Research Article

Analytic Normalized Solutions of 2D Fractional Saint-Venant Equations of a Complex Variable

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Saint-Venant equations describe the flow below a pressure surface in a fluid. We aim to generalize this class of equations using fractional calculus of a complex variable. We deal with a fractional integral operator type Prabhakar operator in the open unit disk. We formulate the extended operator in a linear convolution operator with a normalized function to study some important geometric behaviors. A class of integral inequalities is investigated involving special functions. The upper bound of the suggested operator is computed by using the Fox-Wright function, for a class of convex functions and univalent functions. Moreover, as an application, we determine the upper bound of the generalized fractional 2-dimensional Saint-Venant equations (2D-SVE) of diffusive wave including the difference of bed slope.

1. Introduction

Newly, fractional calculus has expanded considerable attention primarily appreciations to the growing occurrence of investigation mechanisms in the life sciences, allowing for simulations found by fractional operators [1] including differential and integral formulas. Further, the mathematical investigation of fractional calculus has advanced, chief to connections with other mathematical areas such as probability theory, mathematical physics [2], and mathematical biology [3–7] and the investigation of stochastic processes in real cases. In addition, it appears in studies of complex analysis. Now the literature, several different definitions of fractional integrals and derivatives are presented. Some of them such as the Riemann-Liouville integral, the Caputo, and the Riemann-Liouville differential operators are extensively employed in mathematics and physics and actually utilized in applied structures, modeling systems in real cases. While, in complex analysis, especially the theory of geometric functions, the researchers are focusing on Srivastava-Owa integral and

differential operators [8], Tremblay differential operator, and the most recent fractional operator in [9, 10]. A new investigation of the complex ABC-fractional operator is presented to formulate different classes of analytic functions [11]. Some definitions such as the Hilfer and Prabhakar results [12] (differential and integral operators) are essentially the theme of mathematical study.

Our study is aimed to extend the Prabhakar operator [13] to the open unit disk utilizing the class of normalized analytic functions. We formulate this operator in a linear convolution operator to study some important geometric behaviors. A class of integral inequalities is investigated involving special functions. The upper bound of the suggested operator is computed by using the Fox-Wright function, for a class of convex functions and univalent functions, and other studies are illustrated in the sequel.

2. Complex Prabhakar Operator (CPO)

The Prabhakar integral operator is defined for analytic function $\phi(z) \in \mathcal{H}[0, 1] = \{\phi \in \mathcal{U} : \phi_1 z + \phi_2 z^2 + \dots\}$ by the

formula [14–20]

$$P_{\alpha,\beta}^{\gamma,\omega}\phi(z) = \int_0^z (z-\zeta)^{\beta-1} \Xi_{\alpha,\beta}^{\gamma} [\omega(z-\zeta)^{\alpha}] \phi(\zeta) d\zeta = (\phi \cdot \rho_{\alpha,\beta}^{\gamma,\omega})(z),$$

$$(\alpha, \beta, \gamma, \omega \in \mathbb{C}, z \in \mathbb{U} := \{z \in \mathbb{C} : |z| < 1\}, \Re(\alpha), \Re(\beta) > 0), \quad (1)$$

where, [15]

$$\rho_{\alpha,\beta}^{\gamma,\omega}(z) := z^{\beta-1} \Xi_{\alpha,\beta}^{\gamma}(\omega z^{\alpha}),$$

$$\Xi_{\alpha,\beta}^{\gamma}(\chi) = \sum_{n=0}^{\infty} \frac{\Gamma(\gamma+n)}{\Gamma(\gamma)\Gamma(\alpha n + \beta)} \frac{\chi^n}{n!}. \quad (2)$$

For example, let $\phi(z) = z^{\varepsilon-1}$ then in view of [21] Corollary 2.3, we have

$$P_{\alpha,\beta}^{\gamma,\omega} z^{\varepsilon-1} = \int_0^z (z-\zeta)^{\beta-1} \Xi_{\alpha,\beta}^{\gamma} [\omega(z-\zeta)^{\alpha}] (\zeta^{\varepsilon-1}) d\zeta$$

$$= \Gamma(\varepsilon) z^{\beta+\varepsilon-1} \Xi_{\alpha,\beta+\varepsilon}^{\gamma}(\omega z^{\alpha}). \quad (3)$$

The Prabhakar derivative can be computed by the formula [13]

$$D_{\alpha,\beta}^{\gamma,\omega}\phi(z) = \frac{d^k}{dz^k} \left(P_{\alpha,k-\beta}^{-\gamma,\omega}\phi(z) \right), \quad z \in \mathbb{U}. \quad (4)$$

To study the geometric indications of CPO, we introduce the following class of analytic function: a normalized analytic function $\phi(z) \in \Lambda$, $z \in \mathbb{U}$ achieving the power series

$$\phi(z) = z + \sum_{n=2}^{\infty} \phi_n z^n, \quad z \in \mathbb{U}, \quad (5)$$

Two analytic functions f, g are called convoluted, denoting by $f * g$ if and only if

$$(f * g)(z) = \left(\sum_{n=0}^{\infty} a_n z^n \right) * \left(\sum_{n=0}^{\infty} g_n z^n \right) = \sum_{n=0}^{\infty} a_n g_n z^n. \quad (6)$$

Definition 1. Define a new function $\Omega : \mathbb{U} \longrightarrow \mathbb{U}$, such that

$$\Omega_{\alpha,\beta}^{\gamma,\omega}(z) := \left(\frac{\Gamma(\alpha+\beta)}{\gamma\omega^{1/\alpha}} \right) z^{1-\beta} \rho_{\alpha,\beta}^{\gamma,\omega} \left(z^{\frac{1}{\alpha}} \right)$$

$$= \left(\frac{\Gamma(\alpha+\beta)}{\gamma\omega^{1/\alpha}} \right) \Xi_{\alpha,\beta}^{\gamma} \left(\omega^{\frac{1}{\alpha}} \left(z^{\frac{1}{\alpha}} \right)^{\alpha} \right)$$

$$= \left(\frac{\Gamma(\alpha+\beta)}{\gamma\omega^{1/\alpha}} \right) \sum_{n=0}^{\infty} \frac{\Gamma(\gamma+n)}{\Gamma(\gamma)\Gamma(\alpha n + \beta)} \frac{(\omega^{1/\alpha} z)^n}{n!} \quad (7)$$

$$= \left(\frac{\Gamma(\alpha+\beta)}{\gamma\omega^{1/\alpha}} \right) \sum_{n=0}^{\infty} \left(\frac{\Gamma(\gamma+n)\omega^{n/\alpha}}{\Gamma(\gamma)\Gamma(\alpha n + \beta)} \right) \frac{z^n}{n!},$$

$$(\alpha, \beta, \gamma, \omega \in \mathbb{C}, z \in \mathbb{U}, \Re(\alpha), \Re(\beta) > 0).$$

Note that,

$$(\Omega_{1,1}^{1,1})(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} = e^z, \quad (8)$$

where e^z is a transcendental function. Utilizing the functional $\Omega_{\alpha,\beta}^{\gamma,\omega}$, we define the modified complex linear Prabhakar operator

$$\mathbb{P}_{\alpha,\beta}^{\gamma,\omega}\phi(z) := \Omega_{\alpha,\beta}^{\gamma,\omega} * \phi(z), \quad \phi \in \Lambda. \quad (9)$$

We have the following result.

Proposition 2. If $\phi \in \Lambda$, then $\mathbb{P}_{\alpha,\beta}^{\gamma,\omega}\phi(z) = (\Omega_{\alpha,\beta}^{\gamma,\omega} * \phi) \in \Lambda$, where $*$ indicates the convolution product.

