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Research Article

Boundedness for Commutators of Rough p-Adic Hardy Operator on p-Adic Central Morrey Spaces

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In the present article we obtain the boundedness for commutators of rough p-adic Hardy operator on p-adic central Morrey spaces. Furthermore, we also acquire the boundedness of rough p-adic Hardy operator on Lebesgue spaces.

1. Introduction

The classical Hardy operator for a non-negative function $f : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is given as

$$\mathscr{H}f(x) = \frac{1}{x}\int_0^x f(t)dt, \quad x > 0. \tag{1}$$

In [1], Hardy defined the above operator which satisfies

$$\|\mathscr{H}f\|_{L^r(\mathbb{R}^+)} \le \frac{r}{r-1}\|f\|_{L^r(\mathbb{R}^+)}, \quad 1 < r < \infty. \tag{2}$$

The constant $r/(r-1)$ in (2) is sharp. In [2], Faris extended the Hardy operator in $\mathbb{R}^n$ by

$$Hf(\mathbf{x}) = \frac{1}{|\mathbf{x}|^n}\int_{(B(0,|\mathbf{x}|))} f(\mathbf{t})d\mathbf{t}. \tag{3}$$

In this day and age, the Hardy operator has received a relentless consideration, see for example [3–7]. Moreover, the publications [8–12] and the references therein will do world of good to comprehend the Hardy type operators.

The past few years has seen an immense attention towards mathematical physics [13, 14] along with harmonic analysis in the p-adic field [15–23]. Furthermore, the applications of p-adic analysis are seen mainly in string theory [24], quantum gravity [25, 26], quantum mechanics [14] and spring glass theory [27, 28].

Suppose p is a prime number, $r \in \mathbb{Q}$, we introduce the p-adic norm $|r|_p$ by a rule

$$|0|_p = 0, |r|_p = p^{-\alpha}, \tag{4}$$

where the integer $\alpha = \alpha(r)$ is defined by the following notation

$$r = p^\alpha m/n, \tag{5}$$

integers m, n and p are coprime to each other. $|\cdot|_p$ has many properties of a real norm together with

$$|r+s|_p \le \max\left\{|r|_p, |s|_p\right\}. \tag{6}$$

We denote the completion of $\mathbb{Q}$ in the norm $|\cdot|_p$ by $\mathbb{Q}_p$. Any nonzero p-adic number can be written in series form as (see [14]):

$$r = p^\alpha \sum_{i=0}^{\infty} \gamma_i p^i, \tag{7}$$

where $\gamma_i, \alpha \in \mathbb{Z}, \gamma_i \in \mathbb{Z}/p\mathbb{Z}_p, \gamma_0 \neq 0$. The series (7) is convergent as $|p^\alpha \gamma_i p^i|_p = p^{-\alpha-i}$.

The space $\mathbb{Q}_p^n$ contains all n-tuples of $\mathbb{Q}_p$. The norm on this space is

$$|\mathbf{r}|_p = \max_{1\le k\le n} |r_k|_p. \tag{8}$$

Represent by $B_\alpha(\mathbf{a})$ the ball with radius p^α and center at $\mathbf{a}$ and $S_\alpha(\mathbf{a})$ its sphere:

$$B_\alpha(\mathbf{a}) = \left\{\mathbf{r} \in \mathbb{Q}_p^n : |\mathbf{r}-\mathbf{a}|_p \le p^\alpha\right\}, S_\alpha(\mathbf{a}) = \left\{\mathbf{r} \in \mathbb{Q}_p^n : |\mathbf{r}-\mathbf{a}|_p = p^\alpha\right\}. \tag{9}$$

Since $\mathbb{Q}_p^n$ is a locally compact Hausdorff space, then there exists the Haar measure $d\mathbf{x}$ on additive group $\mathbb{Q}_p^n$ and is normalized by

$$\int_{B_0(\mathbf{0})} d\mathbf{x} = |B_0(\mathbf{0})|_H = 1, \tag{10}$$

where $|E|_H$ denotes the Haar measure of a measurable subset E of $\mathbb{Q}_p^n$. Moreover, it is not hard to see that $|B_\gamma(\mathbf{a})|_H = p^{n\gamma}$ and $|S_\gamma(\mathbf{a})|_H = p^{n\gamma}(1-p^{-n})$, for any $\mathbf{a} \in \mathbb{Q}_p^n$.

