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*Research Article*

# Common Fixed Point Results for Generalized $(g-\alpha_{sp},\psi,\varphi)$ Contractive Mappings with Applications

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In this paper, we introduce a new class of $g-\alpha_{sp}$ – admissible mappings and prove some common fixed point theorems involving this new class of mappings which satisfy generalized contractive conditions in the framework of $b$ – metric spaces. We also provide two examples to show the applicability and validity of our results. Meanwhile, we present an application to the existence of solutions to an integral equation by means of one of our results.

## 1. Introduction

The Banach contraction principle [1] is one of the essential pillars of the theory of metric fixed points. Many authors have obtained generalizations, extensions, and applications of their findings by investigating the Banach contraction principle in many directions. One of the most popular and interesting topics among them is the study of new classes of spaces and their fundamental properties.

Czerwik [2] introduced the concept of $b$ – metric space and proved some fixed point theorems of contractive mappings in $b$ – metric space. Subsequently, some authors have studied on the fixed point theorems of a various new type of contractive conditions in $b$ – metric space. Aydi et al. in [3] proved common fixed point results for mappings satisfying a weak $\phi$ – contraction in $b$ – metric spaces. Following the results of Berinde [4], Pacurar [5] obtained the existence and uniqueness of fixed point of $\phi$ – contractions, and Zada et al. [6] got fixed point results satisfying contractive conditions of rational type. In 2019, Hussain et al. studied the existence and uniqueness of a periodic common fixed point for pairs of mappings via rational type contraction in [7]. After that, authors obtained fixed point theorems for $L$ – cyclic $(\alpha,\beta)_s$ – contractions and cyclic $(\alpha,\beta)-(\psi,\phi)_s$ – rational type contractions and discussed the existence of a unique solution to nonlinear fractional differential equations in [8, 9], respectively. Also using rational type contractive conditions, Hussain et al. [10] got the existence and uniqueness of common $n$ – tupled fixed point for a pair of mappings. Using a contractive condition defined by means of a comparison function, [11] established results regarding the common fixed points of two mappings. In 2014, Abbas et al. obtained the results on common fixed points of four mappings in $b$ – metric space in [12].

To generalize the concept of $b$ – metric spaces, Hussain and Shah in [13] introduced the notion of a cone $b$ – metric space, which means that it is a generalization of $b$ – metric spaces and cone metric spaces; they considered topological properties of cone $b$ – metric spaces and obtained some results on KKM mappings in the setting of cone $b$ – metric spaces. In [14], some fixed point results for weakly contractive mappings in ordered partial metric space were obtained. Recently, Samet et al. [15] introduced the concept of $\alpha$ – admissible and $\alpha-\psi$ – contractive mappings and presented fixed point theorems for them. In [16], Jamal et al. used $(\psi,\phi)$ – weak contraction to generalize coincidence point results which are established in the context of partially ordered $b$ – metric spaces. In [17, 18], Zoto et al. studied generalized $\alpha_{sp}$ contractive mappings and $(\alpha-\psi,\phi)$ – contractions in $b$ – metric–like space. Recently, in [16], Jamal et al. used $(\psi, \phi)$ – weak contraction to generalize coincidence point results which are established in the context of partially ordered $b$ – metric spaces. Abu-Donia et al. [19] proved the uniqueness and existence of the fixed points for five mappings from a complete intuitionistic fuzzy 3 – metric space into itself under weak compatible of type $(\alpha)$ and asymptotically regular. For recent development on fixed point theory, we refer to [20–26].

Motivated and inspired by Theorems 27 and 29 in [17], Theorem 3.13 in [18], and Theorem 2.1 in [20], in this paper, our purpose is to introduce the concept of $g - \alpha_{s^p}$ – admissible mappings and obtain a few common fixed point results involving generalized contractive conditions in the framework of $b$ – metric space. Furthermore, we provide examples that elaborated the useability of our results. Meanwhile, we present an application to the existence of solutions to an integral equation by means of one of our results.

## 2. Preliminaries

First of all, we introduce some definitions as follows:

*Definition 1* (see [2]). Let $X$ be a nonempty set and $s \geq 1$ be a given real number. A mapping $d : X \times X \longrightarrow [0,+\infty)$ is said to be a b – metric if and only if, for all x, y, z ∈ X, the following conditions are satisfied:

(i) $d(x, y) = 0$ if and only if $x = y$

(ii) $d(x, y) = d(y, x)$

(iii) $d(x, y) \leq s(d(x, z) + d(y, z))$

Generally, we call $(X, d)$ a $b$ – metric space with parameter $s \geq 1$.

*Remark 2.* We should note that a $b$ – metric space with $s = 1$ is a metric space. We can find several examples of $b$ – metric spaces which are not metric spaces. (see [24]).

*Example 3* (see [20]). Let $(X, \rho)$ be a metric space, and $d(x, y) = (\rho(x, y))^p$, where $p > 1$ is a real number

Then, $d(x, y)$ is a $b$ – metric space with $s = 2^{p-1}$.

*Definition 4* (see [12]). Let $(X, d)$ be a $b$ – metric space with parameter $s \geq 1$. Then, a sequence $\{x_n\}$ in $X$ is said to be:

(i) $b$ – convergent if and only if there exists $x \in X$ such that $d(x_n, x) \longrightarrow 0$ as $n \longrightarrow +\infty$

(ii) a Cauchy sequence if and only if $d(x_n, x_m) \longrightarrow 0$ when $n, m \longrightarrow +\infty$

In general, a $b$ – metric space is called complete if and only if each Cauchy sequence in this space is $b$ – convergent.

*Definition 5* (see [21]). Let $f$ and $g$ be two self-mappings on a nonempty set $X$. If w = fx = gx, for some x ∈ X, then $x$ is said to be the coincidence point of $f$ and $g$, where $w$ is called the point of coincidence of $f$ and $g$. Let $C(f, g)$ denote the set of all coincidence points of $f$ and $g$.

*Definition 6* (see [21]). Let $f$ and $g$ be two self-mappings defined on a nonempty set $X$. Then, $f$ and $g$ are said to be weakly compatible if they commute at every coincidence point, that is, $fx = gx \Rightarrow fgx = gfx$ for every $x \in C(f, g)$.

We need the following lemma to obtain our main results:

**Lemma 7** (see [20]). *Let $(X, d)$ be a $b$ – metric space with parameter $s \geq 1$. Assume that $\{x_n\}$ and $\{y_n\}$ are $b$ – convergent to $x$ and $y$, respectively. Then, we have*

$$\frac{1}{s^2} d(x, y) \leq \liminf_{n \longrightarrow +\infty} d(x_n, y_n) \leq \limsup_{n \longrightarrow +\infty} d(x_n, y_n) \leq s^2 d(x, y). \tag{1}$$

*In particular, if $x = y$, then we have $\lim_{n \longrightarrow +\infty} d(x_n, y_n) = 0$. Moreover, for each $z \in X$, we have*

$$\frac{1}{s} d(x, z) \leq \liminf_{n \longrightarrow +\infty} d(x_n, z) \leq \limsup_{n \longrightarrow +\infty} d(x_n, z) \leq s d(x, z). \tag{2}$$

## 3. Main Results

In this section, we will show the existence and uniqueness of common fixed point for generalized $(g - \alpha_{s^p}, \psi, \varphi)$ contractive mappings in complete $b$ – metric space. Meanwhile, we give two examples to support our results.

*Definition 8.* Let $(X, d)$ be a $b$ – metric space with parameter $s \geq 1$, and let $f, g : X \longrightarrow X$ and $\alpha : X \times X \longrightarrow [0,+\infty)$ be given mappings and $p \geq 1$ be an arbitrary constant. The mapping $f : X \longrightarrow X$ is said to be $g - \alpha_{s^p}$ – admissible if, for all $x, y \in X, \alpha(gx, gy) \geq s^p$ implies $\alpha(fx, fy) \geq s^p$.

*Remark 9.*

(i) Note that, for $g = I$, the definition reduces to an $\alpha_{s^p}$ -admissible mapping in a $b$ – metric space

(ii) For $s = 1$, the definition reduces to the definition of an $\alpha$ – admissible mapping in a metric space

Let $(X, d)$ be a complete $b$ – metric space with parameter $s \geq 1$ and $\alpha : X \times X \longrightarrow [0,+\infty)$ be a function. Then,

$(H_{s^p})$ If $\{x_n\}$ is a sequence in $X$ such that $gx_n \longrightarrow gx$ as $n \longrightarrow +\infty$, then there exists a subsequence $\{gx_{n_k}\}$ of $\{gx_n\}$ with $\alpha(gx_{n_k}, gx) \geq s^p$ for all $k \in N$

$(U_{s^p})$ For all $u, v \in C(f, g)$, we have the condition of $\alpha(gu, gv) \geq s^p$ or $\alpha(gv, gu) \geq s^p$.