Proof. Let $\phi \in \Lambda$, then we have

$$(\Omega_{\alpha,\beta}^{\gamma,\omega} * \phi)(z) = \Omega_{\alpha,\beta}^{\gamma,\omega}(z) * \phi(z)$$

$$= \left(\frac{\Gamma(\alpha+\beta)}{\gamma\omega^{1/\alpha}} \right) \left(\sum_{n=0}^{\infty} \frac{\Gamma(\gamma+n)\omega^{n/\alpha}}{\Gamma(\gamma)\Gamma(\alpha n + \beta)} \frac{z^n}{n!} \right)$$

$$* \left(z + \sum_{n=2}^{\infty} \phi_n z^n \right) = \left(\left(\frac{\Gamma(\alpha+\beta)}{\Gamma(\beta)\gamma\omega^{1/\alpha}} \right) \right.$$

$$\left. + \left(\frac{\Gamma(\alpha+\beta)}{\gamma\omega^{1/\alpha}} \right) \left(\frac{\gamma\omega^{1/\alpha}}{\Gamma(\alpha+\beta)} \right) z + \dots \right)$$

$$* (0 + z + \phi_2 z^2 + \dots) = z + \sum_{n=2}^{\infty} \left(\frac{\Gamma(\alpha+\beta)}{\gamma\omega^{1/\alpha}} \right)$$

$$\cdot \left(\frac{\Gamma(\gamma+n)\omega^{n/\alpha}}{\Gamma(\gamma)\Gamma(\alpha n + \beta)} \right) \frac{\phi_n}{n!} z^n := z + \sum_{n=2}^{\infty} \Phi_n z^n, \quad (10)$$

where

$$\Phi_n = \left(\frac{\Gamma(\alpha+\beta)}{\gamma\omega^{1/\alpha}} \right) \left(\frac{\Gamma(\gamma+n)\omega^{n/\alpha}}{\Gamma(\gamma)\Gamma(\alpha n + \beta)} \right) \frac{\phi_n}{n!}$$

$$= \left(\frac{\Gamma(\alpha+\beta)}{\Gamma(\gamma+1)\omega^{1/\alpha}} \right) \left(\frac{\Gamma(\gamma+n)\omega^{n/\alpha}}{\Gamma(\alpha n + \beta)} \right) \frac{\phi_n}{n!}. \quad (11)$$

Hence, $\mathbb{P}_{\alpha,\beta}^{\gamma,\omega}\phi(z) = (\Omega_{\alpha,\beta}^{\gamma,\omega} * \phi)(z) \in \Lambda$. ?

We need the following concepts to study the $\mathbb{P}_{\alpha,\beta}^{\gamma,\omega}\phi(z)$ geometrically.

Definition 3. A function $\phi \in \Lambda$ is indicated to be starlike including the origin of the linear slice contains the origin to all further point of ϕ in $\phi(z \in \mathbb{U})$. A univalent function (one-one) ϕ is indicated to be convex in $\mathbb{U}$ if the linear slice relating every two points of $\phi(z)$ lies completely in $\phi(z \in \mathbb{U})$. We indicate these classes by $\mathcal{S}^*$ and $\mathcal{C}$ for starlike and convex consistently. Consider the class $\mathcal{P}$ includes all mappings $\wp$ smooth in $\mathbb{U}$ with a positive real part in $\mathbb{U}$ realizing $\wp(0) = 1$

, $\phi'(0) > 0$. Precisely, $\phi \in \mathcal{S}^* \Leftrightarrow z\phi'(z)/\phi(z) \in \mathcal{P}$ and $\phi \in \mathcal{C} \Leftrightarrow 1 + z\phi''(z)/\phi'(z) \in \mathcal{P}$. Regularly, $\Re(z\phi'(z)/\phi(z)) > 0$ for the starlikeness and $\Re(1 + z\phi''(z)/\phi'(z)) > 0$ for the convexity.

Two analytic functions f_1, f_2 in $\mathbb{U}$ are called subordinated [22] denoting by $f_1 < f_2$ or $f_1(z) < f_2(z), z \in \mathbb{U}$, if there occurs an analytic function $\omega, |\omega| \leq |z| < 1$ satisfying

$$f_1(z) = f_2(\omega(z)), \quad z \in \mathbb{U}. \quad (12)$$

Next, lemma can be located in [22], P138-140.

Lemma 4. Let $h \in \Lambda$. Then,

- (a) $h(z) + azh'(z) < (1+a)z + az^2 \Rightarrow h(z) < z$, when $a \in (0, 1/3]$
- (b) $zh'(z)[1 + h(z)] + ah^2(z) < z + (1+a)z^2 \Rightarrow h(z) < z$, when $|1+a| \leq 1/4$
- (c) $[zh'(z) - h(z)]e^{ah(z)} + e^{h(z)} < e^z \Rightarrow h(z) < z$, when $|a-1| \leq \pi/2$
- (d) $zh'(z)(1 + ah(z)) + h(z) < 2z + az^2 \Rightarrow h(z) < z$, when $|a| \leq 1/2$
- (e) $zh'(z)e^{ah(z)} + h(z) < z(1 + aze^{az}) \Rightarrow h(z) < z$, when $|a| \leq 1$
- (f) $h(z) + (zh'(z)/1 + ah(z)) < z \Rightarrow h(z) < z$, when $|a| \leq 1$

and the solution is sharp.

Definition 5. The Fox-Wright function ${}_p\mathcal{W}_q(z)$ (the extension function of hypergeometric function) is formulated by

$$\begin{aligned} & {}_p\mathcal{W}_q \left[\begin{matrix} (x_1, X_1) & (x_2, X_2) & \cdots & (x_p, X_p) \\ (y_1, Y_1) & (y_2, Y_2) & \cdots & (y_q, Y_q) \end{matrix} ; z \right] \\ &= \sum_{n=0}^{\infty} \frac{\Gamma(x_1 + X_1 n) \cdots \Gamma(x_p + X_p n)}{\Gamma(y_1 + Y_1 n) \cdots \Gamma(y_q + Y_q n)} \frac{z^n}{n!}. \end{aligned} \quad (13)$$

And it normalized by

$$\begin{aligned} & {}_p\mathcal{W}_q^* \left[\begin{matrix} (x_1, X_1) & (x_2, X_2) & \cdots & (x_p, X_p) \\ (y_1, Y_1) & (y_2, Y_2) & \cdots & (y_q, Y_q) \end{matrix} ; z \right] \\ &= \frac{\Gamma(y_1) \cdots \Gamma(y_q)}{\Gamma(x_1) \cdots \Gamma(x_p)} \sum_{n=0}^{\infty} \frac{\Gamma(x_1 + X_1 n) \cdots \Gamma(x_p + X_p n)}{\Gamma(y_1 + Y_1 n) \cdots \Gamma(y_q + Y_q n)} \frac{z^n}{n!}. \end{aligned} \quad (14)$$

Note that the series is converged when

$$\triangleright := \sum_{j=1}^q Y_j - \sum_{i=1}^p X_i \geq -1. \quad (15)$$

Moreover, it converges for all finite values z to the entire function provided $\triangleright < -1$. In addition, at the boundary $|z| = 1$, it has the convergence value (see [23])

$$\Re(\triangleleft) := \Re \left(\sum_{j=1}^q y_j - \sum_{i=1}^p x_i + \frac{p-q-1}{2} \right) > 0. \quad (16)$$

The significance of the Fox-Wright function arises regularly from its part in fractional calculus (see [1]). Further fascinating applications correspondingly occur. Wright's original attentiveness in this function was connected to the asymptotic theory of partitions [24]. The formula $\triangleleft$ is generated in [23] by adding a positive parameter $\theta > 0$ as follows:

$$\triangleleft_{\theta} := \sum_{j=1}^q y_j + \theta - \sum_{i=1}^p x_i + \frac{p-q-1}{2}. \quad (17)$$

Based on this generalization, the authors in [24] introduced the following lemma.