Suppose $L^s(\mathbb{Q}_p^n)(1 \le s<\infty)$ is the space of all complex-valued functions f on $\mathbb{Q}_p^n$ such that

$$\|f\|_{L^s(\mathbb{Q}_p^n)} = \left(\int_{\mathbb{Q}_p^n} |f(\mathbf{x})|^s d\mathbf{x}\right)^{1/s} < \infty. \tag{11}$$

In what follows author in [29] introduced the Hardy operator in the p-adic field as for $f \in L_{loc}(\mathbb{Q}_p^n)$, we have

$$H^p f(\mathbf{x}) = \frac{1}{|\mathbf{x}|_p^n} \int_{B(\mathbf{0},|\mathbf{x}|_p)} f(\mathbf{t}) d\mathbf{t}. \tag{12}$$

For better understanding of Hardy type operators in the p-adic field we refer the publications [12, 29–32] and the references therein. From here on, we discuss the rough kernel version of an operator which is also considered an important topic in analysis, see for instance [20, 33–37]. In [10], Fu et al. studied the roughness of Hardy operator in the real field. In the p-adic setting, the rough Hardy operator and its commutator are defined and studied in [20]. Suppose $f : \mathbb{Q}_p^n \longrightarrow \mathbb{R}$, $b : \mathbb{Q}_p^n \longrightarrow \mathbb{R}$ and $\Omega : S_0 \longrightarrow \mathbb{R}$ are measurable mappings, then

$$\begin{aligned} H_\Omega^p f(\mathbf{x}) &= \frac{1}{|\mathbf{x}|_p^n} \int_{B(\mathbf{0},|\mathbf{x}|_p)} \Omega\left(|\mathbf{t}|_p \mathbf{t}\right) f(\mathbf{t}) d\mathbf{t} \\ H_\Omega^{p,b} f(\mathbf{x}) &= \frac{1}{|\mathbf{x}|_p^n} \int_{B(\mathbf{0},|\mathbf{x}|)_p} (b(\mathbf{x}) - b(\mathbf{t}))\Omega\left(|\mathbf{t}|_p \mathbf{t}\right) f(\mathbf{t}) d\mathbf{t}, \end{aligned} \tag{13}$$

respectively, whenever

$$\begin{aligned} &\int_{B(\mathbf{0},|\mathbf{x}|_p)} \left|\Omega\left(|\mathbf{t}|_p \mathbf{t}\right) f(\mathbf{t})\right| d\mathbf{t} < \infty \\ &\int_{B(\mathbf{0},|\mathbf{x}|_p)} \left|b(\mathbf{t})\Omega\left(|\mathbf{t}|_p \mathbf{t}\right) f(\mathbf{t})\right| d\mathbf{t} < \infty. \end{aligned} \tag{14}$$

In [20], authors showed the weighted estimates of $H_\Omega^{p,b}$ on two weighted Herz-Morrey spaces. In the present article, we acquire the $\lambda -$ *central* bounded mean oscillations $(C\dot{M}O^{r,\lambda}(\mathbb{Q}_p^n))$ estimate of $H_\Omega^{p,b}$ on p-adic central Morrey spaces. In addition, we open up our results with a lemma which shows the boundedness of rough p-adic Hardy operator on Lebesgue spaces. Throughout this paper, we have no intention to obtain the best constants in the inequalities. The occurrence of a letter C does not mean a same constant, its value may vary at different positions.

Definition 1 [32]. Suppose $\lambda \in \mathbb{R}$ and $1 < r < \infty$. The p-adic space $\dot{B}^{r,\lambda}(\mathbb{Q}_p^n)$ is defined as follows

$$\|f\|_{\dot{B}^{r,\lambda}(\mathbb{Q}_p^n)} = \sup_{\gamma\in\mathbb{Z}} \left(\frac{1}{|B_\gamma|_H^{1+\lambda r}} \int_{B_\gamma} |f(\mathbf{x})|^r d\mathbf{x}\right)^{1/r} < \infty, \tag{15}$$

where $B_\gamma = B_\gamma(0)$. Interestingly $\dot{B}^{r,\lambda}(\mathbb{Q}_p^n)$ reduces to $\{0\}$ for $-1/r > \lambda$.

