Hindawi
Journal of Function Spaces
Volume 2021, Article ID 4936032, 8 pages
https://doi.org/10.1155/2021/4936032

Research Article

Lie Symmetry Analysis for the General Classes of Generalized Modified Kuramoto-Sivashinsky Equation

Rong Qi,[1] Muhammad Mobeen Munir ,[2] Nazish Younas,[3] Muhammad Idrees,[3] and Jia-Bao Liu [4]

[1]*School of Computer and Data Engineering, Bengbu College of Technology and Business, Bengbu 233000, China*
[2]*Department of Mathematics, University of the Punjab, Quaid-e-Azam Campus, Lahore, Pakistan*
[3]*Department of Mathematics, University of Education, Lahore, Pakistan*
[4]*School of Mathematics and Physics, Anhui Jianzhu University, Hefei 230601, China*

Correspondence should be addressed to Muhammad Mobeen Munir; mmunir@ue.edu.pk

Received 19 August 2021; Accepted 30 September 2021; Published 25 October 2021

Academic Editor: Fanglei Wang

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Lie symmetry analysis of differential equations proves to be a powerful tool to solve or at least reduce the order and nonlinearity of the equation. Symmetries of differential equations is the most significant concept in the study of DE's and other branches of science like physics and chemistry. In this present work, we focus on Lie symmetry analysis to find symmetries of some general classes of KS-type equation. We also compute transformed equivalent equations and some invariant solutions of this equation.

1. Introduction

Symmetry has been a source of inspiration as a powerful tool in the formulation of the laws of the universe. A great number of physical phenomena is transformed into differential equations. Lie symmetry analysis can change the given differential equation into an equivalent form which is easier to solve. In the analysis of differential equations, the symmetry group approach is quite useful. Galois's use of finite groups to solve algebraic equations of degrees two, three, and four, as well as to prove that the general polynomial equation of degrees larger than four could not be solved by radicals, served as the paradigm for this application [1–7]. The symmetry group approach is well-known for its importance in the field of differential equations analysis. Sophus Lie is credited with the invention of group categorization methods and the theoretical basis for the Lie groups.

There are many different methods for computing the symmetries of differential equations. But Lie symmetry analysis is the best because it is a systematic and algorithmic procedure that does not take into account any guesses or approximations. The principal paper on Lie symmetry is [1], in which Lie demonstrated that a linear 2D, 2nd-order PDE admits at most three boundary invariance group. He processed the maximal invariance group of the one-dimensional heat conductivity and used this analysis to compute its explicit solutions. Symmetry reduction is a leading strategy for resolving nonlinear PDEs. Ovisiannikov made a substantial contribution in persisting with these techniques. He presented the strategy of partially invariant solutions [2, 3]. In this work, he gave a methodology that is based on the idea of group called the equivalence group. Gazizov and Ibragimov [8] tracked down the total symmetry analysis of the one-dimensional Black-Scholes model. Shu-Yong and Feng-Xiang, [9] discussed about the connection between the form invariance and Lie symmetry of nonholonomic framework. Buckwar and Luchko [4] initiated the study of symmetry group of scaling transformation for PDEs of fractional order. Yan et al. [6] performed Lie symmetry analysis and fundamental similarity reductions for the coupled Kuramoto-Sivashinsky(KS) equations. Bozhkov and Dimas [10] computed the conversation laws and group classification for generalized 2D KS equation. Nadjafikhah and Ahangari [7] determined the Lie symmetries and reduction for the two-dimensional damped

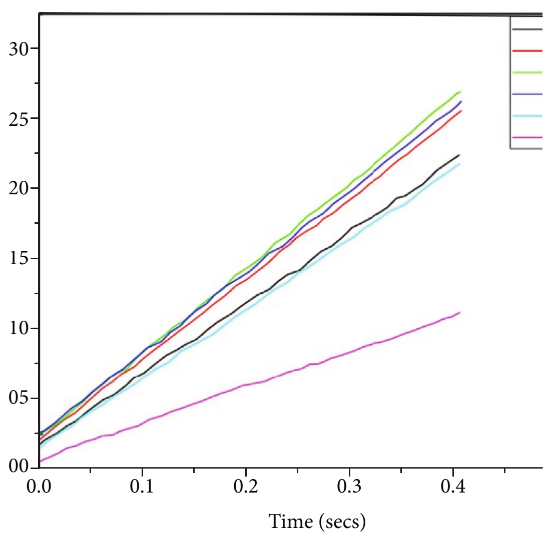

FIGURE 1: Flame front position.