Lemma 6. Assume that $\triangleright = -1, \theta > 0$ and $\Re(\triangleleft_{\theta}) > 0$. Then,

$${}_p\mathcal{W}_q(z) = \Gamma(\theta) \int_0^1 \frac{H(\rho^{-1}t)dt}{t(1-tz\rho^{-1})^{\theta}}, \quad |z| < \rho, \quad (18)$$

where

$$H := \frac{1}{2\pi i} \int_{\mathbb{U}} \Pi(\zeta, \theta) z^{-\zeta} / \Gamma(\zeta + \theta) d\zeta < \infty, \quad (19)$$

is the delta-neutral H function and Π indicates the Fox-Wright coefficients.

Proposition 7. Let $\phi \in \mathcal{C}$ (convex in the open unit disk), then

$$\begin{aligned} & |(\Omega_{\alpha, \beta}^{\gamma, \omega} * \phi)'(z)| \leq {}_1\mathcal{W}_1^* \left[\begin{matrix} (1 + \gamma, 1) \\ (\alpha + \beta, \alpha) \end{matrix} ; \omega^{1/\alpha} r \right], \\ & (\alpha > 0, \quad \beta > 0, \quad |z| := r < 1, \gamma, \omega \in \mathbb{R}_+), \end{aligned} \quad (20)$$

Proof. Since $\phi \in \mathcal{C}$, then for each $n \geq 2$, we have $|\phi_n| \leq 1$.

Then, a computation implies that

$$\begin{aligned}
|(\Omega_{\alpha,\beta}^{\gamma,\omega} * \phi)(z)| &\leq \left(\frac{\Gamma(\alpha+\beta)}{\Gamma(\gamma+1)\omega^{1/\alpha}} \right) \sum_{n=1}^{\infty} \left(\frac{\Gamma(\gamma+n)\omega^{n/\alpha}}{\Gamma(\alpha n+\beta)} \right) \frac{|\phi_n|}{n!} r^n \\
&\leq \left(\frac{\Gamma(\alpha+\beta)}{\Gamma(\gamma+1)\omega^{1/\alpha}} \right) \sum_{n=1}^{\infty} \left(\frac{\Gamma(\gamma+n)\omega^{n/\alpha}}{\Gamma(\alpha n+\beta)} \right) \frac{r^n}{n!} \\
&= \left(\frac{\Gamma(\alpha+\beta)}{\Gamma(\gamma+1)\omega^{1/\alpha}} \right) \sum_{n=1}^{\infty} \left(\frac{\Gamma(\gamma+n)}{\Gamma(\alpha n+\beta)} \right) \frac{(\omega^{1/\alpha}r)^n}{n!} \\
&= \left(\frac{\Gamma(\alpha+\beta)}{\Gamma(\gamma+1)\omega^{1/\alpha}} \right) \sum_{n=0}^{\infty} \left(\frac{\Gamma(\gamma+1+n)}{\Gamma(\alpha n+\alpha+\beta)} \right) \frac{(\omega^{1/\alpha}r)^{n+1}}{(n+1)!} \\
&= (\omega^{1/\alpha}r) \left(\frac{\Gamma(\alpha+\beta)}{\Gamma(\gamma+1)\omega^{1/\alpha}} \right) \sum_{n=0}^{\infty} \left(\frac{\Gamma(1+n)\Gamma(1+\gamma+n)}{\Gamma(2+n)\Gamma(\alpha n+\alpha+\beta)} \right) \frac{(\omega^{1/\alpha}r)^n}{(n)!} \\
&= r \left(\frac{\Gamma(\alpha+\beta)}{\Gamma(\gamma+1)} \right) \sum_{n=0}^{\infty} \left(\frac{\Gamma(1+n)\Gamma(1+\gamma+n)}{\Gamma(2+n)\Gamma(\alpha n+\alpha+\beta)} \right) \frac{(\omega^{1/\alpha}r)^n}{(n)!} \\
&= r \left(\frac{\Gamma(\alpha+\beta)}{\Gamma(\gamma+1)} \right)_2 \mathcal{W}_2 \left[\begin{matrix} (1,1) & (1+\gamma,1) \\ (2,1) & (\alpha+\beta,\alpha) \end{matrix} ; \omega^{1/\alpha}r \right] \\
&= r \left(\frac{\Gamma(2)\Gamma(\alpha+\beta)}{\Gamma(\gamma+1)\Gamma(1)} \right)_2 \mathcal{W}_2 \left[\begin{matrix} (1,1) & (1+\gamma,1) \\ (2,1) & (\alpha+\beta,\alpha) \end{matrix} ; \omega^{1/\alpha}r \right] \\
&= r_2 \mathcal{W}_2^* \left[\begin{matrix} (1,1) & (1+\gamma,1) \\ (2,1) & (\alpha+\beta,\alpha) \end{matrix} ; \omega^{1/\alpha}r \right].
\end{aligned} \tag{21}$$

Now, for the derivative, we have

$$\begin{aligned}
|(\Omega_{\alpha,\beta}^{\gamma,\omega} * \phi)'(z)| &\leq \sum_{n=1}^{\infty} \left(\frac{\Gamma(\alpha+\beta)}{\gamma\omega^{1/\alpha}} \right) \left(\frac{\Gamma(\gamma+n)\omega^{n/\alpha}}{\Gamma(\gamma)\Gamma(\alpha n+\beta)} \right) \frac{n|\phi_n|}{n!} |z|^{\wedge\{n-1\}} \\
&\leq \sum_{n=0}^{\infty} \left(\frac{\Gamma(\alpha+\beta)}{\Gamma(\gamma+1)\omega^{1/\alpha}} \right) \left(\frac{\Gamma(\gamma+n+1)\omega^{(n+1)/\alpha}}{\Gamma(\alpha n+\alpha+\beta)} \right) \frac{r^n}{n!} \\
&= \sum_{n=0}^{\infty} \left(\frac{\Gamma(\alpha+\beta)}{\Gamma(\gamma+1)} \right) \left(\frac{\Gamma(\gamma+n+1)\omega^{n/\alpha}}{\Gamma(\alpha n+\alpha+\beta)} \right) \frac{r^n}{n!} \\
&= \left(\frac{\Gamma(\alpha+\beta)}{\Gamma(\gamma+1)} \right)_1 \mathcal{W}_1 \left[\begin{matrix} (1+\gamma,1) \\ (\alpha+\beta,\alpha) \end{matrix} ; \omega^{1/\alpha}r \right] \\
&= {}_1\mathcal{W}_1^* \left[\begin{matrix} (1+\gamma,1) \\ (\alpha+\beta,\alpha) \end{matrix} ; \omega^{1/\alpha}r \right].
\end{aligned} \tag{22}$$

This completes the proof.

Remark 8.