Definition 2 [32]. Suppose $\lambda < 1/n$ and $1 < r < \infty$. The p-adic space $C\dot{M}O^{r,\lambda}(\mathbb{Q}_p^n)$ is given by

$$\|f\|_{C\dot{M}O^{r,\lambda}(\mathbb{Q}_p^n)} = \sup_{\gamma\in\mathbb{Z}} \left(\frac{1}{|B_\gamma|_H^{1+\lambda r}} \int_{B_\gamma} \left|f(\mathbf{x}) - f_{B_\gamma}\right|^r d\mathbf{x}\right)^{1/r} < \infty, \tag{16}$$

where $f_{B_\gamma} = 1/|B_\gamma|_H \int_{B_\gamma} f(\mathbf{x}) d\mathbf{x}$, $|B_\gamma|_H$ is the Haar measure of B_γ.

Remark 3. If $\lambda = 0$, then $C\dot{M}O^{r,\lambda}(\mathbb{Q}_p^n)$ is reduced to $CMO^r(\mathbb{Q}_p^n)$ (see [29]).

2. Boundedness for Commutators of Rough p-Adic Hardy Operator on Central Morrey Spaces

In the present section $(C\dot{M}O^{r,\lambda}(\mathbb{Q}_p^n))$ estimates of $H_\Omega^{p,b}$ on central Morrey spaces in the p-adic field are obtained. However, to prove the result we need few lemmas.

Lemma 4 [32]. *Let $b \in C\dot{M}O^{r,\lambda}(\mathbb{Q}_p^n)$ and $i, j \in \mathbb{Z}$, $\lambda \geq 0$. Then*

$$\left|b_{B_i} - b_{B_j}\right| \leq p^n |i-j| \|b\|_{C\dot{M}O^{r,\lambda}(\mathbb{Q}_p^n)} \max\left\{|B_i|_H^{\lambda}, |B_j|_H^{\lambda}\right\}. \tag{17}$$

Lemma 5. *Suppose $1 < s < \infty$ and $1/s + 1/s' = 1$. Then the inequality*

$$\|H_\Omega^p f\|_{L^s(\mathbb{Q}_p^n)} \leq C \|f\|_{L^s(\mathbb{Q}_p^n)} \tag{18}$$

holds for all $f \in L_{loc}^s(\mathbb{Q}_p^n)$ and $\Omega \in L^s(S_0)$.

Proof. Firstly, we set

$$\tilde{f}(\mathbf{x}) = \frac{1}{1-p^n} \int_{|\xi_p|=1} f\left(|\mathbf{x}|_p^{-1}\xi\right) d\xi, \mathbf{x} \in \mathbb{Q}_p^n. \tag{19}$$

Obviously $\tilde{f}(\mathbf{x}) = \tilde{f}(|\mathbf{x}|_p^{-1})$. In what follows we take this function a radial function on p-adic Lebesgue space. It is not hard to see that

$$H_\Omega^p\left(\tilde{f}\right)(\mathbf{x}) = H_\Omega^p(f)(\mathbf{x}). \tag{20}$$

In [29], it is shown that $\|\tilde{f}\|_{L^s(\mathbb{Q}_p^n)} \leq \|f\|_{L^s(\mathbb{Q}_p^n)}$. Therefore,

$$\frac{\|H_\Omega^p f\|_{L^s(\mathbb{Q}_p^n)}}{\|f\|_{L^s(\mathbb{Q}_p^n)}} \leq \frac{\left\|H_\Omega^p \tilde{f}\right\|_{L^s(\mathbb{Q}_p^n)}}{\left\|\tilde{f}\right\|_{L^s(\mathbb{Q}_p^n)}}. \tag{21}$$

This implies that $\tilde{f} = f$ providing f is a radial function. Consequently, the norm of an operator H_Ω^p along with its restriction to the function $\hat{f}$ have the same operator norm. So, we assume f to be a radial function in the rest of the proof.