We know that contraction-type mappings are extended in several directions. Since Samet introduced the concept of $\alpha$ – admissible mappings and $\alpha - \psi$ – contractive mapping, some papers have been published to study a series of generalizations. Afterwards, these classes of mappings are used under generalized weakly contractive conditions.

We shall consider the contractive conditions in this section are constructed via auxiliary functions defined with the families $\Psi, \Phi$, respectively:

$$\begin{aligned} &\Psi = \{\psi : [0,+\infty) \longrightarrow [0,+\infty) \text{is an increasing and continuous function}\},\\ &\Phi = \{\varphi : [0,+\infty) \longrightarrow [0,+\infty) \text{is an increasing and continuous function and } \varphi(\mathrm{t}) = 0 \text{ iff } \mathrm{t} = 0\}. \end{aligned} \quad (3)$$

Now, we introduce the notion of rational $(g - \alpha_{s^p}, \psi, \varphi)$ contraction in the setting of $b-$ metric spaces.

*Definition 10.* Let $(X, d)$ be a $b-$ metric space with parameter $s \geq 1$, and let $f, g : X \longrightarrow X$ be two self-mappings. Assume that $\alpha : X \times X \longrightarrow [0,+\infty)$ and $p \geq 1$ is a constant. A mapping $f$ is called a generalized $(g - \alpha_{s^p}, \psi, \varphi)$ contractive mapping, if there exist $\psi \in \Psi, \varphi \in \Phi$ such that

$$\psi(\alpha(gx, gy)d(fx, fy)) \leq \psi(N(x, y)) - \varphi(M(x, y)), \quad (4)$$

for all $x, y \in X$ with $\alpha(x, y) \geq s^p$, where

$$\begin{aligned} N(x, y) = \max \Big\{ & d(gx, gy), d(fx, gx), d(fy, gy),\\ & \cdot \frac{d(gx, fy) + d(fx, gy)}{4s}, \frac{d(fx, gx)d(fy, gy)}{1 + d(fx, fy)},\\ & \cdot \frac{d(fy, gy)[1 + d(fx, gx)]}{1 + d(gx, gy)} \Big\},\\ M(x, y) = \max \Big\{ & d(fx, gy), d(gx, gy), d(fx, gx), d(fy, gy),\\ & \cdot \frac{d(fx, gx)[1 + d(gx, gy)]}{1 + d(fx, gy)}, \frac{d(fx, gx)[1 + d(fx, gx)]}{1 + d(fx, gy)},\\ & \cdot \frac{d(fx, gx)[1 + d(fy, gy)]}{1 + d(fx, gy)} \Big\}. \end{aligned} \quad (5)$$

*Example 11.* Let $X = [0,+\infty)$ and $d(x, y) = (x - y)^2$. Define mappings $f, g : X \longrightarrow X$ by

$$fx = \begin{cases} \dfrac{(x + x^2)}{8}, & x \in [0, 1] \\ 2x, & x > 1 \end{cases}, \text{and } gx = \begin{cases} \dfrac{7(x + x^2)}{8}, & x \in [0, 1] \\ \dfrac{7x}{4}, & x > 1 \end{cases}. \quad (6)$$

Define mappings $\alpha : g(X) \times g(X) \longrightarrow [0,+\infty)$ by

$$\alpha(x, y) = \begin{cases} s^2, & x, y \in \left[0, \dfrac{7}{4}\right] \\ 0, & \text{otherwise} \end{cases}, \quad (7)$$

and $\psi, \varphi : [0,+\infty) \longrightarrow [0,+\infty)$ with $\psi(t) = t/2, \varphi(t) = 64t/585$.

It is clear that $f(X) \subset g(X)$. For $x, y \in X$ such that $\alpha(gx, gy) \geq s^2$, we can know that $gx, gy \in [0, 7/4]$ and this implies that $x, y \in [0, 1]$. By definitions, we obtain $fx, fy \in [0, 7/4]$ and $\alpha(fx, fy) \geq s^2$. That is, $f$ is a $g - \alpha_{s^2} -$ admissible mapping. For all $x, y \in [0, 1]$, we have

$$\begin{aligned} &\psi(\alpha(gx, gy)d(fx, fy))\\ &= \frac{1}{2} \cdot 4 \cdot d(fx, fy)\\ &= 2 \cdot \frac{1}{64} \left((x + x^2) - (y + y^2)\right)^2\\ &\leq \frac{1}{32} \max\left\{ (x + x^2)^2, (y + y^2)^2 \right\},\\ &\psi(N(x, y))\\ &\geq \psi(\max\{d(fx, gx), d(fy, gy)\})\\ &= \frac{9}{32} \max\left\{ (x + x^2)^2 (y + y^2)^2 \right\},\\ &\varphi(M(x, y))\\ &= \varphi\Bigg( \max\Bigg\{ \left(\frac{(x + x^2)}{8} - \frac{7(y + y^2)}{8}\right)^2, \left(\frac{7(x + x^2)}{8} - \frac{7(y + y^2)}{8}\right)^2,\\ &\left(\frac{(x + x^2)}{8} - \frac{7(x + x^2)}{8}\right)^2, \left(\frac{(y + y^2)}{8} - \frac{7(y + y^2)}{8}\right)^2,\\ &\frac{((x + x^2)/8 - 7(x + x^2)/8)^2 \left[1 + (7(x + x^2)/8 - 7(y + y^2)/8)^2\right]}{1 + ((x + x^2)/8 - 7(y + y^2)/8)^2},\\ &\frac{((x + x^2)/8 - 7(x + x^2)/8)^2 \left[1 + ((x + x^2)/8 - 7(x + x^2)/8)^2\right]}{1 + ((x + x^2)/8 - 7(y + y^2)/8)^2},\\ &\frac{((x + x^2)/8 - 7(x + x^2)/8)^2 \left[1 + ((y + y^2)/8 - 7(y + y^2)/8)^2\right]}{1 + ((x + x^2)/8 - 7(y + y^2)/8)^2} \Bigg\} \Bigg)\\ &\leq \varphi\Bigg( \max\Bigg\{ \frac{49}{64} \max\left\{ (x + x^2)^2, (y + y^2)^2 \right\},\\ &\frac{49}{64} \max\left\{ (x + x^2)^2, (y + y^2)^2 \right\}, \frac{9}{16} (x + x^2)^2, \frac{9}{16} (y + y^2)^2,\\ &\frac{9}{16} (x + x^2)^2 \left[1 + \frac{49}{64} \max\left\{ (x + x^2)^2, (y + y^2)^2 \right\}\right],\\ &\frac{9}{16} (x + x^2)^2 \left[1 + \frac{9}{16} (x + x^2)^2\right], \frac{9}{16} (x + x^2)^2 \left[1 + \frac{9}{16} (y + y^2)^2\right] \Bigg\} \Bigg) \end{aligned}$$

$$\leq \frac{64}{585} \cdot \frac{9}{16}$$
$$\cdot \frac{65}{16} \max\left\{ \left(x + x^2\right)^2, \left(y + y^2\right)^2 \right\}$$
$$= \frac{1}{4} \max\left\{ \left(x + x^2\right)^2, \left(y + y^2\right)^2 \right\}. \tag{8}$$

According to the above inequalities, we get that

$$\begin{aligned}\psi(\alpha(gx, gy)d(fx, fy)) &\leq \frac{1}{32} \max\left\{ \left(x + x^2\right)^2, \left(y + y^2\right)^2 \right\} \\ &= \frac{9}{32} \max\left\{ \left(x + x^2\right)^2, \left(y + y^2\right)^2 \right\} \\ &\quad - \frac{1}{4} \max\left\{ \left(x + x^2\right)^2, \left(y + y^2\right)^2 \right\} \\ &\leq \psi(N(x, y)) - \varphi(M(x, y)).\end{aligned} \tag{9}$$

It follows that $f$ is a generalized $(g - \alpha_{s^p}, \psi, \varphi)$ contractive mapping.