- (i) It is clear that the above upper bound of $(\Omega_{\alpha,\beta}^{\gamma,\omega} * \phi)(z)$, $|z| \rightarrow 1$, $\omega = 1$ converges at

$$\begin{aligned}
\Re(\triangleleft) &= \Re \left(\sum_{j=1}^q y_j - \sum_{i=1}^p x_i + \frac{p-q-1}{2} \right) \\
&> 0 \longrightarrow \alpha + \beta - \gamma - 1/2 > 0 \longrightarrow \alpha + \beta > \gamma + 1/2.
\end{aligned} \tag{23}$$

Moreover, the upper bound of $(\Omega_{\alpha,\beta}^{\gamma,\omega} * \phi)(z)$ converges, when

$$\begin{aligned}
\Re(\triangleleft) &= \Re \left(\sum_{j=1}^q y_j - \sum_{i=1}^p x_i + \frac{p-q-1}{2} \right) \\
&> 0 \longrightarrow \alpha + \beta - \gamma - 3/2 > 0 \longrightarrow \alpha + \beta > \gamma + 3/2.
\end{aligned} \tag{24}$$

- (ii) For special case: by Proposition 7 and Lemma 6, we have

$$\begin{aligned}
|(\Omega_{2,\beta}^{\gamma,1} * \phi)(z)| &\leq r \left(\frac{\Gamma(2)\Gamma(2+\beta)}{\Gamma(\gamma+1)\Gamma(1)} \right)_2 \mathcal{W}_2 \left[\begin{matrix} (1,1) & (1+\gamma,1) \\ (2,1) & (2+\beta,2) \end{matrix} ; r \right] \\
&\leq r \left(\frac{\Gamma(2+\beta)}{\Gamma(\gamma+1)} \right) \int_0^1 \frac{H(r^{-1}t)dt}{t(1-tzr^{-1})},
\end{aligned} \tag{25}$$

provided that $5/2 + \beta > \gamma$ and $\theta = 1$.

Proposition 9. Let ϕ be univalent in the open unit disk. Then,

$$\begin{aligned}
|(\Omega_{\alpha,\beta}^{\gamma,\omega} * \phi)(z)| &\leq r_1 \mathcal{W}_1^* \left[\begin{matrix} (1+\gamma,1) \\ (\alpha+\beta,\alpha) \end{matrix} ; \omega^{1/\alpha}r \right], \\
|(\Omega_{\alpha,\beta}^{\gamma,\omega} * \phi)(z)| &\leq \left(r_1 \mathcal{W}_1^* \left[\begin{matrix} (1+\gamma,1) \\ (\alpha+\beta,\alpha) \end{matrix} ; \omega^{1/\alpha}r \right] \right), \\
&(\alpha \neq 0, r < 1, \beta, \gamma \in \mathbb{R}_+).
\end{aligned} \tag{26}$$

Proof. Since ϕ is univalent, then for each $n \geq 2$, we have $|\phi_n| \leq n$. Then a calculation indicates that

$$\begin{aligned}
|(\Omega_{\alpha,\beta}^{\gamma,\omega} * \phi)(z)| &\leq \left(\frac{\Gamma(\alpha+\beta)}{\Gamma(\gamma+1)\omega^{1/\alpha}} \right) \sum_{n=1}^{\infty} \left(\frac{\Gamma(\gamma+n)\omega^{n/\alpha}}{\Gamma(\alpha n+\beta)} \right) \frac{|\phi_n|}{n!} r^n \\
&\leq \left(\frac{\Gamma(\alpha+\beta)}{\Gamma(\gamma+1)\omega^{1/\alpha}} \right) \sum_{n=1}^{\infty} n \left(\frac{\Gamma(\gamma+n)\omega^{n/\alpha}}{\Gamma(\alpha n+\beta)} \right) \frac{r^n}{n!} \\
&= \left(\frac{\Gamma(\alpha+\beta)}{\Gamma(\gamma+1)\omega^{1/\alpha}} \right) \sum_{n=1}^{\infty} \left(\frac{\Gamma(\gamma+n)}{\Gamma(\alpha n+\beta)} \right) \frac{(\omega^{1/\alpha}r)^n}{(n-1)!} \\
&= \left(\frac{\Gamma(\alpha+\beta)}{\Gamma(\gamma+1)\omega^{1/\alpha}} \right) \sum_{n=0}^{\infty} \left(\frac{\Gamma(\gamma+1+n)}{\Gamma(\alpha n+\alpha+\beta)} \right) \frac{(\omega^{1/\alpha}r)^{n+1}}{(n)!} \\
&= r \left(\frac{\Gamma(\alpha+\beta)}{\Gamma(\gamma+1)} \right) \sum_{n=0}^{\infty} \left(\frac{\Gamma(1+\gamma+n)}{\Gamma(\alpha n+\alpha+\beta)} \right) \frac{(\omega^{1/\alpha}r)^n}{(n)!} \\
&= r \left(\frac{\Gamma(\alpha+\beta)}{\Gamma(\gamma+1)} \right)_1 \mathcal{W}_1 \left[\begin{matrix} (1+\gamma,1) \\ (\alpha+\beta,\alpha) \end{matrix} ; \omega^{1/\alpha}r \right] \\
&= r_1 \mathcal{W}_1^* \left[\begin{matrix} (1+\gamma,1) \\ (\alpha+\beta,\alpha) \end{matrix} ; \omega^{1/\alpha}r \right].
\end{aligned} \tag{27}$$

Now, we return to the upper bound of the derivative

$$\begin{aligned}
 |(\Omega_{\alpha,\beta}^{\gamma,\omega} * \phi)(z)| &\leq \sum_{n=1}^{\infty} \left(\frac{\Gamma(\alpha+\beta)}{\gamma\omega^{1/\alpha}} \right) \left(\frac{\Gamma(\gamma+n)\omega^{n/\alpha}}{\Gamma(\gamma)\Gamma(\alpha n+\beta)} \right) \frac{n\phi_n}{n!} |z|^{n-1} \\
 &\leq \sum_{n=1}^{\infty} \left(\frac{\Gamma(\alpha+\beta)}{\gamma\omega^{1/\alpha}} \right) \left(\frac{\Gamma(\gamma+n)\omega^{n/\alpha}}{\Gamma(\gamma)\Gamma(\alpha n+\beta)} \right) \frac{n|\phi_n|}{n!} r^{n-1} \\
 &\leq \sum_{n=1}^{\infty} \left(\frac{\Gamma(\alpha+\beta)}{\gamma\omega^{1/\alpha}} \right) \left(\frac{\Gamma(\gamma+n)\omega^{n/\alpha}}{\Gamma(\gamma)\Gamma(\alpha n+\beta)} \right) \frac{n}{(n-1)!} r^{n-1} \\
 &= \sum_{n=0}^{\infty} \left(\frac{\Gamma(\alpha+\beta)}{\Gamma(\gamma+1)} \right) \left(\frac{\Gamma(\gamma+n+1)}{\Gamma(\alpha n+\beta+\alpha)} \right) \frac{(n+1)}{n!} (\omega^{1/\alpha} r)^n \\
 &= \left(\frac{\Gamma(\alpha+\beta)}{\Gamma(\gamma+1)} \right) \sum_{n=0}^{\infty} \left(\frac{\Gamma(1+\gamma+n)}{\Gamma(\alpha n+\alpha+\beta)} \right) \frac{(n+1)}{(n)!} (\omega^{1/\alpha} r)^n \\
 &= \left(\frac{\Gamma(\alpha+\beta)}{\Gamma(\gamma+1)} \right) \left(r_1 \mathcal{W}_1 \left[\begin{matrix} (1+\gamma, 1) \\ (\alpha+\beta, \alpha) \end{matrix}; \omega^{1/\alpha} r \right] \right)' \\
 &= \left(r_1 \mathcal{W}_1^* \left[\begin{matrix} (1+\gamma, 1) \\ (\alpha+\beta, \alpha) \end{matrix}; \omega^{1/\alpha} r \right] \right)'.
 \end{aligned} \tag{28}$$

This completes the proof. $\square$

More integral inequality results will consider in the following theorem.