Kuramoto-Sivashinsky ((2D) DKS) equation. Najafikhah and Ahangari also computed Lie symmetry of 2D generalized Kuramoto-Sivashinsky (KS) equation in [11]. The one-dimensional modified KS-type equation is

$$u_t + u_{xx} + u_{xxxx} + (\lambda - 1)u_x^2 - \sigma u_{xx}^2 = 0, \tag{1}$$

Chou [12] determined the solution of the cauchy problem for the MKS equation and also computed the solvability with the help of the blow up theorem.

In the present paper, we deal with the generalized modified one-dimensional Kuramoto-Sivashinsky (GMKS) type equation and determine the symmetry algebra by using Lie symmetry analysis. In particular, we want to find the optimal system and similarity solutions corresponding to some special cases of GMKS equation. The GMKS type equation is given as

$$f(u)u_t + u_{xx} + u_{xxxx} + (\lambda - 1)u_x^2 - \sigma u_{xx}^2 = 0. \tag{2}$$

We seek the Lie symmetry algebras for this GMKS equations for $f(u) = u^n$, $f(u) = e^{nu}$, and $f(u) = e^{u^n}$ where λ and σ are arbitrary constant and $\lambda \neq 1$. For $\lambda = 1$, equation (2) is proposed in [13]. Its second derivative satisfies an equation of Cahn-Hilliard type in [14]. This equation has various applications as physical models in biofluids, mechanics, and liquids. In equation (2), u is the velocity function, x is space parameter, and t is time variable. This equation can also be derived from a model in the continuity equation by fitting a suitable function [15]. Actually, the Kuramoto-Sivashinsky equation gives the change of the position of a flame front (Figure 1). It shows the flame front position against time for horizontally propagating methane flame, the movement of a fluid going down a vertical wall, or a spatially uniform oscillating chemical reaction in a homogeneous medium [16]. This equation is also helpful to display solitary pulses in a falling slender film [17]. Figure 2 shows the schematic representation of the flow showing a film flowing vertically down, subjected to an electric field imposed across electrodes separated by a distance d.

2. Lie Symmetry of Generalized Modified Kuramoto-Sivashinsky Equation

In this part, we compute our main results.

Consider one parameter local Lie group of transformation for the independent factors x, t and dependent factor u as follows:

$$\begin{aligned} x^* &= x + \delta\alpha(x,t,u) + O(\delta^2), \\ t^* &= t + \delta\beta(x,t,u) + O(\delta^2), \\ u^* &= u + \delta\gamma(x,t,u) + O(\delta^2), \end{aligned} \tag{3}$$

in which $\delta \in \mathbb{R}$ is the parameter.

Proposition 1. *For all $n \geq 1$, $n \in N$, the algebra of symmetries of*

$$u^n u_t + u_{xx} + u_{xxxx} + (\lambda - 1)u_x^2 - \sigma u_{xx}^2 = 0, \tag{4}$$

is 2-dimensional Abelian Lie algebra.

Proof. The general infinitesimal generator (symmetries) is

$$H = \alpha(x,t,u)\partial_x + \beta(x,t,u)\partial_t + \gamma(x,t,u)\partial_u. \tag{5}$$

The derivation of nth prolongation of H

$$pr^n H = H + \sum_{i=1}^{q}\sum_{P}\gamma_i^P(x,u^n)\left(\frac{\partial}{\partial u_P^i}\right), \tag{6}$$

interprets the relating jet space $Q^n \subset X \times U^n$, where q is a dependent variables, and $P = (P_1, P_2, \cdots P_k)$ with $1 \leq P_k \leq p$, $1 \leq k \leq n$ and

$$\gamma_i^P(x,u^n) = D_P\left(\gamma_i - \sum_{l=1}^{p}\alpha^l u^i{}_l\right) + \sum_{l=1}^{p}\alpha^l u^i{}_{P,l}, \tag{7}$$

where p is an independent variable and $u^i{}_l = \partial u^i/\partial u^l$ and $u^i{}_{P,l} = \partial u^i{}_P/\partial x^l$.