**Theorem 12.** *Let $(X, d)$ be a complete $b -$ metric space with parameter $s \geq 1$ and let $f, g : X \longrightarrow X$ be given self-mappings on $X$ such that $f(X) \subset g(X)$. Also, $g(X)$ is a closed subset of $X$, and $\alpha : X \times X \longrightarrow [0,+\infty)$ is a given mapping. If the following conditions are satisfied:*

*(i) $f$ is a $g - \alpha_{s^p} -$ admissible mapping*

*(ii) $f$ is a generalized $(g - \alpha_{s^p}, \psi, \varphi)$ contractive mapping*

*(iii) there is $x_0 \in X$ with $\alpha(gx_0, fx_0) \geq s^p$*

*(iv) properties $(H_{s^p})$ and $(U_{s^p})$ are satisfied*

*(v) $\alpha$ has a transitive property type $s^p$, that is, for $x, y, z \in X$*

$$\alpha(x, y) \geq s^p \text{ and } \alpha(y, z) \geq s^p \Rightarrow \alpha(x, z) \geq s^p. \tag{10}$$

*Then, $f$ and $g$ have a unique point of coincidence in $X$. Moreover, $f$ and $g$ have a unique common fixed point provided that $f$ and $g$ are weakly compatible.*

*Proof.* According to condition (3), there exists an $x_0 \in X$ such that $\alpha(gx_0, fx_0) \geq s^p$. Define the sequences $\{x_n\}$ and $\{y_n\}$ in $X$ by $y_n = fx_n = gx_{n+1}$ for all $n \in N$. If $y_n = y_{n+1}$ for some $n \in N$, then we have $y_n = y_{n+1} = fx_{n+1} = gx_{n+1}$ and it is easy to see that $f$ and $g$ have a point of coincidence. Without loss of generality, assume that $y_n \neq y_{n+1}$ for all $n \in N$. By the condition (1), we get

$$\begin{aligned}\alpha(gx_0, gx_1) &= \alpha(gx_0, fx_0) \geq s^p, \\ \alpha(gx_1, gx_2) &= \alpha(fx_0, fx_1) \geq s^p, \\ \alpha(gx_2, gx_3) &= \alpha(fx_1, fx_2) \geq s^p.\end{aligned} \tag{11}$$

Therefore, by induction, we obtain $\alpha(gx_n, gx_{n+1}) = \alpha(y_{n-1}, y_n) \geq s^p$ for all $n \in N$. It follows from (4) that

$$\begin{aligned}\psi(d(y_n, y_{n+1})) &\leq \psi(s^p d(y_n, y_{n+1})) \\ &\leq \psi(\alpha(gx_n, gx_{n+1})d(fx_n, fx_{n+1})) \\ &\leq \psi(N(x_n, x_{n+1})) - \varphi(M(x_n, x_{n+1})),\end{aligned} \tag{12}$$

where

$$\begin{aligned}N(x_n, x_{n+1}) &= \max\left\{ d(y_{n-1}, y_n), d(y_n, y_{n-1}), d(y_{n+1}, y_n), \right. \\ &\quad \cdot \frac{d(y_{n-1}, y_{n+1}) + d(y_n, y_n)}{4s}, \frac{d(y_{n-1}, y_n)d(y_{n+1}, y_n)}{1 + d(y_n, y_{n+1})}, \\ &\quad \left. \cdot \frac{d(y_{n+1}, y_n)[1 + d(y_n, y_{n-1})]}{1 + d(y_{n-1}, y_n)} \right\} \\ &\leq \max\left\{ d(y_{n-1}, y_n), d(y_n, y_{n+1}), \right. \\ &\quad \left. \cdot \frac{s[d(y_{n-1}, y_n) + d(y_n, y_{n+1})]}{4s} \right\} \\ &= \max\left\{ d(y_{n-1}, y_n), d(y_n, y_{n+1}) \right\},\end{aligned} \tag{13}$$

$$\begin{aligned}M(x_n, x_{n+1}) &= \max\left\{ d(y_n, y_n), d(y_{n-1}, y_n), d(y_n, y_{n-1}), d(y_{n+1}, y_n), \right. \\ &\quad \cdot \frac{d(y_n, y_{n-1})[1 + d(y_{n-1}, y_n)]}{1 + d(y_n, y_n)}, \frac{d(y_n, y_{n-1})[1 + d(y_n, y_{n-1})]}{1 + d(y_n, y_n)}, \\ &\quad \left. \cdot \frac{d(y_n, y_{n-1})[1 + d(y_{n+1}, y_n)]}{1 + d(y_n, y_n)} \right\}.\end{aligned} \tag{14}$$

If we assume that, for some $n \in N$,

$$d(y_n, y_{n+1}) \geq d(y_n, y_{n-1}) > 0, \tag{15}$$

then from inequalities (13) and (14), we have

$$N(x_n, x_{n+1}) \leq d(y_n, y_{n+1}), \tag{16}$$

$$M(x_n, x_{n+1}) \geq \max\{d(y_n, y_{n+1}), d(y_n, y_{n-1})\} = d(y_n, y_{n+1}). \tag{17}$$

Using (12), (16), and (17), one can obtain that

$$\begin{aligned}\psi(d(y_n, y_{n+1})) &\leq \psi(N(x_n, x_{n+1})) - \varphi(M(x_n, x_{n+1})) \\ &\leq \psi(d(y_n, y_{n+1})) - \varphi(d(y_n, y_{n+1})),\end{aligned} \tag{18}$$

which gives $\varphi(d(y_n, y_{n+1})) \leq 0$ and then $y_n = y_{n+1}$, a contradiction. It follows that $d(y_n, y_{n+1}) < d(y_n, y_{n-1})$, that is, $\{d(y_n, y_{n+1})\}$ is a nonincreasing sequence and so there exists $r \geq 0$ such that

$$\lim_{n \longrightarrow +\infty} d(y_n, y_{n+1}) = r. \tag{19}$$

By virtue of (13) and (14) again, we have

$$N(x_n, x_{n+1}) \leq d(y_n, y_{n-1}),$$
$$M(x_n, x_{n+1}) \geq d(y_n, y_{n-1}). \quad (20)$$

It follows that

$$\begin{aligned}\psi(d(y_n, y_{n+1})) &\leq \psi(N(x_n, x_{n+1})) - \varphi(M(x_n, x_{n+1})) \\ &\leq \psi(d(y_n, y_{n-1})) - \varphi(d(y_n, y_{n-1})).\end{aligned} \quad (21)$$

Now suppose that $r > 0$, then taking the limit as $n \longrightarrow +\infty$ in above inequality, we have $\psi(r) \leq \psi(s^p r) \leq \psi(r) - \varphi(r)$, which gives a contradiction. Hence,

$$\lim_{n\longrightarrow+\infty} d(y_n, y_{n+1}) = 0. \quad (22)$$

Next, we aim to prove that $\{y_n\}$ is a Cauchy sequence. Suppose on the contrary that, $\lim_{n,m\longrightarrow+\infty} d(y_n, y_m) \neq 0$, then there exists $\varepsilon > 0$ for which one can find sequences $\{y_{m_k}\}$ and $\{y_{n_k}\}$ of $\{y_n\}$ satisfying $n_k$ is the smallest index for which $n_k > m_k > k$,

$$\varepsilon \leq d\left(y_{m_k}, y_{n_k}\right), \quad (23)$$

$$d\left(y_{m_k}, y_{n_k-1}\right) < \varepsilon. \quad (24)$$

In view of the triangle inequality, we have

$$\begin{aligned}\varepsilon \leq d\left(y_{m_k}, y_{n_k}\right) &\leq sd\left(y_{m_k}, y_{n_k-1}\right) + sd\left(y_{n_k-1}, y_{n_k}\right) \\ &< s\varepsilon + sd\left(y_{n_k}, y_{n_k-1}\right).\end{aligned} \quad (25)$$

Taking the upper limit as $k \longrightarrow +\infty$ in the above inequality and using (22), we have

$$\varepsilon \leq \limsup_{k\longrightarrow+\infty} d\left(y_{m_k}, y_{n_k}\right) \leq s\varepsilon. \quad (26)$$

Also,

$$d\left(y_{m_k}, y_{n_k}\right) \leq sd\left(y_{m_k}, y_{n_k-1}\right) + sd\left(y_{n_k-1}, y_{n_k}\right), \quad (27)$$

$$d\left(y_{m_k}, y_{n_k}\right) \leq sd\left(y_{m_k}, y_{m_k-1}\right) + sd\left(y_{m_k-1}, y_{n_k}\right), \quad (28)$$

$$d\left(y_{m_k-1}, y_{n_k}\right) \leq sd\left(y_{m_k-1}, y_{m_k}\right) + sd\left(y_{m_k}, y_{n_k}\right). \quad (29)$$