Theorem 10. Consider $\phi \in \Lambda$ and the operator $(\Omega_{\alpha,\beta}^{\gamma,\omega} * \phi)(z)$, where $\alpha \neq 0, z \in \mathbb{U}, \beta, \gamma \in \mathbb{R}_+$. If one of the following subordination inequalities hold

$$\begin{aligned}
 (\Omega_{\alpha,\beta}^{\gamma,\omega} * \phi)(z) + z(\Omega_{\alpha,\beta}^{\gamma,\omega} * \phi)(z) &< \sigma z^2 + 2z, \quad |\sigma| \leq \frac{1}{2}, \\
 (\Omega_{\alpha,\beta}^{\gamma,\omega} * \phi)(z) + z(\Omega_{\alpha,\beta}^{\gamma,\omega} * \phi)(z) e^{\sigma z} &< (1 + \sigma z e^{\sigma z})z, \quad |\sigma| \leq 1, \\
 (\Omega_{\alpha,\beta}^{\gamma,\omega} * \phi)(z) + \frac{z(\Omega_{\alpha,\beta}^{\gamma,\omega} * \phi)(z)}{1 + \sigma(\Omega_{\alpha,\beta}^{\gamma,\omega} * \phi)(z)} &< \frac{z(1 + \sigma z)}{1 + \sigma z}, \quad |\sigma| \leq 1, \\
 (z(\Omega_{\alpha,\beta}^{\gamma,\omega} * \phi)(z) - (\Omega_{\alpha,\beta}^{\gamma,\omega} * \phi)(z)) e^{\sigma(\Omega_{\alpha,\beta}^{\gamma,\omega} * \phi)(z)} &+ e^{\sigma(\Omega_{\alpha,\beta}^{\gamma,\omega} * \phi)(z)} < e^z, \quad |\sigma - 1| \leq \frac{\pi}{2}, \\
 (\Omega_{\alpha,\beta}^{\gamma,\omega} * \phi)(z) + \sigma z(\Omega_{\alpha,\beta}^{\gamma,\omega} * \phi)(z) &< (1 + \sigma z) + \sigma z^2, \quad 0 < |\sigma| \leq \frac{1}{3}, \\
 (\Omega_{\alpha,\beta}^{\gamma,\omega} * \phi)(z) + z(\Omega_{\alpha,\beta}^{\gamma,\omega} * \phi)(z) &< 2z, \\
 (\Omega_{\alpha,\beta}^{\gamma,\omega} * \phi)(z) \left[1 + \frac{z(\Omega_{\alpha,\beta}^{\gamma,\omega} * \phi)(z)}{(\Omega_{\alpha,\beta}^{\gamma,\omega} * \phi)(z)} \right] &< 2z, \\
 (\Omega_{\alpha,\beta}^{\gamma,\omega} * \phi)(z) \left[1 + z \left(\frac{(\Omega_{\alpha,\beta}^{\gamma,\omega} * \phi)(z)}{(\Omega_{\alpha,\beta}^{\gamma,\omega} * \phi)(z)} \right)^2 \right] &< 2z,
 \end{aligned} \tag{29}$$

then $\Psi(z) := 1/e^{\sigma z} \int_0^z (\Omega_{\alpha,\beta}^{\gamma,\omega} * \phi)(\zeta) e^{\sigma \zeta} d\zeta \in \mathcal{H}[0, 1]$ and

$$\left| \frac{1}{e^{\sigma z}} \int_0^z (\Omega_{\alpha,\beta}^{\gamma,\omega} * \phi)(\zeta) e^{\sigma \zeta} d\zeta \right| < \frac{1}{2-|\sigma|}, \quad |\sigma| < 2. \tag{30}$$

Proof. Suppose that the operator $(\Omega_{\alpha,\beta}^{\gamma,\omega} * \phi)$ achieves one of the subordination inequalities (a)-(h) then, in view of Propo-

sition 2 and results in Lemma 4, we have

$$(\Omega_{\alpha,\beta}^{\gamma,\omega} * \phi)(z) < z, \quad z \in \mathbb{U}. \tag{31}$$

Consequently, we obtain the upper bound inequality

$$|(\Omega_{\alpha,\beta}^{\gamma,\omega} * \phi)(z)| < 1, \quad z \in \mathbb{U}. \tag{32}$$

The function $e^{\sigma z}$ achieves the real inequality

$$\Re \left(\frac{z(e^{\sigma z})'}{e^{\sigma z}} + 2 \right) = \Re(\sigma z + 2) > 0, \tag{33}$$

provided that $|\sigma| < 2$. Moreover, we have superior inequality

$$\begin{aligned}
 \chi &= \sup_{|z|<1} \left\{ \frac{|e^{\sigma z}|}{|z(e^{\sigma z})' + 2e^{\sigma z}|} \right\} = \sup_{|z|<1} \left\{ \frac{|e^{\sigma z}|}{|z\sigma e^{\sigma z} + 2e^{\sigma z}|} \right\} \\
 &\leq \frac{1}{2-|\sigma|} < \infty, \quad |\sigma| < 2.
 \end{aligned} \tag{34}$$

Hence, in view of [22] [Corollary 4.3a.2, P210], we conclude that

$$\left| \frac{1}{e^{\sigma z}} \int_0^z (\Omega_{\alpha,\beta}^{\gamma,\omega} * \phi)(\zeta) e^{\sigma \zeta} d\zeta \right| \leq \frac{|z|^2}{2-|\sigma|}, \quad |\sigma| < 2, \tag{35}$$

which yields

$$\left| \frac{1}{e^{\sigma z}} \int_0^z (\Omega_{\alpha,\beta}^{\gamma,\omega} * \phi)(\zeta) e^{\sigma \zeta} d\zeta \right| < \frac{1}{2-|\sigma|}, \tag{36}$$

($|\sigma| < 2, z \in \mathbb{U}$),

hence the proof. $\square$

Now, we investigate another integral inequality involving the operator $(\Omega_{\alpha,\beta}^{\gamma,\omega} * \phi)$ where $\phi \in \Lambda$.

Theorem 11. Consider $\phi \in \Lambda$ and the operator $(\Omega_{\alpha,\beta}^{\gamma,\omega} * \phi)(z)$, where $\alpha, \beta, \gamma, \omega \in \mathbb{C}, z \in \mathbb{U}, \Re(\alpha), \Re(\beta) > 0$. If one of the subordination inequalities in Theorem 10 holds, then

$$\begin{aligned}
 G(z) &:= \frac{1}{e^{\sigma z}} \int_0^z (\Omega_{\alpha,\beta}^{\gamma,\omega} * \phi)(\zeta) g(\zeta) \zeta^{\sigma-1} d\zeta \in \mathcal{H}[0, 1], \\
 \left| \frac{1}{e^{\sigma z}} \int_0^z (\Omega_{\alpha,\beta}^{\gamma,\omega} * \phi)(\zeta) g(\zeta) d\zeta \right| &< \frac{|g(z)|}{1-|\sigma|}, \quad |\sigma| < 1,
 \end{aligned} \tag{37}$$

where $g \in \mathcal{H}[1, 1], g \neq 0$.