The fourth-order prolongation of H is

$$\begin{aligned} Pr^{(4)}H = H &+ \gamma^x\frac{\partial}{\partial u_x} + \gamma^t\frac{\partial}{\partial u_t} + \gamma^{xx}\frac{\partial}{\partial u_{xx}} + \gamma^{xt}\frac{\partial}{\partial u_{xt}} \\ &+ \gamma^{tt}\frac{\partial}{\partial u_{tt}} + \gamma^{xxx}\frac{\partial}{\partial u_{xxx}} + \gamma^{xxt}\frac{\partial}{\partial u_{xxt}} + \gamma^{xtt}\frac{\partial}{\partial u_{xtt}} \\ &+ \gamma^{ttt}\frac{\partial}{\partial u_{ttt}} + \gamma^{xxxx}\frac{\partial}{\partial u_{xxxx}} + \gamma^{xxxt}\frac{\partial}{\partial u_{xxxt}} \\ &+ \gamma^{xxtt}\frac{\partial}{\partial u_{xxtt}} + \gamma^{xttt}\frac{\partial}{\partial u_{xttt}} + \gamma^{tttt}\frac{\partial}{\partial u_{tttt}}, \end{aligned} \tag{8}$$

$$\begin{aligned} pr^4\left[u^n u_t + u_{xx} + u_{xxxx} + (\lambda - 1)u_x^2 - \sigma u_{xx}^2\right] &\equiv 0 \mod (2), \\ nu^{n-1}\gamma^t + \gamma^{xx} + \gamma^{xxxx} + 2(\lambda - 1)u_x\gamma^x - 2\sigma u_{xx}\gamma^{xx} &\equiv 0 \mod (2). \end{aligned} \tag{9}$$

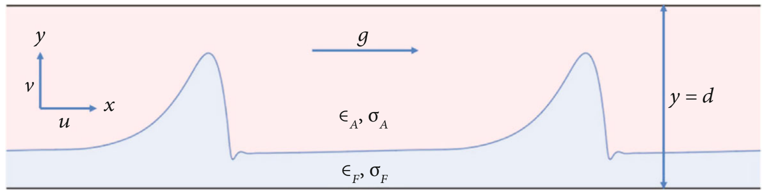

Figure 2: Schematic representation of the flow showing a film flowing vertically down.

Table 1: Commutator table.

[.., ..]	H_1	H_2
H_1	0	0
H_2	0	0

Table 2: Adjoint table.

ad	H_1	H_2
H_1	H_1	H_2
H_2	H_1	H_2

We can calculate γ^t,γ^x, γ^{xx}, and γ^{xxxx} from equation (7) such that

$$\begin{aligned}\gamma^t &= D_t(\gamma - \alpha u_x - \beta u_t) + \alpha u_{xt} + \beta u_{tt},\\ \gamma^x &= D_x(\gamma - \alpha u_x - \beta u_t) + \alpha u_{xx} + \beta u_{xt},\\ \gamma^{xx} &= D_x^2(\gamma - \alpha u_x - \beta u_t) + \alpha u_{xxx} + \beta u_{xxt},\\ \gamma^{xxxx} &= D_x^4(\gamma - \alpha u_x - \beta u_t) + \alpha u_{xxxxx} + \beta u_{xxxxt},\end{aligned} \quad (10)$$

where D_x and D_t are total derivatives.□ □

Putting all the above in equation (9) and eliminating u_t by using the relation $u_t = 1/u^n(\sigma u_{xx}^2 - (\lambda - 1)u_x^2 - u_{xx} - u_{xxxx})$, we get a polynomial equation containing the different differentials of u. Equating the coefficient of u to zero, which are some derivatives of α, β, and γ, it gives the total set of determining equations.

$$\alpha_x = \alpha_t = \alpha_u = 0, \beta_x = \beta_t = \beta_u = 0, \gamma = 0. \quad (11)$$

This gives

$$\alpha(x, t, u) = 0, \beta(x, t, u) = 0, \gamma(x, t, u) = 0. \quad (12)$$

This implies that the Lie group (algebra) of infinitesimal generators of equaution (2) is comprised of two vector fields:

$$\begin{aligned}H_1 &= \partial_x,\\ H_2 &= \partial_t.\end{aligned} \quad (13)$$

The commutator table of the Lie group for equation (2) is given as in Table 1,

The adjoint table of infinitesimal symmetries for equation (2) is given as in Table 2,

In this case, we have only two different basis for a Lie algebra of symmetries.