From (24) and (27), we obtain

$$\frac{\varepsilon}{s} \leq \limsup_{k\longrightarrow+\infty} d\left(y_{m_k}, y_{n_k-1}\right) \leq \varepsilon. \quad (30)$$

Using (28) and (29), we have

$$\frac{\varepsilon}{s} \leq \limsup_{k\longrightarrow+\infty} d\left(y_{m_k-1}, y_{n_k}\right) \leq s^2\varepsilon. \quad (31)$$

Similarly,

$$\begin{aligned}&d\left(y_{m_k-1}, y_{n_k-1}\right) \leq sd\left(y_{m_k-1}, y_{m_k}\right) + sd\left(y_{m_k}, y_{n_k-1}\right), \\ &d\left(y_{m_k}, y_{n_k}\right) \leq sd\left(y_{m_k}, y_{m_k-1}\right) + s^2 d\left(y_{m_k-1}, y_{n_k-1}\right) + s^2 d\left(y_{n_k-1}, y_{n_k}\right),\end{aligned} \quad (32)$$

so there is

$$\frac{\varepsilon}{s^2} \leq \limsup_{k\longrightarrow+\infty} d\left(y_{m_k-1}, y_{n_k-1}\right) \leq s\varepsilon. \quad (33)$$

In view of the definition of $N(x, y)$, one can deduce that

$$\begin{aligned}N\left(x_{m_k}, x_{n_k}\right) = \max \Bigg\{ &d\left(y_{m_k-1}, y_{n_k-1}\right), d\left(y_{m_k}, y_{m_k-1}\right), d\left(y_{n_k}, y_{n_k-1}\right), \\ &\cdot \frac{d\left(y_{m_k-1}, y_{n_k}\right) + d\left(y_{m_k}, y_{n_k-1}\right)}{4s}, \\ &\cdot \frac{d\left(y_{m_k-1}, y_{m_k}\right) d\left(y_{n_k}, y_{n_k-1}\right)}{1 + d\left(y_{m_k}, y_{n_k}\right)}, \\ &\cdot \frac{d\left(y_{n_k}, y_{n_k-1}\right)\left[1 + d\left(y_{m_k}, y_{m_k-1}\right)\right]}{1 + d\left(y_{m_k-1}, y_{n_k-1}\right)} \Bigg\},\end{aligned} \quad (34)$$

which yields that

$$\limsup_{k\longrightarrow+\infty} N\left(x_{m_k}, x_{n_k}\right) \leq \max \left\{ \varepsilon s, 0, 0, \frac{\varepsilon s^2 + \varepsilon}{4s}, 0, 0 \right\} = \varepsilon s. \quad (35)$$

Similarly, we obtain

$$\begin{aligned}M\left(x_{m_k}, x_{n_k}\right) = \max \Bigg\{ &d\left(y_{m_k}, y_{n_k-1}\right), d\left(y_{m_k-1}, y_{n_k-1}\right), d\left(y_{m_k}, y_{m_k-1}\right), d\left(y_{n_k}, y_{n_k-1}\right), \frac{d\left(y_{m_k}, y_{m_k-1}\right)\left[1 + d\left(y_{m_k-1}, y_{n_{k-1}}\right)\right]}{1 + d\left(y_{m_k}, y_{n_k-1}\right)}, \\ &\cdot \frac{d\left(y_{m_k}, y_{m_k-1}\right)\left[1 + d\left(y_{m_k}, y_{m_k-1}\right)\right]}{1 + d\left(y_{m_k}, y_{n_k-1}\right)}, \frac{d\left(y_{m_k}, y_{m_k-1}\right)\left[1 + d\left(y_{n_k}, y_{n_k-1}\right)\right]}{1 + d\left(y_{m_k}, y_{n_k-1}\right)} \Bigg\}.\end{aligned} \quad (36)$$

So there is

$$\liminf_{k\longrightarrow+\infty} M(x_{m_k}, x_{n_k}) \geq \max\left\{\frac{\varepsilon}{s}, \frac{\varepsilon}{s^2}, 0, 0, 0, 0, 0\right\} \geq \frac{\varepsilon}{s^2},$$
$$\liminf_{k\longrightarrow+\infty} M(x_{m_k}, x_{n_k}) \leq \max\{\varepsilon, s\varepsilon, 0, 0, 0, 0, 0\} = s\varepsilon, \tag{37}$$

that is,

$$\frac{\varepsilon}{s^2} \leq \liminf_{k\longrightarrow+\infty} M(x_{m_k}, x_{n_k}) \leq s\varepsilon. \tag{38}$$

Using the transitive property type $s^p$ of $\alpha$, we get

$$\alpha(x_{m_k}, x_{n_k}) \geq s^p. \tag{39}$$

Applying (4) with $x = x_{n_k}$ and $y = x_{m_k}$,we get

$$\begin{aligned}\psi\left(d\left(y_{m_k}, y_{n_k}\right)\right) &\leq \psi\left(s^p d\left(f x_{m_k}, f x_{n_k}\right)\right)\\ &\leq \psi\left(\alpha\left(g x_{m_k}, g x_{n_k}\right) d\left(f x_{m_k}, f x_{n_k}\right)\right)\\ &\leq \psi\left(N\left(x_{m_k}, x_{n_k}\right)\right) - \varphi\left(M\left(x_{m_k}, x_{n_k}\right)\right).\end{aligned} \tag{40}$$

By (35) and (38), we have

$$\begin{aligned}\psi(s\varepsilon) \leq \psi(s^p\varepsilon) &\leq \psi\left(s^p \limsup_{k\longrightarrow+\infty} d\left(f x_{m_k}, f x_{n_k}\right)\right)\\ &\leq \psi\left(\limsup_{k\longrightarrow+\infty} N\left(x_{m_k}, x_{n_k}\right)\right) - \varphi\left(\liminf_{k\longrightarrow+\infty} M\left(x_{m_k}, x_{n_k}\right)\right)\\ &\leq \psi(s\varepsilon) - \varphi\left(\liminf_{k\longrightarrow+\infty} M\left(x_{m_k}, x_{n_k}\right)\right),\end{aligned} \tag{41}$$

which implies that

$$\liminf_{k\longrightarrow+\infty} M(x_{m_k}, x_{n_k}) = 0, \tag{42}$$

a contradiction to (38). Therefore, $\{y_n\}$ is a Cauchy sequence in $X$. The completeness of $X$ ensures that there exists a $u \in X$ such that

$$\lim_{n\longrightarrow+\infty} d(y_n, u) = \lim_{n\longrightarrow+\infty} d(f x_n, u) = \lim_{n\longrightarrow+\infty} d(g x_{n+1}, u) = 0. \tag{43}$$

Since $g(X)$ is closed, we have $u \in g(X)$. It follows that one can choose a $z \in X$ such that $u = gz$, and we can write (43) as

$$\lim_{n\longrightarrow+\infty} d(y_n, gz) = \lim_{n\longrightarrow+\infty} d(f x_n, gz) = \lim_{n\longrightarrow+\infty} d(g x_{n+1}, gz) = 0. \tag{44}$$

The property $(H_{s^p})$ yields that there exists a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ so that $\alpha(y_{n_k-1}, gz) \geq s^p$ for all $k \in N$. If $fz \neq gz$, applying contractive condition (4) with $x = x_{n_k}$ and $y = z$, we have

$$\begin{aligned}\psi\left(d\left(y_{n_k}, fz\right)\right) = \psi\left(d\left(f x_{n_k}, fz\right)\right) &\leq \psi\left(s^p d\left(f x_{n_k}, fz\right)\right)\\ &\leq \psi\left(\alpha\left(g x_{n_k}, gz\right) d\left(f x_{n_k}, fz\right)\right)\\ &\leq \psi\left(N\left(x_{n_k}, z\right)\right) - \varphi\left(M\left(x_{n_k}, z\right)\right),\end{aligned} \tag{45}$$

where

$$\begin{aligned}N(x_{n_k}, z) = \max\Bigg\{ & d\left(y_{n_k-1}, gz\right), d\left(y_{n_k}, y_{n_k-1}\right), d(fz, gz),\\ & \cdot \frac{d\left(y_{n_k-1}, fz\right) + d\left(y_{n_k}, gz\right)}{4s},\\ & \cdot \frac{d\left(y_{n_k}, y_{n_k-1}\right) d(fz, gz)}{1 + d\left(y_{n_k}, fz\right)},\\ & \cdot \frac{d(fz, gz)\left[1 + d\left(y_{n_k}, y_{n_k-1}\right)\right]}{1 + d\left(y_{n_k-1}, gz\right)}\Bigg\},\end{aligned}$$

$$\begin{aligned}M(x_{n_k}, z) = \max\Bigg\{ & d\left(y_{n_k}, gz\right), d\left(y_{n_k-1}, gz\right), d\left(y_{n_k}, y_{n_k-1}\right), d(fz, gz),\\ & \cdot \frac{d\left(y_{n_k}, y_{n_k-1}\right)\left[1 + d\left(y_{n_k-1}, gz\right)\right]}{1 + d\left(y_{n_k}, gz\right)},\\ & \cdot \frac{d\left(y_{n_k}, y_{n_k-1}\right)\left[1 + d\left(y_{n_k}, y_{n_k-1}\right)\right]}{1 + d\left(y_{n_k}, gz\right)},\\ & \cdot \frac{d\left(y_{n_k}, y_{n_k-1}\right)\left[1 + d(fz, gz)\right]}{1 + d\left(y_{n_k}, gz\right)}\Bigg\}.\end{aligned} \tag{46}$$