Proof. Suppose that the operator $(\Omega_{\alpha,\beta}^{\gamma,\omega} * \phi)$ has one of the subordination inequalities (a)-(h) in Theorem 10 then, in view of Proposition 2, we have

$$(\Omega_{\alpha,\beta}^{\gamma,\omega} * \phi)(z) < z, \quad z \in \mathbb{U}, \tag{38}$$

which implies that

$$|(\Omega_{\alpha,\beta}^{\gamma,\omega} * \phi)(z)| < 1, \quad z \in \mathbb{U}. \quad (39)$$

As in Theorem 10, the function $e^{\sigma z}$ admits the real inequality

$$\Re \left(\frac{z(e^{\sigma z})'}{e^{\sigma z}} + 1 \right) = \Re(\sigma z + 1) > 0, \quad (40)$$

provided that $|\sigma| < 1$. Moreover, we have superior inequality

$$\begin{aligned} \nu &= \sup_{|z|<1} \left\{ \frac{|g(z)|}{|z(e^{\sigma z})' + (1+\sigma)e^{\sigma z}|} \right\} = \sup_{|z|<1} \left\{ \frac{|g(z)|}{|z\sigma e^{\sigma z} + e^{\sigma z}|} \right\} \\ &= \sup_{|z|<1} \left\{ \frac{|g(z)|}{1-|\sigma|} \right\}, \quad |\sigma| < 1 < \infty, \end{aligned} \quad (41)$$

Thus, in view of [22] [Theorem 4.3a, P207], we indicate that then $G(z) \in \mathcal{H}[0, 1]$ and

$$\left| \frac{1}{e^{\sigma z}} \int_0^z (\Omega_{\alpha,\beta}^{\gamma,\omega} * \phi)(\zeta) g(\zeta) d\zeta \right| < \frac{|g(z)|}{1-|\sigma|}, \quad |\sigma| < 1, \quad (42)$$

where $g \in \mathcal{H}[1, 1]$, $g \neq 0$, hence the proof. ?

In addition, we have the following result by replacing $e^{\sigma z}$ by $ze^{\sigma z}$:

Theorem 12. Consider $\phi \in \Lambda$ and the operator $(\Omega_{\alpha,\beta}^{\gamma,\omega} * \phi)(z)$, where $\alpha \neq 0, z \in \mathbb{U}, \gamma \in \mathbb{R}_+$. If one of the subordination inequalities in Theorem 10 holds, then

$$\begin{aligned} L(z) &:= \frac{1}{z^\kappa e^{\sigma z}} \int_0^z (\Omega_{\alpha,\beta}^{\gamma,\omega} * \phi)(\zeta) (1 + \sigma\zeta) e^{\sigma\zeta} \zeta^{\kappa-1} d\zeta \in \mathcal{H}[0, 1], \\ \left| \frac{1}{z^\kappa e^{\sigma z}} \int_0^z (\Omega_{\alpha,\beta}^{\gamma,\omega} * \phi)(\zeta) (1 + \sigma\zeta) e^{\sigma\zeta} \zeta^{\kappa-1} d\zeta \right| &< \left\{ \frac{1+|\sigma|}{1-|\sigma|+\kappa} \right\}, \quad \kappa > 0, |\sigma| \leq 1. \end{aligned} \quad (43)$$

Proof. Suppose that the operator $(\Omega_{\alpha,\beta}^{\gamma,\omega} * \phi)$ has one of the subordination inequalities (a)-(h) then, in view of Proposition 2 and results in [22], P138-140, we have

$$(\Omega_{\alpha,\beta}^{\gamma,\omega} * \phi)(z) < z, \quad z \in \mathbb{U}, \quad (44)$$

which leads to

$$|(\Omega_{\alpha,\beta}^{\gamma,\omega} * \phi)(z)| < 1, \quad z \in \mathbb{U}. \quad (45)$$

The function $ze^{\sigma z}$ admits the following properties

$$\begin{aligned} ze^{\sigma z} &\in \Lambda, \\ \frac{(ze^{\sigma z})'(ze^{\sigma z})}{z} &= e^{2\sigma z} + \sigma ze^{2\sigma z} \neq 0, \quad |\sigma| \leq 1; \\ \Re \left(\frac{z(ze^{\sigma z})'}{(ze^{\sigma z})} + \kappa \right) &= \Re(1 + \sigma z + \kappa) > 0. \end{aligned} \quad (46)$$

Moreover, we have superior inequality

$$\begin{aligned} S &= \sup_{|z|<1} \left\{ \frac{|z(ze^{\sigma z})'|}{|z(ze^{\sigma z})' + \kappa(ze^{\sigma z})|} \right\} = \sup_{|z|<1} \left\{ \frac{1 + \sigma z}{1 + \sigma z + \kappa} \right\} \\ &= \sup_{|z|<1} \left\{ \frac{1+|\sigma|}{1-|\sigma|+\kappa} \right\}, \quad |\sigma| < 1, \kappa > 0 < \infty. \end{aligned} \quad (47)$$

Thus, in view of [22]-[Corollary 4.3a.1, P208], $L(z) \in \mathcal{H}[0, 1]$ and

$$\begin{aligned} \left| \frac{1}{z^\kappa e^{\sigma z}} \int_0^z (\Omega_{\alpha,\beta}^{\gamma,\omega} * \phi)(\zeta) (1 + \sigma\zeta) e^{\sigma\zeta} \zeta^{\kappa-1} d\zeta \right| &< \left\{ \frac{1+|\sigma|}{1-|\sigma|+\kappa} \right\}, \\ (\kappa > 0, \quad |\sigma| \leq 1, \quad z \in \mathbb{U}, \phi \in \Lambda). \end{aligned} \quad (48)$$

This completes the proof. ?

Theorem 13. Consider that ϕ is convex univalent function in the open unit disk and the operator $(\Omega_{\alpha,\beta}^{\gamma,\omega} * \phi)(z)$, where $\alpha \neq 0, z \in \mathbb{U}, \beta, \gamma \in \mathbb{R}_+$. Then,

$$\Psi(z) \in \mathcal{H}[0, 1], \quad (49)$$

with

$$\begin{aligned} \left| \frac{1}{e^{\sigma z}} \int_0^z (\Omega_{\alpha,\beta}^{\gamma,\omega} * \phi)(\zeta) e^{\sigma\zeta} d\zeta \right| &< \frac{{}_2\mathcal{W}_2^* \left[\begin{matrix} (1, 1) & (1 + \gamma, 1) \\ (2, 1) & (\alpha + \beta, \alpha) \end{matrix}; \omega^{1/\alpha} \right]}{2-|\sigma|}, \quad |\sigma| < 2. \end{aligned} \quad (50)$$

$G(z) \in \mathcal{H}[0, 1]$ with

$$\begin{aligned} \left| \frac{1}{e^{\sigma z}} \int_0^z (\Omega_{\alpha,\beta}^{\gamma,\omega} * \phi)(\zeta) g(\zeta) d\zeta \right| &< \frac{|g(z)| {}_2\mathcal{W}_2^* \left[\begin{matrix} (1, 1) & (1 + \gamma, 1) \\ (2, 1) & (\alpha + \beta, \alpha) \end{matrix}; \omega^{1/\alpha} \right]}{1-|\sigma|}, \quad |\sigma| < 1, \end{aligned} \quad (51)$$

where $g \in \mathcal{H}[1, 1]$, $g \neq 0$.

$L(z) \in \mathcal{H}[0, 1]$ with

$$\begin{aligned} & \left| \frac{1}{z^\kappa e^{\sigma z}} \int_0^z \left(\Omega_{\alpha, \beta}^{\gamma, \omega} * \phi \right) (\zeta) (1 + \sigma \zeta) e^{\sigma \zeta} \zeta^{\kappa-1} d\zeta \right| \\ & \quad (1 + |\sigma|)_2 \mathcal{W}_2^* \left[\begin{matrix} (1, 1) & (1 + \gamma, 1) \\ (2, 1) & (\alpha + \beta, \alpha) \end{matrix} ; \omega^{1/\alpha} \right] \\ & < \frac{1 + |\sigma| + \kappa}{(\kappa > 0, \quad |\sigma| \leq 1, \quad z \in \mathbb{U})}, \end{aligned} \quad (52)$$

Proof. Let ϕ be convex univalent in $\mathbb{U}$. Then, in view of Proposition 7, we have

$$\left| \left(\Omega_{\alpha, \beta}^{\gamma, \omega} * \phi \right) (z) \right| < r_2 \mathcal{W}_2^* \left[\begin{matrix} (1, 1) & (1 + \gamma, 1) \\ (2, 1) & (\alpha + \beta, \alpha) \end{matrix} ; \omega^{1/\alpha} r \right]. \quad (53)$$