Hence, this shows that the group of symmetries of equation (2) is two dimensional and tables ensure that it is abelian.

Proposition 2. *For all $n > 1$ and $n \in N$, the group of symmetries of*

$$e^{u^n} u_t + u_{xx} + u_{xxxx} + (\lambda - 1)u_x^2 - \sigma u_{xx}^2 = 0, \quad (14)$$

is two-dimensional abelian.

Proof. The infinitesimal generator is

$$X = \alpha(x, t, u)\partial x + \beta(x, t, u)\partial t + \gamma(x, t, u)\partial u. \quad (15)$$

In order to find the symmetry group of equation (14), we have to apply invariance condition that is $X^{(4)}(5) \equiv 0$ mod (6) on equation .(14) where $X^{(4)}$ is the fourth-order prolongation of X given as

$$\begin{aligned}Pr^{(4)}X = X &+ \gamma^x \frac{\partial}{\partial u_x} + \gamma^t \frac{\partial}{\partial u_t} + \gamma^{xx} \frac{\partial}{\partial u_{xx}} + \gamma^{xt} \frac{\partial}{\partial u_{xt}}\\ &+ \gamma^{tt} \frac{\partial}{\partial u_{tt}} + \gamma^{xxx} \frac{\partial}{\partial u_{xxx}} + \gamma^{xxt} \frac{\partial}{\partial u_{xxt}} + \gamma^{xtt} \frac{\partial}{\partial u_{xtt}}\\ &+ \gamma^{ttt} \frac{\partial}{\partial u_{ttt}} + \gamma^{xxxx} \frac{\partial}{\partial u_{xxxx}} + \gamma^{xxxt} \frac{\partial}{\partial u_{xxxt}}\\ &+ \gamma^{xxtt} \frac{\partial}{\partial u_{xxtt}} + \gamma^{xttt} \frac{\partial}{\partial u_{xttt}} + \gamma^{tttt} \frac{\partial}{\partial u_{tttt}}.\end{aligned} \quad (16)$$

After applying an invariance condition on equation (14), we get

$$\begin{aligned}&nu^{n-1}e^{u^n}\gamma^t + \gamma^{xx} + \gamma^{xxxx} + 2(\lambda - 1)u_x\gamma^x\\ &\quad - 2\sigma u_{xx}\gamma^{xx} \equiv 0.\text{mod}\ (2).\end{aligned} \quad (17)$$

We can calculate γ^t, γ^x, γ^{xx}, and γ^{xxxx} from equation (7) such that

$$\begin{aligned}\gamma^t &= D_t(\gamma - \alpha u_x - \beta u_t) + \alpha u_{xt} + \beta u_{tt},\\ \gamma^x &= D_x(\gamma - \alpha u_x - \beta u_t) + \alpha u_{xx} + \beta u_{xt},\\ \gamma^{xx} &= D_x^2(\gamma - \alpha u_x - \beta u_t) + \alpha u_{xxx} + \beta u_{xxt},\\ \gamma^{xxxx} &= D_x^4(\gamma - \alpha u_x - \beta u_t) + \alpha u_{xxxxx} + \beta u_{xxxxt},\end{aligned} \quad (18)$$

where D_x and D_t are total derivatives.□ □

Table 3: Commutator table.