It is easy to show that

$$\begin{aligned}\limsup_{k\longrightarrow+\infty} N(x_{n_k}, z) &\leq \max\left\{0, 0, d(gz, fz), \frac{sd(fz, gz)}{4s}, 0, d(gz, fz)\right\}\\ &= d(gz, fz),\end{aligned}$$
$$\liminf_{k\longrightarrow+\infty} M(x_{n_k}, z) = \max\{0, 0, 0, d(fz, gz), 0, 0, 0\} = d(fz, gz). \tag{47}$$

Taking the upper limit as $k \longrightarrow +\infty$ in (45), we have

$$\begin{aligned}\psi(d(gz, fz)) \leq \psi\left(s^{p-1} d(gz, fz)\right) &= \psi\left(s^p \frac{1}{s} d(gz, fz)\right)\\ &\leq \psi\left(s^p \limsup_{k\longrightarrow+\infty} d\left(f x_{n_k}, fz\right)\right) \leq \psi\left(\limsup_{k\longrightarrow+\infty} N\left(x_{n_k}, z\right)\right)\\ &- \varphi\left(\liminf_{k\longrightarrow+\infty} M\left(x_{n_k}, z\right)\right) \leq \psi(d(gz, fz)) - \varphi(d(gz, fz)),\end{aligned} \tag{48}$$

which implies that

$$d(fz, gz) = 0. \tag{49}$$

That is, $fz = gz$. Therefore, $u = fz = gz$ is a point of coincidence for $f$ and $g$. By using contractive condition (4) and the property $(U_{s^p})$, one can conclude that the point of coincidence is unique. Assume on the contrary that, there exist $z, z' \in C(f, g)$ and $z \neq z'$. According to the property of $(U_{s^p})$, without loss of generality, we assume that

$$\alpha\left(gz, gz'\right) \geq s^p. \tag{50}$$

Applying (4) with $x = z$ and $y = z'$, we obtain that

$$d\left(fz, fz'\right) = 0, \tag{51}$$

that is, $fz = fz'$. By the weak compatibility of $f$ and $g$, it is easy to show that $z$ is a unique common fixed point. This completes the proof.

*Remark 13.* It is obvious that the mappings defined in Example 11 satisfy the conditions of Theorem 12, so $f$ and $g$ have a unique common fixed point 0.

In Theorem 12, put $\psi(t) = t$, one can get the following result.

**Corollary 14.** *Let $(X, d)$ be a complete $b-$metric space with parameter $s \geq 1$ and let $f, g : X \longrightarrow X$ be given self-mappings on $X$ with $f(X) \subset g(X)$. Also, $g(X)$ is a closed subset of $X$, and $\alpha : X \times X \longrightarrow [0,+\infty)$ is a given mapping. If the following conditions are satisfied:*

*(i) $f$ is a $g - \alpha_{s^p} -$ admissible mapping*

*(ii) there is function $\varphi \in \Phi$ such that*

$$\alpha(gx, gy)d(fx, fy) \leq N(x, y) - \varphi(M(x, y)), \tag{52}$$

*where $N(x, y), M(x, y)$ are same as Theorem 12,*

*(iii) there exists $x_0 \in X$ with $\alpha(gx_0, fx_0) \geq s^p$*

*(iv) properties $(H_{s^p})$ and $(U_{s^p})$ are satisfied*

*(v) $\alpha$ has a transitive property type $s^p$, that is, for $x, y, z \in X$*

$$\alpha(x, y) \geq s^p \text{ and } \alpha(y, z) \geq s^p \Rightarrow \alpha(x, z) \geq s^p. \tag{53}$$

*Then, $f$ and $g$ have a unique point of coincidence in $X$. Moreover, $f$ and $g$ have a unique common fixed point provided that $f$ and $g$ are weakly compatible.*

**Theorem 15.** *Let $(X, d)$ be a complete $b-$metric space with parameter $s \geq 1$, and let $f, g : X \longrightarrow X$ be given self-mappings on $X$ with $f(X) \subset g(X)$. Also, $g(X)$ is a closed subset of $X$, and $\alpha : X \times X \longrightarrow [0,+\infty)$ is a given mapping. Suppose that the following conditions are satisfied:*

*(i) $f$ is a $g - \alpha_{s^p} -$ admissible mapping*

*(ii) there are functions $\psi \in \Psi$ and $\varphi \in \Phi$ such that for all $x, y \in X$*

$$\psi(\alpha(gx, gy)d(fx, fy)) \leq \psi(L(x, y)) - \varphi(M(x, y)), \tag{54}$$

*where $M(x, y)$ is same as Theorem 12 and*

$$L(x, y) = \max\left\{d(fx, gy), d(fx, gx), d(fy, gy), \cdot \frac{d(gx, gy) + d(fx, gy)}{2s}\right\}. \tag{55}$$

*(iii) there exists $x_0 \in X$ with $\alpha(gx_0, fx_0) \geq s^p$*

*(iv) properties $(H_{s^p})$ and $(U_{s^p})$ are satisfied*

*(v) $\alpha$ has a transitive property type $s^p$, that is, for $x, y, z \in X$*

$$\alpha(x, y) \geq s^p \text{ and } \alpha(y, z) \geq s^p \Rightarrow \alpha(x, z) \geq s^p, \tag{56}$$

*then $f$ and $g$ have a unique point of coincidence in $X$. Moreover, $f$ and $g$ have a unique common fixed point provided that $f$ and $g$ are weakly compatible.*

*Proof.* It is the same as the proof of Theorem 12, we also define the sequences $\{x_n\}$ and $\{y_n\}$ in $X$ by $y_n = fx_n = g x_{n+1}$ for $n \in N$ and suppose that $y_n \neq y_{n+1}$ for each $n \in N$, so one can get that

$$\alpha(y_{n-1}, y_n) = \alpha(gx_n, gx_{n+1}) \geq s^p. \tag{57}$$

It follows from (54) that

$$\begin{aligned} \psi(d(y_n, y_{n+1})) &\leq \psi(s^p d(y_n, y_{n+1})) \\ &\leq \psi(\alpha(gx_n, gx_{n+1})d(fx_n, fx_{n+1})) \\ &\leq \psi(L(x_n, x_{n+1})) - \varphi(M(x_n, x_{n+1})), \end{aligned} \tag{58}$$

where

$$\begin{aligned} L(x_n, x_{n+1}) &= \max\left\{d(y_n, y_n), d(y_n, y_{n-1}), d(y_{n+1}, y_n), \cdot \frac{d(y_{n-1}, y_n) + d(y_n, y_n)}{2s}\right\} \\ &\leq \max\left\{d(y_{n+1}, y_n), d(y_n, y_{n-1})\right\}, \end{aligned} \tag{59}$$

$$M(x_n, x_{n+1}) = \max \left\{ d(y_n, y_n), d(y_{n-1}, y_n), d(y_n, y_{n-1}), d(y_{n+1}, y_n), \cdot \frac{d(y_n, y_{n-1})[1 + d(y_{n-1}, y_n)]}{1 + d(y_n, y_n)}, \cdot \frac{d(y_n, y_{n-1})[1 + d(y_n, y_{n-1})]}{1 + d(y_n, y_n)}, \cdot \frac{d(y_n, y_{n-1})[1 + d(y_{n+1}, y_n)]}{1 + d(y_n, y_n)} \right\}. \tag{60}$$