Consequently, by assuming $r \rightarrow 1$, we obtain

$$\left| \left(\Omega_{\alpha, \beta}^{\gamma, \omega} * \phi \right) (z) \right| < {}_2\mathcal{W}_2^* \left[\begin{matrix} (1, 1) & (1 + \gamma, 1) \\ (2, 1) & (\alpha + \beta, \alpha) \end{matrix} ; \omega^{1/\alpha} \right]. \quad (54)$$

By the proof of Theorem 10, we conclude that (A). Similarly, by using the proof in Theorems 11 and 12, we have (B) and (C), respectively. This ends the proof. $\square$

Theorem 14. Consider that ϕ is univalent function in the open unit disk and the operator $(\Omega_{\alpha, \beta}^{\gamma, \omega} * \phi)(z)$, where $\alpha \neq 0, z \in \mathbb{U}$, $\beta, \gamma \in \mathbb{R}_+$. Then

$\Psi(z) \in \mathcal{H}[0, 1]$ with

$$\left| \frac{1}{e^{\sigma z}} \int_0^z \left(\Omega_{\alpha, \beta}^{\gamma, \omega} * \phi \right) (\zeta) e^{\sigma \zeta} d\zeta \right| < \frac{{}_1\mathcal{W}_1^* \left[\begin{matrix} (1 + \gamma, 1) \\ (\alpha + \beta, \alpha) \end{matrix} ; \omega^{1/\alpha} \right]}{2 - |\sigma|}, \quad |\sigma| < 2. \quad (55)$$

$G(z) \in \mathcal{H}[0, 1]$ with

$$\left| \frac{1}{e^{\sigma z}} \int_0^z \left(\Omega_{\alpha, \beta}^{\gamma, \omega} * \phi \right) (\zeta) g(\zeta) d\zeta \right| < \frac{|g(z)| {}_1\mathcal{W}_1^* \left[\begin{matrix} (1 + \gamma, 1) \\ (\alpha + \beta, \alpha) \end{matrix} ; \omega^{1/\alpha} \right]}{1 - |\sigma|}, \quad |\sigma| < 1, \quad (56)$$

where $g \in \mathcal{H}[1, 1]$, $g \neq 0$.

$L(z) \in \mathcal{H}[0, 1]$ with

$$\begin{aligned} & \left| \frac{1}{z^\kappa e^{\sigma z}} \int_0^z \left(\Omega_{\alpha, \beta}^{\gamma, \omega} * \phi \right) (\zeta) (1 + \sigma \zeta) e^{\sigma \zeta} \zeta^{\kappa-1} d\zeta \right| \\ & \quad (1 + |\sigma|)_1 \mathcal{W}_1^* \left[\begin{matrix} (1 + \gamma, 1) \\ (\alpha + \beta, \alpha) \end{matrix} ; \omega^{1/\alpha} \right] \\ & < \frac{1 + |\sigma| + \kappa}{(\kappa > 0, \quad |\sigma| \leq 1, \quad z \in \mathbb{U})}. \end{aligned} \quad (57)$$

Proof. Let ϕ be convex univalent in $\mathbb{U}$. Then, in view of Proposition 9, we have

$$\left| \left(\Omega_{\alpha, \beta}^{\gamma, \omega} * \phi \right) (z) \right| < r_1 \mathcal{W}_1^* \left[\begin{matrix} (1 + \gamma, 1) \\ (\alpha + \beta, \alpha) \end{matrix} ; \omega^{1/\alpha} r \right]. \quad (58)$$

Consequently, by assuming $r \rightarrow 1$, we have

$$\left| \left(\Omega_{\alpha, \beta}^{\gamma, \omega} * \phi \right) (z) \right| < {}_1\mathcal{W}_1^* \left[\begin{matrix} (1 + \gamma, 1) \\ (\alpha + \beta, \alpha) \end{matrix} ; \omega^{1/\alpha} \right]. \quad (59)$$

By the proof of Theorem 10, we conclude that (A). Similarly, by using the proof in Theorems 11 and 12, we have (B) and (C), respectively. This ends the proof. $\square$

In the next result, we discuss the starlikeness of the operator $(\Omega_{\alpha, \beta}^{\gamma, \omega} * \phi)$.

Theorem 15. Consider the operator $(\Omega_{\alpha, \beta}^{\gamma, \omega} * \phi)$, $\phi \in \Lambda$.

(A) If $|(\Omega_{\alpha, \beta}^{\gamma, \omega} * \phi)(z) / e^{\sigma z} (\sigma z + 1) - 1| < 0.04$, where $|\sigma| < 1/2(\sqrt{13} - 3) \approx 0.3027$ then $(\Omega_{\alpha, \beta}^{\gamma, \omega} * \phi) \in \mathcal{S}^*$

(B) If $|z\phi'(z)/\phi(z) - 1| < 0.374$ and $\sup \{ |(\Omega_{\alpha, \beta}^{\gamma, \omega} * \phi)(z) / (\Omega_{\alpha, \beta}^{\gamma, \omega} * \phi)(z) | \} \equiv |\sigma|$, $|\sigma| < 1$ then

$$\int_0^z \left(\Omega_{\alpha, \beta}^{\gamma, \omega} * \phi \right) (z) \phi'(z) dz \in \mathcal{S}^*. \quad (60)$$

(C) If $| (\Omega_{\alpha, \beta}^{\gamma, \omega} * \phi)(z) | < 2/\sqrt{5}$ then $(\Omega_{\alpha, \beta}^{\gamma, \omega} * \phi)(z) \in \mathcal{S}^*$

Proof. For part (A), assume that $g(z) = ze^{\sigma z}$ then

$$\sup \left\{ \left| \frac{g'(z)}{g(z)} \right| \right\} = \sup \left\{ \left| \frac{2\sigma + z\sigma^2}{1 + z\sigma} \right| \right\}. \quad (61)$$

Consequently, we have

$$\sup \left\{ \left| \frac{2\sigma + z\sigma^2}{1 + z\sigma} \right| \right\} \leq \frac{2|\sigma| + |\sigma|^2}{1 - |\sigma|} < 1, \quad |\sigma| < 1. \quad (62)$$

The value $|\sigma| < 1/2(\sqrt{13} - 3) \approx 0.3027$ implies that $2|\sigma| + |\sigma|^2 / 1 - |\sigma| \approx 0.98 < 1$. Since

$$\left| \frac{\left(\Omega_{\alpha, \beta}^{\gamma, \omega} * \phi \right) (z)}{e^{\sigma z} (\sigma z + 1)} - 1 \right| < 0.04, \quad (63)$$

then in view of [22]-[Theorem 5.5c, P296], we conclude that $(\Omega_{\alpha, \beta}^{\gamma, \omega} * \phi) \in \mathcal{S}^*$.

For part (B), since

$$\left| \frac{z\phi'(z)}{\phi(z)} - 1 \right| < 0.374, \quad (64)$$

where the number 0.374 is a solution of the equation $(1+x)e^x = 2$ then by [22] [Theorem 5.5g, P299], we have $|\phi'(z) - 1| < 1$. Moreover, in terms of $|\sigma|$, we have

$$\begin{aligned} |\phi'(z) - 1| &< \frac{(2-|\sigma|)\sqrt{2(2-2|\sigma|)+1-|\sigma|}}{(2-|\sigma|)^2+1} < 1, |\sigma| < 1, \\ \lim_{|\sigma| \rightarrow 1} \frac{(2-|\sigma|)\sqrt{2(2-2|\sigma|)+1-|\sigma|}}{(2-|\sigma|)^2+1} &\approx 0. \end{aligned} \quad (65)$$

by [22] [Theorem 5.5d, P298] then

$$\int_0^z (\Omega_{\alpha,\beta}^{\gamma,\omega} * \phi)(z) \phi'(z) dz \in \mathcal{S}^*. \quad (66)$$

The last part immediately comes from [22] [Corollary 5.5.a, P294]. This ends the proof. ?