By the change of p-adic variables $\mathbf{t} = |\mathbf{x}|_p^{-1}\mathbf{y}$, we have

$$\begin{aligned}\|H_\Omega^p f\|_{L^s(\mathbb{Q}_p^n)} &= \left(\int_{\mathbb{Q}_p^n} \left|\frac{1}{|\mathbf{x}|_p^n} \int_{B(\mathbf{0},|\mathbf{x}|_p)} \Omega\left(|\mathbf{t}|_p \mathbf{t}\right) f(\mathbf{t}) d\mathbf{t}\right|^s d\mathbf{x}\right)^{1/s} \\ &= \left(\int_{\mathbb{Q}_p^n} \left|\int_{B(\mathbf{0},1)} \Omega(|\mathbf{y}|_p \mathbf{y}) f\left(|\mathbf{x}|_p^{-1}\mathbf{y}\right) d\mathbf{y}\right|^s d\mathbf{x}\right)^{1/s}.\end{aligned} \tag{22}$$

Now by using Minkowski's inequality and Hölder's inequality $(1/s + 1/s' = 1)$, we get

$$\begin{aligned}\|H_\Omega^p f\|_{L^s(\mathbb{Q}_p^n)} &\leq \int_{B(\mathbf{0},1)} \Omega\left(|\mathbf{y}|_p \mathbf{y}\right) \left(\int_{\mathbb{Q}_p^n} \left|f\left(|\mathbf{y}|_p^{-1}\mathbf{x}\right)\right|^s d\mathbf{x}\right)^{1/s} d\mathbf{y} \\ &\leq \left(\int_{B(\mathbf{0},1)} \Omega\left(|\mathbf{y}|_p \mathbf{y}\right) |\mathbf{y}|_p^{-n/s} d\mathbf{y}\right) \|f\|_{L^s(\mathbb{Q}_p^n)} \\ &= \left(\sum_{j=-\infty}^{0} \int_{S_j} \Omega(p^j \mathbf{y}) p^{-nj/s} d\mathbf{y}\right) \|f\|_{L^s(\mathbb{Q}_p^n)} \\ &\leq \sum_{j=-\infty}^{0} p^{-jn/s} \left(\int_{S_j} \left|\Omega(p^j \mathbf{y})\right|^s d\mathbf{y}\right)^{1/s} \left(\int_{S_j} dy\right)^{1/s'} \\ &\quad \cdot \|f\|_{L^s(\mathbb{Q}_p^n)}.\end{aligned} \tag{23}$$

We handle the first part of sum as follows

$$\int_{S_j} \left|\Omega(p^j \mathbf{y})\right|^s d\mathbf{y} = \int_{|\mathbf{z}|_p=1} |\Omega(\mathbf{z})|^s p^{jn} d\mathbf{z} = C p^{jn}. \tag{24}$$

Hence inequality (23) takes the following form

$$\|H_\Omega^p f\|_{L^s(\mathbb{Q}_p^n)} \leq C \|f\|_{L^s(\mathbb{Q}_p^n)}, \tag{25}$$

which completes the proof of a lemma.

Now, we turn towards our key result.

Theorem 6. *Suppose $1 < r_1 < \infty$, $r_1' < r_2 < \infty$, $n(1/r_2 - 1/r_1) < n/r_1$, $1/r_1 + 1/r_2 = 1/r$, $-1/r_1 < \lambda_1 < 0$, $\lambda = \lambda_1 + \lambda_2$ and $0 \leq \lambda_2 < 1/n$. If $r_1' < s < \infty$, then the below inequality*

$$\left\|H_\Omega^{p,b} f\right\|_{\dot{B}^{r,\lambda}(\mathbb{Q}_p^n)} \leq C \|f\|_{\dot{B}^{r,\lambda_1}(\mathbb{Q}_p^n)}, \tag{26}$$

holds for $b \in CMO^{\max\{r_2, sr_1'/(s-r_1'), \lambda_2\}}(\mathbb{Q}_p^n)$ and $\Omega \in L^s(S_0)$.