If we assume that, for some $n \in N$

$$d(y_n, y_{n+1}) \geq d(y_{n-1}, y_n) > 0, \tag{61}$$

then according to inequalities (59) and (60), we obtain

$$\begin{aligned} L(x_n, x_{n+1}) &\leq d(y_{n+1}, y_n), \\ M(x_n, x_{n+1}) &\geq d(y_{n+1}, y_n). \end{aligned} \tag{62}$$

In view of (58), we get

$$\begin{aligned} \psi(d(y_n, y_{n+1})) &\leq \psi(L(x_n, x_{n+1})) - \varphi(M(x_n, x_{n+1})) \\ &\leq \psi(d(y_n, y_{n+1})) - \varphi(d(y_n, y_{n+1})), \end{aligned} \tag{63}$$

which implies that $d(y_n, y_{n+1}) = 0$, a contradiction to $d(y_n, y_{n+1}) > 0$. It follows that $d(y_n, y_{n+1}) < d(y_n, y_{n-1})$. Hence, $\{d(y_n, y_{n+1})\}$ is a nonincreasing sequence. Consequently, the limit of the sequence is a nonnegative number, say $r \geq 0$. That is, $\lim_{n \longrightarrow +\infty} d(y_n, y_{n+1}) = r$.

By (59) and (60), we have

$$\begin{aligned} L(x_n, x_{n+1}) &\leq d(y_n, y_{n-1}), \\ M(x_n, x_{n+1}) &\geq d(y_n, y_{n-1}). \end{aligned} \tag{64}$$

So,

$$\begin{aligned} \psi(d(y_n, y_{n+1})) &\leq \psi(L(\mathrm{x}_n, x_{n+1})) - \varphi(M(x_n, x_{n+1})) \\ &\leq \psi(d(y_n, y_{n-1})) - \varphi(d(y_n, y_{n-1})). \end{aligned} \tag{65}$$

If $r > 0$, then letting $n \longrightarrow +\infty$ in above inequality, we obtain that $\psi(r) = \psi(r) - \varphi(r)$, which implies that $r = 0$, i.e.,

$$\lim_{n \longrightarrow +\infty} d(y_n, y_{n+1}) = 0. \tag{66}$$

Now, we prove that $\{y_n\}$ is a Cauchy sequence. If not, as the proof of Theorem 12, there exists $\varepsilon > 0$ for which one can find sequences $\{y_{m_k}\}$ and $\{y_{n_k}\}$ of $\{y_n\}$ so that $n_k$ is the smallest index for which $n_k > m_k > k$, and the following inequalities hold:

$$\varepsilon \leq \limsup_{k \longrightarrow +\infty} d\left(y_{m_k}, y_{n_k}\right) \leq s\varepsilon, \tag{67}$$

$$\frac{\varepsilon}{s} \leq \limsup_{k \longrightarrow +\infty} d\left(y_{m_k}, y_{n_k - 1}\right) \leq \varepsilon, \tag{68}$$

$$\frac{\varepsilon}{s} \leq \limsup_{k \longrightarrow +\infty} d\left(y_{m_k - 1}, y_{n_k}\right) \leq s^2\varepsilon, \tag{69}$$

$$\frac{\varepsilon}{s^2} \leq \limsup_{k \longrightarrow +\infty} d\left(y_{m_k - 1}, y_{n_k - 1}\right) \leq s\varepsilon. \tag{70}$$

In view of the definitions of $L(x, y)$ and $M(x, y)$, we have

$$\begin{gathered} L\left(x_{m_k}, x_{n_k}\right) = \max \left\{ d\left(y_{m_k}, y_{n_k - 1}\right), d\left(y_{m_k - 1}, y_{m_k}\right), d\left(y_{n_k}, y_{n_k - 1}\right), \frac{d\left(y_{m_k - 1}, y_{n_k - 1}\right) + d\left(y_{m_k}, y_{n_k - 1}\right)}{2s} \right\}. \\ M\left(x_{m_k}, x_{n_k}\right) = \max \left\{ d\left(y_{m_k}, y_{n_k - 1}\right), d\left(y_{m_k - 1}, y_{n_k - 1}\right), d\left(y_{m_k - 1}, y_{m_k}\right), d\left(y_{n_k}, y_{n_k - 1}\right), \cdot \frac{d\left(y_{m_k}, y_{m_k - 1}\right)\left[1 + d\left(y_{m_k - 1}, y_{n_k - 1}\right)\right]}{1 + d\left(y_{m_k}, y_{n_k - 1}\right)}, \frac{d\left(y_{m_k}, y_{m_k - 1}\right)\left[1 + d\left(y_{m_k}, y_{m_k - 1}\right)\right]}{1 + d\left(y_{m_k}, y_{n_k - 1}\right)}, \cdot \frac{d\left(y_{m_k}, y_{m_k - 1}\right)\left[1 + d\left(y_{n_k}, y_{n_k - 1}\right)\right]}{1 + d\left(y_{m_k}, y_{n_k - 1}\right)} \right\}. \end{gathered} \tag{71}$$

Letting $k \longrightarrow +\infty$ and using (67)–(70), one can obtain

$$\limsup_{k\longrightarrow+\infty} L\left(x_{m_k}, x_{n_k}\right) \le \max\left\{\varepsilon, 0, 0, \frac{s\varepsilon + s}{2s}\right\} = \varepsilon. \tag{72}$$

Similarly, we get that

$$\begin{aligned}\liminf_{k\longrightarrow+\infty} M\left(x_{m_k}, x_{n_k}\right) &\le \max\{\varepsilon, s\varepsilon, 0, 0, 0, 0, 0\} = s\varepsilon,\\ \liminf_{k\longrightarrow+\infty} M\left(x_{m_k}, x_{n_k}\right) &\ge \max\left\{\frac{\varepsilon}{s}, \frac{\varepsilon}{s^2}, 0, 0, 0, 0, 0\right\} = \frac{\varepsilon}{s^2}.\end{aligned} \tag{73}$$

That is,

$$\frac{\varepsilon}{s^2} \le \liminf_{k\longrightarrow+\infty} M\left(x_{m_k}, x_{n_k}\right) \le s\varepsilon. \tag{74}$$

Using the transitive property type $s^p$ of $\alpha$, we have

$$\alpha\left(x_{m_k}, x_{n_k}\right) \ge s^p. \tag{75}$$

Taking $x = x_{n_k}$ and $y = x_{m_k}$ in (54), one can deduce that

$$\begin{aligned}\psi\left(d\left(y_{m_k}, y_{n_k}\right)\right) &\le \psi\left(s^p d\left(f x_{m_k}, f x_{n_k}\right)\right)\\ &\le \psi\left(\alpha\left(g x_{m_k}, g x_{n_k}\right) d\left(f x_{m_k}, f x_{n_k}\right)\right)\\ &\le \psi\left(L\left(x_{m_k}, x_{n_k}\right)\right) - \varphi\left(M\left(x_{m_k}, x_{n_k}\right)\right).\end{aligned} \tag{76}$$

Therefore,

$$\begin{aligned}\psi(\varepsilon) \le \psi(s^p\varepsilon) &\le \psi\left(s^p \limsup_{k\longrightarrow+\infty} d\left(f x_{m_k}, f x_{n_k}\right)\right)\\ &\le \psi\left(\limsup_{k\longrightarrow+\infty} L\left(x_{m_k}, x_{n_k}\right)\right) - \varphi\left(\liminf_{k\longrightarrow+\infty} M\left(x_{m_k}, x_{n_k}\right)\right)\\ &\le \psi(\varepsilon) - \varphi\left(\liminf_{k\longrightarrow+\infty} M\left(x_{m_k}, x_{n_k}\right)\right).\end{aligned} \tag{77}$$

It follows that $\liminf_{k\longrightarrow+\infty} M(x_{m_k}, x_{n_k}) = 0$, and which gives a contradiction to (74). Hence,

$$\lim_{n,m\longrightarrow+\infty} d(y_n, y_m) = 0. \tag{78}$$

From the completeness of $X$ and the closure of $g(X)$, there exists $u \in X$ such that

$$\lim_{n\longrightarrow+\infty} d(y_n, u) = \lim_{n\longrightarrow+\infty} d(f x_n, u) = \lim_{n\longrightarrow+\infty} d(g x_{n+1}, u) = 0. \tag{79}$$

