. *Let (X, S) be a complete S_b-metric space and $A : X \longrightarrow X$ be a continuous mapping. If there exists $\delta > 0$ for every $\varepsilon > 0$ such that*

$$\varepsilon \leq N(\theta, \phi, \psi) < \varepsilon + \delta \tag{67}$$

implies

$$S(A\theta, A\phi, A\psi) < \frac{\varepsilon}{b}, \tag{68}$$

where

$$N(\theta, \phi, \psi) = \max\left\{S(\theta, \phi, \psi), \frac{1}{3}[S(\theta, \theta, A\theta) + S(\phi, \phi, A\phi) + S(\psi, \psi, A\psi)]\right\} \tag{69}$$

for all $\theta, \phi, \psi \in X$. Then, A has a unique fixed point.

The Meir-Keeler contraction can be stated on S_b-metric spaces as follows.

Corollary 31. *Let (X, S) be a complete S_b-metric space and $A : X \longrightarrow X$ be a continuous Meir-Keeler mapping. If there exists $\delta > 0$ for every $\varepsilon > 0$ such that*

$$\varepsilon \leq S(\theta, \phi, \psi) < \varepsilon + \delta \tag{70}$$

becomes

$$S(A\theta, A\phi, A\psi) < \frac{\varepsilon}{b} \tag{71}$$

for all $\theta, \phi, \psi \in X$. Then A has a unique fixed point.

4. Conclusion

In this article, we define Meir-Keeler contraction in S_b-metric spaces using the concept of α-admissible mapping. Further, we define generalized α_s-Meir-Keeler contraction. Using these definitions of contractive mappings, we prove theorems for the existence and uniqueness of fixed points. We show that obtained results are potential generalizations of various results in the literature.

Data Availability

No data is used in this research.

Conflicts of Interest

The authors declare not having competing interests.

Authors' Contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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