[.., ..]	X_1	X_2
X_1	0	0
X_2	0	0

Table 4: Adjoint table.

ad	X_1	X_2
X_1	X_1	X_2
X_2	X_1	X_2

After putting all the above in equation (17), and eliminating u_t by using the relation $u_t = 1/e^{u^n}(\sigma u_{xx}^2 - (\lambda - 1)u_x^2 - u_{xx} - u_{xxxx})$, we get a polynomial equation containing the different differentials of u. Equating the coefficient of u to zero, which are some derivatives of α, β, and γ, it gives the total set of determining equations, as given by

$\alpha_x = \alpha_t = \alpha_u = 0$, $\beta_x = \beta_t = \beta_u = 0$, $\gamma = 0$,

that gives

$$\alpha(x, t, u) = 0, \beta(x, t, u) = 0, \gamma(x, t, u) = 0. \tag{19}$$

This implies that the Lie group (algebra) of infinitesimal generators of equation (9) comprises two vector fields. Following, Table 3 gives the commutator table as

$$\begin{aligned} X_1 &= \partial_x, \\ X_2 &= \partial_t. \end{aligned} \tag{20}$$

The commutator table of the Lie group for equation (14) is given in Table 3,

The adjoint table of infinitesimal symmetries for equation (14) is given in Table 4,

In this case, we have only two different basis for Lie algebra.

Hence, this shows that the group of symmetries of equation (14) is two-dimensional abelian.

2.1. Symmetry Algebra for $e^{u^n} u_t + u_{xx} + u_{xxxx} + (\lambda - 1)u_x^2 - \sigma u_{xx}^2 = 0$ When $n = 1$. For $n = 1$, the equation is

$$\Delta : e^u u_t + u_{xx} + u_{xxxx} + (\lambda - 1)u_x^2 - \sigma u_{xx}^2 = 0. \tag{21}$$

The general infinitesimal generator is

$$V = \tau(x, t, u)\partial x + \mu(x, t, u)\partial t + \nu(x, t, u)\partial u. \tag{22}$$

In order to find the symmetry algebra, we have to apply invariance condition that is

$$V^{(n)}(\Delta) \equiv 0 \bmod (\Delta), \tag{23}$$

on equation (21) where $V^{(4)}$ is the fourth-order prolongation of V such that

$$\begin{aligned} Pr^{(4)}V = V &+ v^x \frac{\partial}{\partial u_x} + v^t \frac{\partial}{\partial u_t} + v^{xx} \frac{\partial}{\partial u_{xx}} + v^{xt} \frac{\partial}{\partial u_{xt}} \\ &+ v^{tt} \frac{\partial}{\partial u_{tt}} + v^{xxx} \frac{\partial}{\partial u_{xxx}} + v^{xxt} \frac{\partial}{\partial u_{xxt}} + v^{xtt} \frac{\partial}{\partial u_{xtt}} \\ &+ v^{ttt} \frac{\partial}{\partial u_{ttt}} + v^{xxxx} \frac{\partial}{\partial u_{xxxx}} + v^{xxxt} \frac{\partial}{\partial u_{xxxt}} \\ &+ v^{xxtt} \frac{\partial}{\partial u_{xxtt}} + v^{xttt} \frac{\partial}{\partial u_{xttt}} + v^{tttt} \frac{\partial}{\partial u_{tttt}}. \end{aligned} \tag{24}$$

After applying invariance condition (23) on equation (21)

$$ue^u v^t + v^{xx} + v^{xxxx} + 2(\lambda - 1)u_x v^x - 2\sigma u_{xx} v^{xx} \equiv 0 \bmod (\Delta), \tag{25}$$

where v^t,v^x, v^{xx}, and v^{xxxx} from equation (7) such that

$$\begin{aligned} v^t &= D_t(\nu - \tau u_x - \mu u_t) + \tau u_{xt} + \mu u_{tt}, \\ v^x &= D_x(\nu - \tau u_x - \mu u_t) + \tau u_{xx} + \mu u_{xt}, \\ v^{xx} &= D_x^2(\nu - \tau u_x - \mu u_t) + \tau u_{xxx} + \mu u_{xxt}, \\ v^{xxxx} &= D_x^4(\nu - \tau u_x - \mu u_t) + \tau u_{xxxxx} + \mu u_{xxxxt}, \end{aligned} \tag{26}$$

where D_x and D_t are total derivative.

Putting all the above in equation (25), we eliminateu_t by using the relation $u_t = 1/e^u(\sigma u_{xx}^2 - (\lambda - 1)u_x^2 - u_{xx} - u_{xxxx})$ and get a polynomial equation containing the different differentials of u. Equating the coefficient of u to zero, which are some derivatives of τ, μ, and ν, it gives the total set of determining equations.