Theorem 16. Consider the operator $(\Omega_{\alpha,\beta}^{\gamma,\omega} * \phi)$, $\phi \in \Lambda$.

(A) If the following inequality holds

$$\frac{z(\Omega_{\alpha,\beta}^{\gamma,\omega} * \phi)(z)}{(\Omega_{\alpha,\beta}^{\gamma,\omega} * \phi)(z)} < e^z - 1, \quad z \in \mathbb{U}, \quad (67)$$

then

$$(\Omega_{\alpha,\beta}^{\gamma,\omega} * \phi)(z) < \exp \left(\int_0^z (e^\zeta - 1) \zeta^{-1} d\zeta \right), \quad z \in \mathbb{U}, \quad (68)$$

(B) If the subordination occurs

$$\frac{z(\Omega_{\alpha,\beta}^{\gamma,\omega} * \phi)(z)}{(\Omega_{\alpha,\beta}^{\gamma,\omega} * \phi)(z)} - 1 < e^z - 1, \quad z \in \mathbb{U}, \quad (69)$$

then

$$\frac{(\Omega_{\alpha,\beta}^{\gamma,\omega} * \phi)(z)}{z} < \exp \left(\int_0^z (e^\zeta - 1) \zeta^{-1} d\zeta \right), \quad z \in \mathbb{U}, \quad (70)$$

(C) If the next relation exists

$$z(\Omega_{\alpha,\beta}^{\gamma,\omega} * \phi)(z) < e^z - 1, \quad z \in \mathbb{U}, \quad (71)$$

then

$$(\Omega_{\alpha,\beta}^{\gamma,\omega} * \phi)(z) < \int_0^z (e^\zeta - 1) \zeta^{-1} d\zeta. \quad (72)$$

Proof. It is clear that the function

$$f(z) = e^z - 1 = z + \frac{z^2}{2} + \frac{z^3}{6} + \frac{z^4}{24} + \frac{z^5}{120} + O(z^6), \quad (73)$$

satisfies $f(0) = 0$ and it is convex in the open unit disk. Consequently, it is starlike. By Proposition 2, the operator $(\Omega_{\alpha,\beta}^{\gamma,\omega} * \phi)(z) \in \Lambda$ and hence $(\Omega_{\alpha,\beta}^{\gamma,\omega} * \phi)(0) = 1$ that is $(\Omega_{\alpha,\beta}^{\gamma,\omega} * \phi)(z) \in \mathcal{H}[1, 1]$. Similarly for the function

$$\frac{(\Omega_{\alpha,\beta}^{\gamma,\omega} * \phi)(z)}{z} \in \mathcal{H}[1, 1]. \quad (74)$$

Thus, in view of [22]-[Corollary 3.1d.1, P76], we have the desired results.

For the last part (C), $z(\Omega_{\alpha,\beta}^{\gamma,\omega} * \phi)(z) \in \mathcal{H}[0, 1]$; thus, in virtue of [22]-[Theorem 3.1d, P76], where $a = 0$, we conclude the last subordination. ?

2.1. Fractional Saint-Venant Equations. By using the fractional calculus of the construction, we formulate the fractional 2D-Saint-Venant equations utilizing the functional convolution operator $\Omega_{\alpha,\beta}^{\gamma,\omega} * \phi$, $\phi \in \Lambda$, and $z \in \mathbb{U}$.

Example 1. We investigate the upper bound of the two-dimensional Saint-Venant equations (2D-SVE) of diffusive wave (this equation has measured the level of the water). This equation simply presents the formula

$$\frac{d\Theta(z)}{dz} - \Delta(z) = 0, \quad (75)$$

where Θ is the height deviation of the horizontal pressure surface at two-dimensional position $z = x + iy$ and $\Delta(\zeta)$ represents the difference of bed slope. By using the convolution operator, we generalize 2D-SVE into the form

$$(\Omega_{\alpha,\beta}^{\gamma,\omega} * \phi)(z) - \Delta(z) = 0, \quad z \in \mathbb{U}. \quad (76)$$

Multiplying both sides of Eq. (76) by z and let

$$\Delta(z) = \frac{e^z - 1}{z} = 1 + \frac{z}{2} + \frac{z^2}{6} + \frac{z^3}{24} + \frac{z^4}{120} + \frac{z^5}{720} + O(z^6), \quad (77)$$

we have

$$z(\Omega_{\alpha,\beta}^{\gamma,\omega} * \phi)(z) = e^z - 1, \quad z \in \mathbb{U}. \quad (78)$$

(a) The first row represents the function $f(z) = (e^z) - 1$ and the second row indicates the solution (79) of the extended 2D-SVE

(b) The solution $f(z) := (\Omega_{\alpha, \beta}^{\gamma, \omega} * \phi)(z)$

FIGURE 1: Saint-Venant equations (2D-SVE) of diffusive wave.

Thus, in view of Theorem 16-(C), we conclude the upper solution of Eq. (78) is given by (see Figure 1, second row)

$$(\Omega_{\alpha,\beta}^{\gamma,\omega} * \phi)(z) < \int_0^z (e^\zeta - 1) \zeta^{-1} d\zeta. \quad (79)$$

Hence, we obtain

$$\begin{aligned} (\Omega_{\alpha,\beta}^{\gamma,\omega} * \phi)(z) &= c + Ei(z) - \log(z) = c + 0.577 + z + \frac{z^2}{4} \\ &\quad + \frac{z^3}{18} + \frac{z^4}{96} + \frac{z^5}{600} + \frac{z^6}{4320} + O(z^7) \\ &\quad - i\pi \left[\frac{(\arg(z) + \pi)}{(2\pi)} \right], \end{aligned} \quad (80)$$

where c is a constant and

$$Ei(z) = 0.577 + \ln(z) + z + \dots, \quad (81)$$

indicates the exponential integral. Assuming that

$$c = i\pi \left[\frac{(\arg(z) + \pi)}{(2\pi)} \right] - 0.577, \quad (82)$$

we get

$$(\Omega_{\alpha,\beta}^{\gamma,\omega} * \phi)(z) = z + \frac{z^2}{4} + \frac{z^3}{18} + \frac{z^4}{96} + \frac{z^5}{600} + \frac{z^6}{4320} + O(z^7) \in \Lambda. \quad (83)$$

By the convexity of e^z (see [22]-P139), we confirm that the solution is normalized analytic convex in $\cup$. Note that the term $(\Omega_{\alpha,\beta}^{\gamma,\omega} * \phi)(z)$ is called the convective acceleration term. Figure 1 shows the behavior of solutions of 2D-SVE of diffusive wave.

3. Conclusion

From above, we have extended the Prabhakar operator in the open unit disk. We formulated it in a linear convolution operator with a normalized function. A class of integral inequalities is investigated involving special functions. The upper bound of the suggested operator is computed by using the Fox-Wright function, for a class of convex functions and univalent functions. Moreover, we applied the operator to generalize the 2D-SVE. A solution of the extended 2D-SVE is computed by using recent result (Theorem 16).

For future works, one can consider extra studies in the geometric function theory by considering the operator in different classes of analytic functions, such as normalized functions, harmonic functions, and meromorphic functions.

Data Availability

Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

Conflicts of Interest

The authors declare no conflict of interest.

Authors' Contributions

All authors contributed equally and significantly to writing this article. All authors read and agreed to the published version of the manuscript.

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