Proof. We suppose $f \in \dot{B}^{r_1,\lambda_1}(\mathbb{Q}_p^n)$. We also take $\gamma \in \mathbb{Z}$ and without any brevity we consider $\|b\|_{CMO^{\max\{r_2, sr_1'/(s-r_1'), \lambda_2\}}(\mathbb{Q}_p^n)} = 1$. Applying Minkowski's inequality to have

$$\begin{aligned}&\left(\frac{1}{|B_\gamma|_H^{1+\lambda r}} \int_{B_\gamma} \left|H_\Omega^{p,b} f(\mathbf{x})\right|^r d\mathbf{x}\right)^{1/r} \\ &= \left(\frac{1}{|B_\gamma|_H^{1+\lambda r}} \int_{B_\gamma} \left|\frac{1}{|\mathbf{x}|_p^n} \int_{B(\mathbf{0},|\mathbf{x}|_p)} \Omega(|\mathbf{t}|_p \mathbf{t}) f(\mathbf{t})(b(\mathbf{x}) - b(\mathbf{t})) d\mathbf{t}\right|^r d\mathbf{x}\right)^{1/r} \\ &\leq \left(\frac{1}{|B_\gamma|_H^{1+\lambda r}} \int_{B_\gamma} \left|\frac{1}{|\mathbf{x}|_p^n} \int_{B(\mathbf{0},|\mathbf{x}|_p)} \Omega(|\mathbf{t}|_p \mathbf{t}) f(\mathbf{t}) \left(b(\mathbf{x}) - b_{B_\gamma}\right) d\mathbf{t}\right|^r d\mathbf{x}\right)^{1/r} \\ &\quad + \left(\frac{1}{|B_\gamma|_H^{1+\lambda r}} \int_{B_\gamma} \left|\frac{1}{|\mathbf{x}|_p^n} \int_{B(\mathbf{0},|\mathbf{x}|_p)} \Omega(|\mathbf{t}|_p \mathbf{t}) f(\mathbf{t}) \left(b(\mathbf{t}) - b_{B_\gamma}\right) d\mathbf{t}\right|^r d\mathbf{x}\right)^{1/r} \\ &= I + II.\end{aligned} \tag{27}$$

For the evaluation of I, we make use of Lemma (5) which shows that H_Ω^p is bounded from $L^r(\mathbb{Q}_p^n)$ to $L^r(\mathbb{Q}_p^n)$, $(1 < r < \infty)$. By Hölder's inequality $(1 = r/r_1 + r/r_2)$, we have

$$\begin{aligned} I &\le |B_\gamma|_H^{-1/r-\lambda}\left(\int_{B_\gamma}\left|b(\mathbf{x}) - b_{B_\gamma}\right|^{r_2} d\mathbf{x}\right)^{1/r_2}\left(\int_{B_\gamma}\left|H_\Omega^p f(\mathbf{x})\right|^{r_1} d\mathbf{x}\right)^{1/r_1} \\ &\le |B_\gamma|_H^{-1/r-\lambda}\left(\int_{B_\gamma}\left|b(\mathbf{x}) - b_{B_\gamma}\right|^{r_2} d\mathbf{x}\right)^{1/r_2}\left(\int_{B_\gamma}|f(\mathbf{x})|^{r_1} d\mathbf{x}\right)^{1/r_1} \\ &= C\|f\|_{\dot{B}^{r_1,\lambda_1}(\mathbb{Q}_p^n)}. \end{aligned} \tag{28}$$