$\tau_x = \tau_t = \tau_u = 0$, $\mu_x = \mu_t = \mu_u = 0$, $\nu = 0$,

that gives

$$\tau(x, t, u) = c_3, \mu(x, t, u) = c_1 t + c_2, \nu(x, t, u) = c_1. \tag{27}$$

This implies that the Lie group (algebra) of infinitesimal generators of equation (21) is comprised of three vector fields:

$$V_1 = \partial_u + t\partial_t, \tag{28}$$

$$V_2 = \partial_t, \tag{29}$$

$$V_3 = \partial_x. \tag{30}$$

The commutator table of the Lie group for equation (21) is given in Table 5,

The adjoint table of infinitesimal symmetries for equation (21) is given in Table 6,

In this case, we have three different Lie algebras.

Theorem 3. *The algebra of symmetries of Kuramoto-Sivashinsky type equation is*

$$e^{u^n} u_t + u_{xx} + u_{xxxx} + (\lambda - 1)u_x^2 - \sigma u_{xx}^2 = 0, \tag{31}$$

Table 5: Commutator table.

$[..,..]$	V_1	V_2	V_3
V_1	0	$-V_2$	0
V_2	V_2	0	0
V_3	0	0	0

Table 6: Adjoint table.

$[ad]$	V_1	V_2	V_3
V_1	V_1	$V_2 e^{\varepsilon}$	V_3
V_2	$V_1 - \varepsilon V_2$	V_2	V_3
V_3	V_1	V_2	V_3

where it is two-dimensional abelian for all $n > 1$, $n \varepsilon N$ and three-dimensional nonabelian for $n = 1$.

Proof. The proof follows easily using Propositions 1 and 2.□

□

Theorem 4. *If $G_s^i(x,t,u)$ be the one parameter group generated by equation (28) then*

$$\begin{aligned} G_s^1(x,t,u) &= (x, e^s t, u), \\ G_s^2(x,t,u) &= (x, t+s, u), \\ G_s^3(x,t,u) &= (x+s, t, u). \end{aligned} \tag{32}$$

There will be a family of solutions to each one parameter subgroups of the full symmetry group of a system called group invariant solutions.

Theorem 5. *If $u = f(x,t)$ is a solution of equation (21), so are the functions*

$$\begin{aligned} \tilde{u}^1 &= f(x, e^{-s} t), \\ \tilde{u}^2 &= f(x, t-s), \\ \tilde{u}^3 &= f(x-s, t), \end{aligned} \tag{33}$$

where $\tilde{u}^i = G_s^i * f(x,t)$, $i = 1,2,3$ and $s << 1$ is any positive number.

Proof. The one parameter Lie group of equation (21) is

$$G_s^1 : (x,t,u) \longrightarrow (x, e^s t, u), \tag{34}$$

with the infinitesimal generator

$$V_1 = t\partial t + \partial u, \tag{35}$$

if $\tilde{u}^1(x,t)$ is any function then it transformed by G_s^1 as

$$\begin{aligned} \tilde{u}^1 &= u, \\ \tilde{u}^1 &= f(x,t), \end{aligned} \tag{36}$$

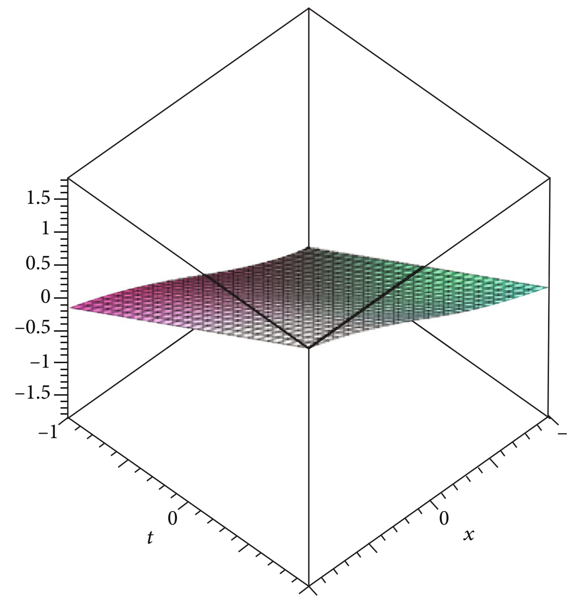

Figure 3: For $u^1(x,t) = \sin(x) + e^{-s}(t)$, $s = 0.00001$.