It follows that one can choose a $z \in X$ such that $u = gz$, and write the above equality as

$$\lim_{n\longrightarrow+\infty} d(y_n, gz) = \lim_{n\longrightarrow+\infty} d(f x_n, gz) = \lim_{n\longrightarrow+\infty} d(g x_{n+1}, gz) = 0. \tag{80}$$

In view of the property $(H_{s^p})$, one can get a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ with $\alpha(y_{n_k-1}, gz) \ge s^p$ for all $k \in N$. If $fz \ne gz$, taking $x = x_{n_k}$ and $y = z$ in contractive condition (54), we have

$$\begin{aligned}\psi\left(d\left(y_{n_k}, fz\right)\right) &= \psi\left(d\left(f x_{n_k}, fz\right)\right) \le \psi\left(s^p d\left(f x_{n_k}, fz\right)\right)\\ &\le \psi\left(\alpha\left(g x_{n_k}, gz\right) d\left(f x_{n_k}, fz\right)\right)\\ &\le \psi\left(L\left(x_{n_k}, z\right)\right) - \varphi\left(M\left(x_{n_k}, z\right)\right),\end{aligned} \tag{81}$$

where

$$\begin{aligned}L\left(x_{n_k}, z\right) &= \max\left\{d\left(y_{n_k}, gz\right), d\left(y_{n_k}, y_{n_k-1}\right), d(fz, gz), \frac{d\left(y_{n_k-1}, gz\right) + d\left(y_{n_k}, gz\right)}{2s}\right\}.\\ M\left(x_{n_k}, z\right) &= \max\left\{d\left(y_{n_k}, gz\right), d\left(y_{n_k-1}, gz\right), d\left(y_{n_k}, y_{n_k-1}\right), d(fz, gz), \frac{d\left(y_{n_k}, y_{n_k-1}\right)\left[1 + d\left(y_{n_k-1}, gz\right)\right]}{1 + d\left(y_{n_k}, gz\right)},\right.\\ &\quad\left.\cdot \frac{d\left(y_{n_k}, y_{n_k-1}\right)\left[1 + d\left(y_{n_k}, y_{n_k-1}\right)\right]}{1 + d\left(y_{n_k}, gz\right)}, \frac{d\left(y_{n_k}, y_{n_k-1}\right)[1 + d(fz, gz)]}{1 + d\left(y_{n_k}, gz\right)}\right\}.\end{aligned} \tag{82}$$

Consequently,

$$\limsup_{k\longrightarrow+\infty} L\left(x_{n_k}, z\right) \leq \max\{0, 0, d(fz, gz), 0\} = d(fz, gz).$$
$$\liminf_{k\longrightarrow+\infty} M\left(x_{n_k}, z\right) = d(fz, gz). \tag{83}$$

Taking the upper limit as $k \longrightarrow +\infty$ in (81), we get

$$\begin{aligned}\psi(d(gz, fz)) &\leq \psi\left(s^{p-1} d(gz, fz)\right) = \psi\left(s^p \frac{1}{s} d(gz, fz)\right) \\ &\leq \psi\left(s^p \limsup_{k\longrightarrow+\infty} d\left(fx_{n_k}, fz\right)\right) \leq \psi\left(\limsup_{k\longrightarrow+\infty} L\left(x_{n_k}, z\right)\right) \\ &\quad - \varphi\left(\liminf_{k\longrightarrow+\infty} M\left(x_{n_k}, z\right)\right) \leq \psi(d(gz, fz)) \\ &\quad - \varphi(d(gz, fz)).\end{aligned} \tag{84}$$

It follows that $d(fz, gz) = 0$. That is, $u = fz = gz$ is a point of coincidence for $f$ and $g$. Using the same technique in the proof of Theorem 12, one can complete the proof.

*Example 16.* Let $X = [0, +\infty)$ and $d(x, y) = (x - y)^2$. Define mappings $f, g : X \longrightarrow X$ by

$$fx = \begin{cases} \dfrac{x}{64}, & x \in [0, 1] \\ e^x - e + \dfrac{1}{2}, & x > 1 \end{cases}, \text{ and } gx = \begin{cases} \dfrac{x}{2}, & x \in [0, 1] \\ e^{2x} - e^2 + \dfrac{1}{2}, & x > 1 \end{cases}. \tag{85}$$

Define mappings $\alpha : g(X) \times g(X) \longrightarrow [0, +\infty)$ by

$$\alpha(x, y) = \begin{cases} s^2, & x, y \in \left[0, \dfrac{1}{2}\right], \\ 0, & \text{otherwise} \end{cases} \tag{86}$$

and $\psi, \varphi : [0, +\infty) \longrightarrow [0, +\infty)$ with $\psi(t) = t, \varphi(t) = 3828t/4805$.

It is clear that $f(X) \subset g(X)$ and $g(X)$ is closed. For $x, y \in X$ such that $\alpha(gx, gy) \geq s^2$, we can know that $gx, gy \in [0, 1/2]$ and which implies that $x, y \in [0, 1]$. It follows that $fx, fy \in [0, 1/2]$ and $\alpha(fx, fy) \geq s^2$, that is, $f$ is a $g - \alpha_{s^p}$ – admissible mapping.

For $x, y \in [0, 1]$, we have

$$\begin{aligned}\psi(\alpha(gx, gy) d(fx, fy)) &= 4 \cdot \left(\frac{x}{64} - \frac{y}{64}\right)^2 \leq \frac{4}{64^2} \max\left\{x^2, y^2\right\}, \\ \psi(L(x, y)) &\geq \psi(\max\{d(fx, gx), d(fy, gy)\}) = \left(\frac{31}{64}\right)^2 \max\left\{x^2, y^2\right\}, \\ \varphi(M(x, y)) &= \varphi\left(\max\left\{\left(\frac{x}{64} - \frac{y}{2}\right)^2, \left(\frac{x}{2} - \frac{y}{2}\right)^2, \left(\frac{x}{64} - \frac{x}{2}\right)^2, \left(\frac{y}{64} - \frac{y}{2}\right)^2, \frac{(x/64 - x/2)^2\left[1 + (x/2 - y/2)^2\right]}{1 + (x/64 - y/2)^2},\right.\right. \\ &\quad \left.\left. \cdot \frac{(x/64 - x/2)^2\left[1 + (x/2 - x/64)^2\right]}{1 + (x/64 - y/2)^2}, \frac{(x/64 - x/2)^2\left[1 + (y/2 - y/64)^2\right]}{1 + (x/64 - y/2)^2}\right\}\right) \\ &\leq \varphi\left(\max\left\{\frac{1}{4}\max\left\{x^2, y^2\right\}, \frac{1}{4}\max\left\{x^2, y^2\right\}, \left(\frac{31}{64}\right)^2 \max\left\{x^2, y^2\right\}, \left(\frac{31}{64}\right)^2 \max\left\{x^2, y^2\right\},\right.\right. \\ &\quad \left.\left. \cdot \left(\frac{31}{64}\right)^2 \max\left\{x^2, y^2\right\} \cdot \frac{5}{4}, \left(\frac{31}{64}\right)^2 \max\left\{x^2, y^2\right\} \cdot \frac{5057}{4096}, \left(\frac{31}{64}\right)^2 \max\left\{x^2, y^2\right\} \cdot \frac{5057}{4096}\right\}\right) \\ &= \frac{3828}{4805} \cdot \left(\frac{31}{64}\right)^2 \max\left\{x^2, y^2\right\} \cdot \frac{5}{4}.\end{aligned} \tag{87}$$

Obviously, we conclude

$$\begin{aligned}\psi(\alpha(gx, gy) d(fx, fy)) &\leq \frac{4}{64^2} \max\left\{x^2, y^2\right\} = \left(\frac{31}{64}\right)^2 \max\left\{x^2, y^2\right\} - \frac{3828}{4805} \cdot \left(\frac{31}{64}\right)^2 \max\left\{x^2, y^2\right\} \cdot \frac{5}{4} \\ &\leq \psi(L(x, y)) - \varphi(M(x, y)).\end{aligned} \tag{88}$$

It follows that all conditions of Theorem 15 are satisfied. It is obvious that 0 is the unique common fixed point of $f$ and $g$.