In order to estimate II, we proceed as follows

$$\begin{aligned} II^r &\le \frac{1}{|B_\gamma|_H^{1+\lambda r}}\int_{B_\gamma}\left|\frac{1}{|\mathbf{x}|_p^n}\int_{B(\mathbf{0},|\mathbf{x}_p)}\Omega(|\mathbf{t}|_p\mathbf{t})f(\mathbf{t})\left(b(\mathbf{t}) - b_{B_\gamma}\right)d\mathbf{t}\right|^r d\mathbf{x} \\ &\le \frac{1}{|B_\gamma|_H^{1+\lambda r}}\sum_{k=-\infty}^{\gamma}\int_{S_k}p^{-knr}\left(\int_{B(\mathbf{0},p^k)}|\Omega\left(|\mathbf{t}|_p\mathbf{t}\right)f(\mathbf{t})\left(b(\mathbf{t}) - b_{B_\gamma}\right)d\mathbf{t}|\right)^r d\mathbf{x} \\ &= \frac{C}{|B_\gamma|_H^{1+\lambda r}}\sum_{k=-\infty}^{\gamma}p^{kn(1-r)}\left(\sum_{j=-\infty}^{k}\int_{S_j}|\Omega(p^j\mathbf{t})f(\mathbf{t})\left(b(\mathbf{t}) - b_{B_\gamma}\right)d\mathbf{t}|\right)^r \\ &\le \frac{C}{|B_\gamma|_H^{1+\lambda r}}\sum_{k=-\infty}^{\gamma}p^{kn(1-r)}\left(\sum_{j=-\infty}^{k}\int_{S_j}|\Omega(p^j\mathbf{t})f(\mathbf{t})\left(b(\mathbf{t}) - b_{B_j}\right)d\mathbf{t}|\right)^r \\ &\quad + \frac{C}{|B_\gamma|_H^{1+\lambda r}}\sum_{k=-\infty}^{\gamma}p^{kn(1-r)}\left(\sum_{j=-\infty}^{k}\int_{S_j}|\Omega(p^j\mathbf{t})f(\mathbf{t})\left(b_{B_j} - b_{B_\gamma}\right)d\mathbf{t}|\right)^r \\ &= II_1 + II_2. \end{aligned} \tag{29}$$

For $j, k \in \mathbb{Z}$ with $j \le k$, we have

$$\int_{S_j}\left|\Omega(p^j\mathbf{t})\right|^s d\mathbf{t} = \int_{|\mathbf{z}|_p=1}|\Omega(\mathbf{z})|^s p^{jn} d\mathbf{z} \le Cp^{kn}. \tag{30}$$

To evaluate II_1, we apply Hölder's inequality together with (30) to get

$$\begin{aligned} II_1 &\le \frac{C}{|B_\gamma|_H^{1+\lambda r}}\sum_{k=-\infty}^{\gamma}p^{kn(1-r)}\Bigg[\sum_{j=-\infty}^{k}\left(\int_{S_j}\left|\Omega(p^j\mathbf{t})\right|^s d\mathbf{t}\right)^{1/s} \\ &\quad \times\left(\int_{S_j}|f(\mathbf{t})|^{r_1}d\mathbf{t}\right)^{1/r_1}\left(\int_{S_j}\left|b(\mathbf{t}) - b_{B_j}\right|^{r_2}d\mathbf{t}\right)^{1/r_2}\Bigg]^r \\ &\le \frac{C}{|B_\gamma|_H^{1+\lambda r}}\|f\|_{\dot{B}^{r,\lambda_1}(\mathbb{Q}_p^n)}\sum_{k=-\infty}^{\gamma}p^{kn(1-r+r/s)}\left\{\sum_{j=-\infty}^{k}|B_j|^{1/r_1+\lambda_1+1/r_2+\lambda_2}\right\}^r \\ &\le \frac{C}{|B_\gamma|_H^{1+\lambda r}}\|f\|_{\dot{B}^{r,\lambda_1}(\mathbb{Q}_p^n)}\sum_{k=-\infty}^{\gamma}p^{kn(1+\lambda r)} = \frac{C}{|B_\gamma|_H^{1+\lambda r}}\|f\|_{\dot{B}^{r,\lambda_1}(\mathbb{Q}_p^n)}p^{\gamma n(1+\lambda r)} \\ &= C\|f\|_{\dot{B}^{r,\lambda_1}(\mathbb{Q}_p^n)}. \end{aligned} \tag{31}$$

The convergence of above series is eminent from $\lambda_1 + \lambda_2 + 1/r_1 + 1/r_2 \ge \lambda_1 + 1 - 1/s > -1/r + 1 - 1/s = 1/r_1{}' - 1/s > 0$.