now

$$(\tilde{x}, \tilde{t}) = (x, e^{-s} t), \tag{37}$$

therefore,

$$\tilde{u}^1 = f(x, e^{-s} t). \tag{38}$$

The graph for $\tilde{u}^1 = f(x, e^{-s} t)$ is given in Figure 3. The one parameter Lie group of equation (21) is

$$G_s^2 : (x,t,u) \longrightarrow (x, t-s, u), \tag{39}$$

with the infinitesimal generator

$$V_2 = t\partial t, \tag{40}$$

if $\tilde{u}^2(x,t)$ is any function then it transformed by G_s^2 as

$$\begin{aligned} \tilde{u}^2 &= u, \\ \tilde{u}^2 &= f(x,t), \end{aligned} \tag{41}$$

now

$$(\tilde{x}, \tilde{t}) = (x, t-s), \tag{42}$$

therefore

$$\tilde{u}^2 = f(x, t-s). \tag{43}$$

The graph for $\tilde{u}^2 = f(x, t-s)$ is given in Figure 4. The one parameter Lie group of equation (21) is

$$G_s^3 : (x,t,u) \longrightarrow (x-s, t, u), \tag{44}$$

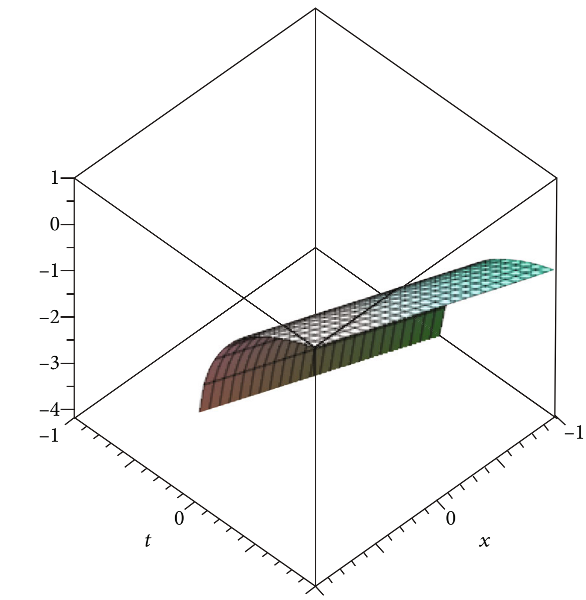

Figure 4: For $u^1(x,t)=\log(x+t)-s, s=0.00001$.

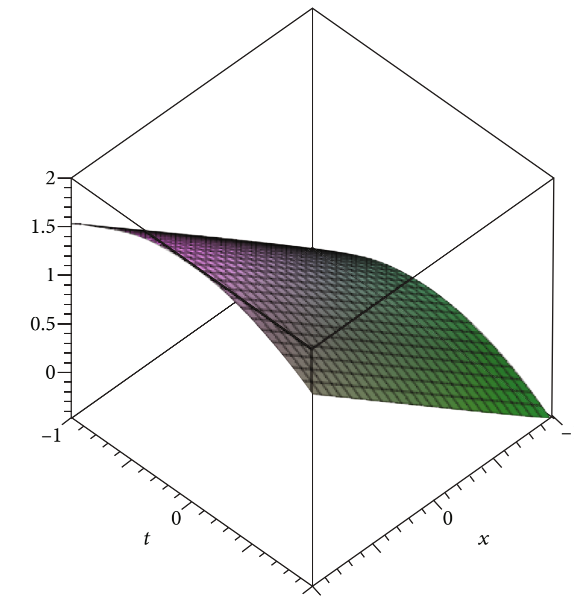

Figure 5: For $u^3(x,t)=(x-s)+\cos(t), s=0.00001$.