*Remark 17.* Taking $S = T = I_x$ in Theorem 2.1 of [12], Roshan et al. give that the existence of common fixed point for mappings $f, g$ such that

$$d(fx, gy) \leq \frac{q}{s^4} \max \left\{ d(x, y), d(fx, x), d(gy, y), \frac{1}{2}(d(x, fy) + d(fx, y)) \right\}, \quad (89)$$

where $q \in (0, 1)$ is a constant. Suppose all hypotheses in Example 16 are true. For $x = 0, y \in (0, 1/2)$, it is easy to calculate that

$$\begin{aligned} d(fx, gy) &= \frac{y^2}{4} > \frac{y^2}{16} \geq \frac{q}{16} \max \left\{ y^2, 0, \frac{y^2}{4}, \frac{y^2}{2 \cdot 64^2} + \frac{y^2}{2} \right\} \\ &= \frac{q}{s^4} \max \left\{ d(x, y), d(fx, x), d(gy, y), \frac{1}{2}(d(x, fy) + d(fx, y)) \right\}, \end{aligned} \quad (90)$$

which implies that Theorem 2.1 of [12] cannot be applied to testify the existence of common fixed points of the mappings $f$ and $g$ in $X$.

If $\psi(t) = t$ and $\varphi(t) = t$ in Theorem 15, we get the following result immediately:

**Corollary 18.** *Let $(X, d)$ be a complete $b-$metric space with parameter $s \geq 1$ and let $f, g : X \longrightarrow X$ be given self-mappings on $X$ such that $f(X) \subset g(X)$. Also, $g(X)$ is a closed subset of $X$, and $\alpha : X \times X \longrightarrow [0, +\infty)$ is a given mapping. If the following conditions are satisfied:*

*(i) $f$ is a $g - \alpha_{s^p}$ – admissible mapping,*

*(ii) for $x, y \in X$*

$$\alpha(gx, gy)d(fx, fy) \leq L(x, y) - M(x, y), \quad (91)$$

*where $L(x, y), M(x, y)$ are same as Theorem 15,*

*(iii) there exists $x_0 \in X$ with $\alpha(gx_0, fx_0) \geq s^p$*

*(iv) properties $(H_{s^p})$ and $(U_{s^p})$ are satisfied*

*(v) $\alpha$ has a transitive property type $s^p$, that is, for $x, y, z \in X$*

$$\alpha(x, y) \geq s^p \text{ and } \alpha(y, z) \geq s^p \Rightarrow \alpha(x, z) \geq s^p. \quad (92)$$

*Then, $f$ and $g$ have a unique point of coincidence in $X$. Moreover, if $f$ and $g$ are weakly compatible, then $f$ and $g$ have a unique common fixed point.*

Let $g = I$, $\psi(t) = t$, and $\varphi(t) = Lt$ ($L > 0$ is a constant), we obtain that

**Theorem 19.** *Let $(X, d)$ be a complete $b-$metric space with parameter $s \geq 1$ and $f : X \longrightarrow X$ be a given self-mapping on $X$. Let $\alpha : X \times X \longrightarrow [0, +\infty)$ be a given mapping. If the following conditions are satisfied:*

*(i) $f$ is a $\alpha_{s^p}$ – admissible mapping*

*(ii) for $x, y \in X$*

$$\alpha(x, y)d(fx, fy) \leq (1 - L)K^*(x, y), \quad (93)$$

*where*

$$K^*(x, y) = \max \{d(x, y), d(fx, x), d(fy, y), d(fx, y)\}, L \in (0, 1), \quad (94)$$

*(iii) there exists $x_0 \in X$ with $\alpha(x_0, fx_0) \geq s^p$*

*(iv) properties $(H_{s^p})$ and $(U_{s^p})$ are satisfied when $g = I$*

*(v) $\alpha$ has a transitive property type $s^p$, that is, for $x, y, z \in X$*

$$\alpha(x, y) \geq s^p \text{ and } \alpha(y, z) \geq s^p \Rightarrow \alpha(x, z) \geq s^p. \quad (95)$$

*Then, $f$ has a unique fixed point.*

*Proof.* The proof of Theorem 19 is similar to that of Theorem 15, we omit it.

*Remark 20.* Since a $b-$metric space is a metric space when $s = 1$, so our results can be viewed as the generalization and the extension of comparable results.

## 4. Application

In this section, we will use Theorem 19 to show that there is a solution to the integral equation:

$$x(t) = \int_0^T G(t, r, x(r))dr. \quad (96)$$

Let $X = C([0, T])$ be the set of real continuous functions defined on $[0, T]$. The standard metric given by

$$\rho(x, y) = \sup_{t \in [0,T]} |x(t) - y(t)| \text{for all } x, y \in X. \quad (97)$$

Now for $p \geq 1$, we define

$$d(x, y) = (\rho(x, y))^p = \sup_{t \in [0,T]} |x(t) - y(t)|^p \text{ for all } x, y \in X. \quad (98)$$

It is obvious that $(X, d)$ is a complete $b-$metric space with $s = 2^{p-1}$.

Consider the mapping $f : X \longrightarrow X$ defined by

$$fx(t) = \int_0^T G(t, r, x(r))dr, \tag{99}$$

and let $\xi : R \times R \longrightarrow R$ be a given function.

**Theorem 21.** *Consider equation (96) and suppose that*

*(i)* $G : [0, T] \times [0, T] \times R \longrightarrow R^+$ *is continuous*

*(ii) there exists* $x_0 \in X$ *such that* $\xi(x_0(t), fx_0(t)) \geq 0$ *for all* $t \in [0, T]$

*(iii) for all* $t \in [0, T]$ *and* $x, y \in X$, $\xi(x(t), y(t)) \geq 0$ *implies* $\xi(fx(t), fy(t)) \geq 0$

*(iv) properties* $(H_{s^p})$ *and* $(U_{s^p})$ *are satisfied when* $g = I$

*(v) there exists a continuous function* $\gamma : [0, T] \times [0, T] \longrightarrow R^+$ *such that*

$$\sup_{t\in[0,T]} \int_0^T \gamma(t, r)dr \leq 1, \tag{100}$$

*(vi) there exists a constant* $L \in (0, 1)$ *such that for* $(t, r) \in [0, T] \times [0, T]$

$$|G(t, r, x(r)) - G(t, r, y(r))| \leq \sqrt[p]{\frac{1-L}{s^p}}\gamma(t, r)|x(r) - y(r)|. \tag{101}$$

*Then, the integral equation (96) has a unique solution* $x \in X$.

*Proof.* Define $\alpha : X \times X \longrightarrow [0,+\infty)$ by

$$\alpha(x, y) = \begin{cases} s^p, & \text{if } \xi(x(t), y(t)) \geq 0 \\ 0, & \text{otherwise.} \end{cases} \tag{102}$$

It is easy to prove that $f$ is $\alpha_{s^p}$-admissible. For $x, y \in X$, by virtue of assumptions (1)–(6), we have

$$\begin{aligned} s^p d(fx(t), fy(t)) &= s^p \sup_{t\in[0,T]} |fx(t) - fy(t)|^p \\ &= s^p \sup_{t\in[0,T]} \left| \int_0^T G(t, r, x(r))dr - \int_0^T G(t, r, y(r))dr \right|^p \\ &\leq s^p \sup_{t\in[0,T]} \left( \int_0^T |G(t, r, x(r)) - G(t, r, y(r))|dr \right)^p \\ &\leq s^p \sup_{t\in[0,T]} \left( \int_0^T \sqrt[p]{\frac{1-L}{s^p}}\gamma(t, r)|x(r) - y(r)|dr \right)^p \\ &\leq s^p \sup_{t\in[0,T]} \left( \int_0^T \sqrt[p]{\frac{1-L}{s^p}}\gamma(t, r)dr \right)^p \sup_{t\in[0,T]} |x(t) - y(t)|^p \\ &\leq (1 - L)K^*(x(t), y(t)), \end{aligned} \tag{103}$$

which implies that

$$\alpha(x(t), y(t))d(fx(t), fy(t)) \leq (1 - L)K^*(x(t), y(t)). \tag{104}$$

Therefore, all the conditions of Theorem 19 hold. As a result, the mapping $f$ has a unique fixed point $x \in X$, which is a solution of the integral equation (96).

## 5. Conclusions

In this manuscript, we introduced a new class of $g - \alpha_{s^p} -$ admissible mappings and obtained common fixed point theorems for generalized $(g - \alpha_{s^p}, \psi, \varphi)$ contractive mappings in the framework of $b -$ metric space. Further, we provided examples that elaborated the useability of our results. As an application of our result, we obtained a solution to an integral equation. The obtained results will be helpful for the variational iteration method, so we are going to study this topic in future.

## Data Availability

No data were used to support this study.

## Conflicts of Interest

The authors declare that they have no conflicts of interest regarding the publication of this paper.

## Authors' Contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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