For II_2, we use Lemma 4, inequality (30) and Hölder's inequality to obtain

$$\begin{aligned} II_2 &= \frac{C}{|B_\gamma|_H^{1+\lambda r}}\sum_{k=-\infty}^{\gamma}p^{kn(1-r)}\left(\sum_{j=-\infty}^{k}\int_{S_j}|\Omega(p^j\mathbf{t})f(\mathbf{t})\left(b_{B_j} - b_{B_\gamma}\right)d\mathbf{t}|\right)^r \\ &\le \frac{C}{|B_\gamma|_H^{1+\lambda r}}\sum_{k=-\infty}^{\gamma}p^{kn(1-r)}\left[\sum_{j=-\infty}^{k}\int_{S_j}\Omega(p^j\mathbf{t})f(\mathbf{t})(\gamma - j)|B_\gamma|_H^{\lambda_2}d\mathbf{t}\right]^r \\ &= \frac{C}{|B_\gamma|_H^{1+\lambda r}}\sum_{k=-\infty}^{\gamma}p^{kn(1-r)}\left[\sum_{j=-\infty}^{k}(\gamma - j)\int_{S_j}\Omega(p^j\mathbf{t})f(\mathbf{t})d\mathbf{t}\right]^r \\ &\le \frac{C}{|B_\gamma|_H^{1+\lambda_1 r}}\sum_{k=-\infty}^{\gamma}p^{kn(1-r)}\Bigg[\sum_{j=-\infty}^{k}(\gamma - j)\left(\int_{S_j}\left|\Omega(p^j\mathbf{t})\right|^s d\mathbf{t}\right)^{1/s} \\ &\quad \times\left(\int_{S_j}|f(\mathbf{t})|^{r_1}d\mathbf{t}\right)^{1/r_1}\left(\int_{S_j}d\mathbf{t}\right)^{1/r_1{}'-1/s}\Bigg]^r \\ &\le \frac{C}{|B_\gamma|_H^{1+\lambda_1 r}}\|f\|_{\dot{B}^{r,\lambda_1}(\mathbb{Q}_p^n)}\sum_{k=-\infty}^{\gamma}p^{kn(1-r+r/s)}\left[\sum_{j=-\infty}^{k}(\gamma - j)|B_j|^{\lambda_1+1-1/s}\right]^r \\ &\le \frac{C}{|B_\gamma|_H^{1+\lambda_1 r}}\|f\|_{\dot{B}^{r,\lambda_1}(\mathbb{Q}_p^n)}\sum_{k=-\infty}^{\gamma}p^{kn(1-r+r/s)}(\gamma - k)^r|B_k|^{(\lambda_1+1-1/s)r} \\ &= \frac{C}{|B_\gamma|_H^{1+\lambda_1 r}}\|f\|_{\dot{B}^{r,\lambda_1}(\mathbb{Q}_p^n)}\sum_{k=-\infty}^{\gamma}(\gamma - k)^r p^{knr(1/r+\lambda_1)} \\ &= \frac{C}{|B_\gamma|_H^{1+\lambda_1 r}}\|f\|_{\dot{B}^{r,\lambda_1}(\mathbb{Q}_p^n)}p^{\gamma nr(1/r+\lambda_1)} = C\|f\|_{\dot{B}^{r,\lambda_1}(\mathbb{Q}_p^n)}, \end{aligned} \tag{32}$$

where we notice that $0 < \lambda_1 + 1 - 1/s$ together with $\lambda_1 + 1/r_1 + 1/r_2 > 1/r_2 > 0 = \lambda_1 + 1/r$. From (28), (31) and (32), we get

$$\left\|H_\Omega^{p,b}f\right\|_{\dot{B}^{r,\lambda_1}(\mathbb{Q}_p^n)} \le C\|f\|_{\dot{B}^{r,\lambda_1}(\mathbb{Q}_p^n)}. \tag{33}$$

3. Conclusion

We mainly focused on the boundedness for commutators of rough p-adic Hardy operator on p-adic central Morrey spaces. Besides, we also obtained the boundedness of rough p-adic Hardy operator on p-adic Lebesgue spaces.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflict of interest.

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