with the infinitesimal generator

$$V_3 = t\partial x, \tag{45}$$

if $\tilde{u}^3(x,t)$ is any function then it transformed by G_s^3 as

$$\begin{aligned}\tilde{u}^3 &= u,\\ \tilde{u}^3 &= f(x,t),\end{aligned} \tag{46}$$

now

$$(\tilde{x},\tilde{t}) = (x-s,t), \tag{47}$$

therefore

$$\tilde{u}^3 = f(x-s,t). \tag{48}$$

The graph for $\tilde{u}^3 = f(x-s,t)$ as a solution is given in Figure 5.☐ ☐

3. Optimal System of Subalgebras

This is remarkable that the Lie symmetry technique assumes a significant part to determine the solutions of PDEs as well as performing the symmetric reductions. Every combination (should be linear) of infinitesimal symmetries(generators) is a result of another infinitesimal symmetry(generator). As any transformation in the full symmetry groups plot a solution to another, it is sufficient to determine the invariant solution which are not related by transformations in the full symmetry group; this prompted the Optimal system [18, 19].

Theorem 6. *A 1D optimal system of equation (21) is given by those generated by*

$$\begin{aligned}Y_1 &= V_1\\ Y_2 &= V_2,\\ Y_3 &= V_3,\\ Y_4 &= V_1 + V_3,\\ Y_5 &= V_3 - V_1.\end{aligned} \tag{49}$$

Proof. Since the combination of vector field (infinitesimal generator) is also a vector field. Consider a linear combination V of V_1, V_2, and V_3,

$$V = \sum_{i=1}^{3} = b_i V_i, \tag{50}$$

a nonzero vector field. Here, for proof, we will improve as many of the coefficient $b_i{}'s$ as possible by using adjoint application on V.☐ ☐

Case 1. Firstly assume that $b_3 \neq 0$ then

$$V = b_1 V_1 + b_2 V_2 + V_3, \tag{51}$$

acting on V with $Adj(\exp(b_2/b_1)V_2)$ by using the adjoint table (adjoint Table 3)

$$V' = Adj\left(\exp\frac{b_2}{b_1}V_2\right)V = b_1 V_1 + V_3. \tag{52}$$

When $b_1 > 0$, then we get Y_4.
When $b_1 < 0$, then we get Y_5.
When $b_1 = 0$, then we get Y_3.

Case 2. Let $b_3 = 0$ and $b_1 = 0$,

$$V = b_2 V_2, \tag{53}$$

when $b_1 = 1$ then we get Y_2, Let $b_3 = 0$, $b_2 = 0$ and b_1, then we get Y_1 There is no any more cases for consultation and the proof is complete.

4. Lie Invariants and Similarity Solutions

We can discover that the invariants correlate with the infinitesimal symmetries (28); they can be determined by solving the equations (by using characteristic method). For $V_2 = \partial t$, the characteristic equation is $dx/0 = dt/1 = du/0$ and the corresponding invariants of this system $x = r$ and $u = w$.

We obtain a similar solution of the form $w = w(r)$, and we put it into equation (21) to obtain the form of the function w, and then, we conclude that $w = w(r) = w(x)$ solution of the following differential equation as similarity reduce equation:

$$w_{rr} + w_{rrrr} + (\lambda - 1)w_r^2 - \sigma w_{rr}^2 = 0. \tag{54}$$

For other example, take $V_3 = \partial x$; the characteristic equation for this has the form $dx/1 = dt/0 = du/0$ so the corresponding invariants are $t = r$ and $u = w$.

Taking into account the last invariants, the following similarity solution is obtained $w = w(r) = w(t)$ where the solution satisfied the similarity reduce equation:

$$e^w w_r = 0. \tag{55}$$

5. Conclusions

The present paper addresses Lie symmetries for some general cases of modified one-dimensional Kuramoto-Sivashinky equation (MKS) as well as its similarity solutions using a symmetry operator. In Section 2, we discussed general results for Lie algebras for some general cases of MKS and provide a comparison between them and obtained some general results. In Section 3, we find the optimal system for (MKS). In the last section, we obtained similarity solutions and Lie invariants.

Remarks. It is worthmentioning that $f(u)$ can be any arbitrary function. For other similar functions chosen as $f(u)$, the procedure for symmetry analysis can be very tedious and symmetry algebra can be different.

Data Availability

Data sharing is not applicable to this article as no dataset was generated or analyzed during the current study

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

This research is partially supported by the Higher Education Commission, Pakistan.

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